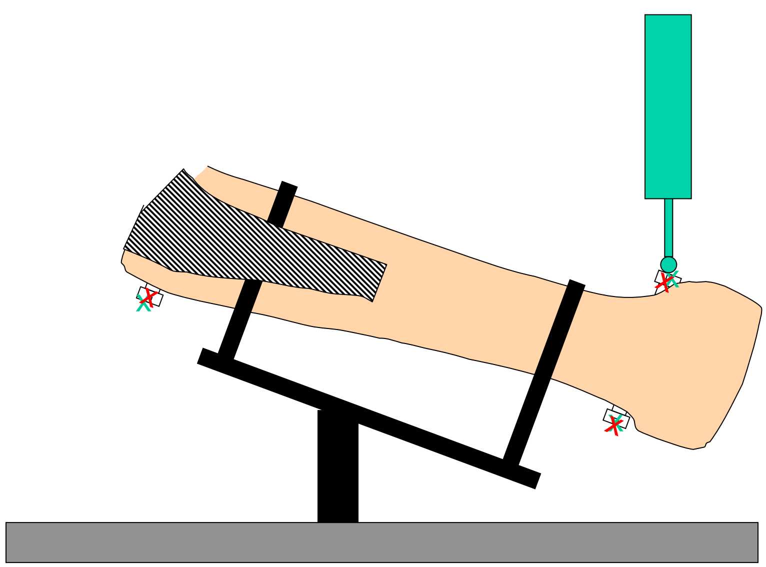
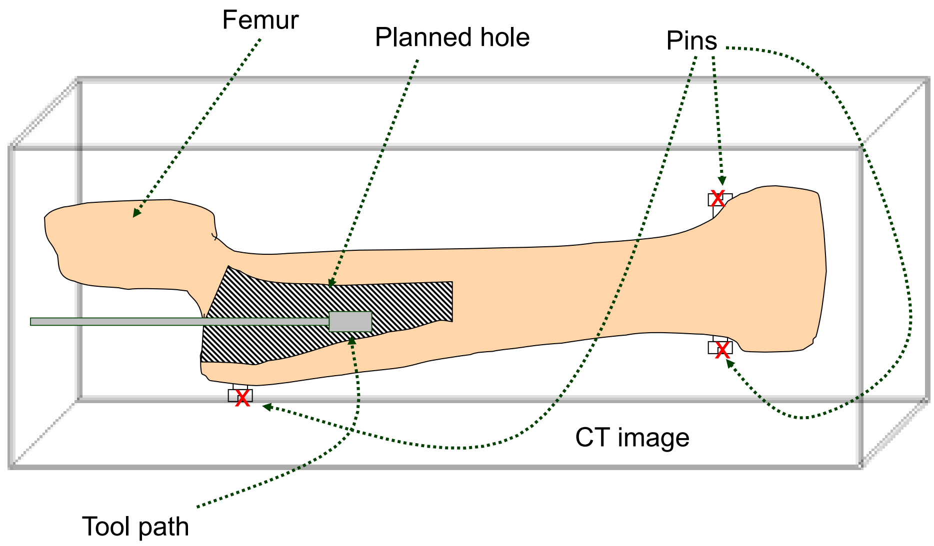
Cartesian Coordinates, Points, and Transformations

CIS - 600.445

Russell Taylor





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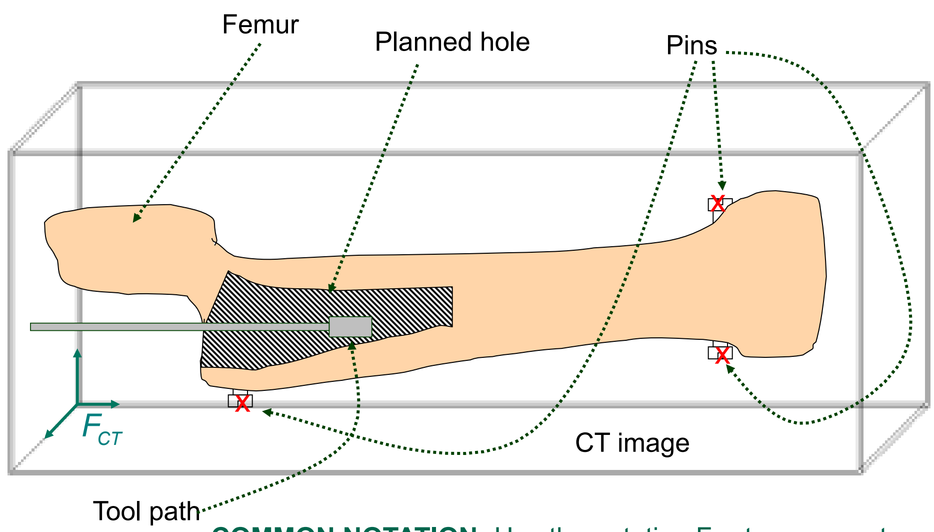

COMMON NOTATION: Use the notation F_{obj} to represent a coordinate system or the position and orientation of an object (relative to some unspecified coordinate system). Use $F_{x,y}$ to mean position and orientation of y relative to x.

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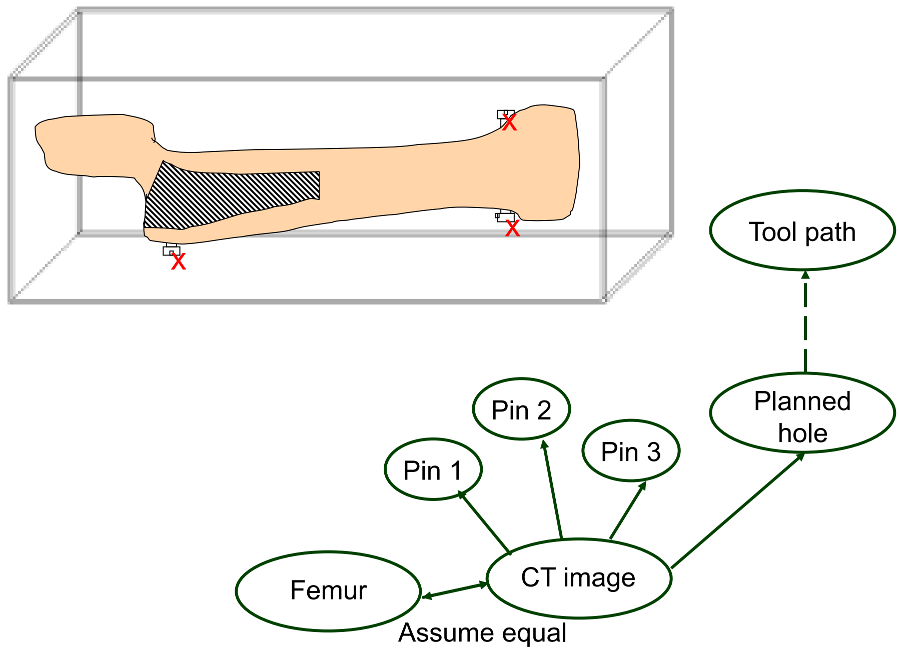
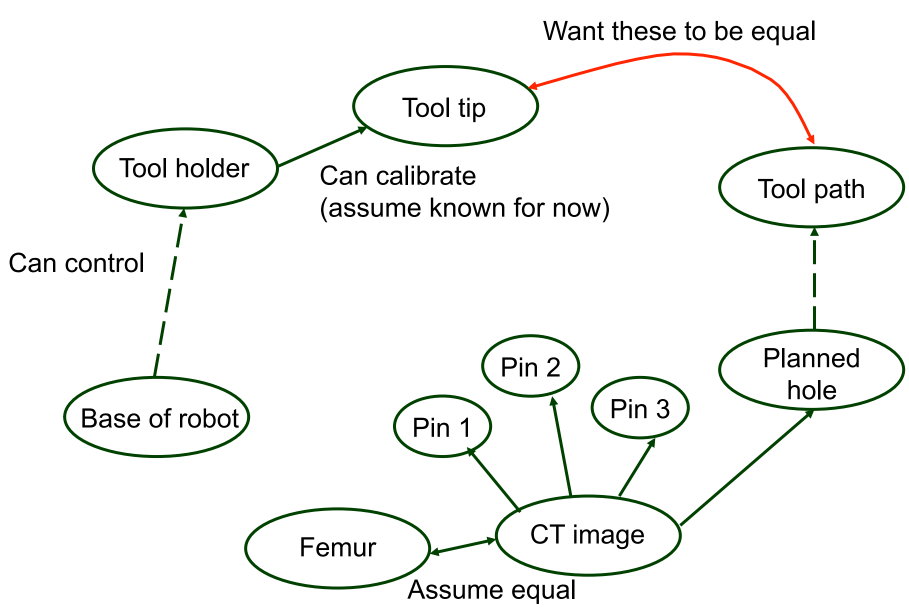
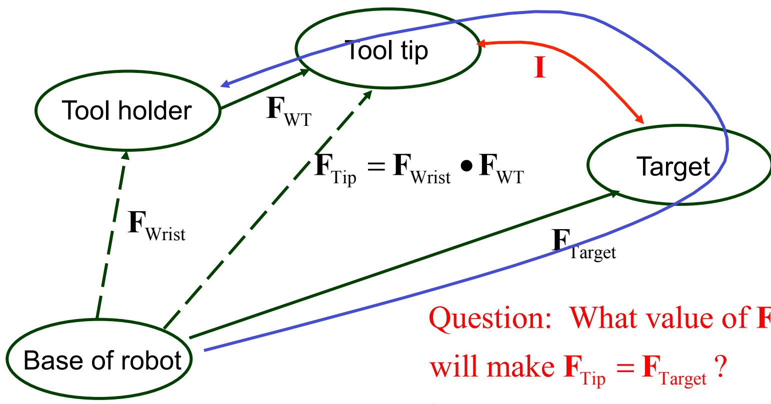


Tool path
Planned hole
Pin 2
Pin 3
Pin 1
CT image
Femur
Assume equal



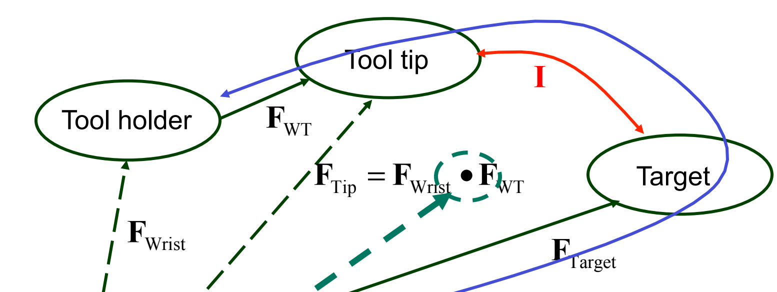
Want these to be equal
Tool tip
Tool holder
Can calibrate
(assume known for now)
Tool path
Can control
Planned hole
Pin 2
Pin 3
Base of robot
Pin 1
CT image
Femur
Assume equal



Question: What value of $\mathbf{F}_{Wrist}$ will make $\mathbf{F}_{Tip} = \mathbf{F}_{Target}$?

Answer:

$$\mathbf{F}_{Wrist} = \mathbf{F}_{Target} \bullet \mathbf{I} \bullet \mathbf{F}_{WT}^{-1}$$
$$= \mathbf{F}_{Target} \bullet \mathbf{F}_{WT}^{-1}$$



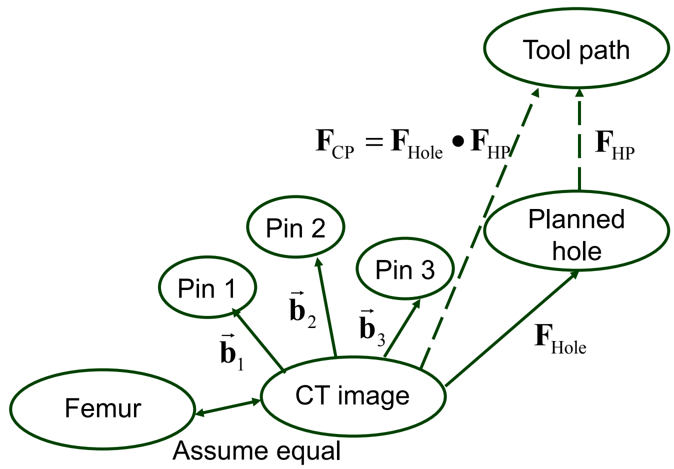
Notational Note

We use the notation $\mathbf{A} \bullet \mathbf{B}$ to represent composition or transformation. Where the context is clear, we may also use **AB** for the same thing.

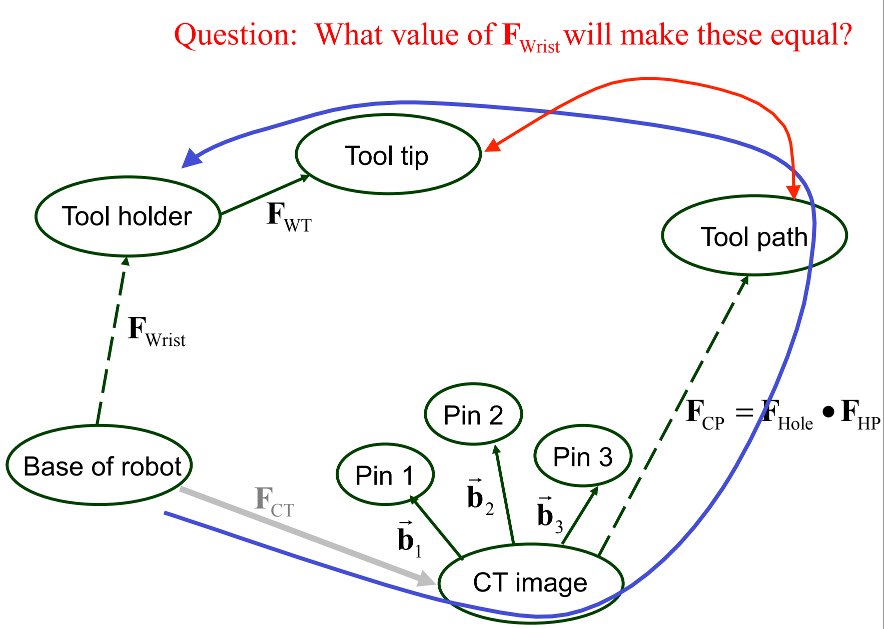
Question: What value of $\mathbf{F}_{Wrist}$ will make $\mathbf{F}_{Tip} = \mathbf{F}_{Target}$?

Answer:

$$\mathbf{F}_{Wrist} = \mathbf{F}_{Target} \bullet \mathbf{I} \bullet \mathbf{F}_{WT}^{-1}$$
$$= \mathbf{F}_{Target} \bullet \mathbf{F}_{WT}^{-1}$$

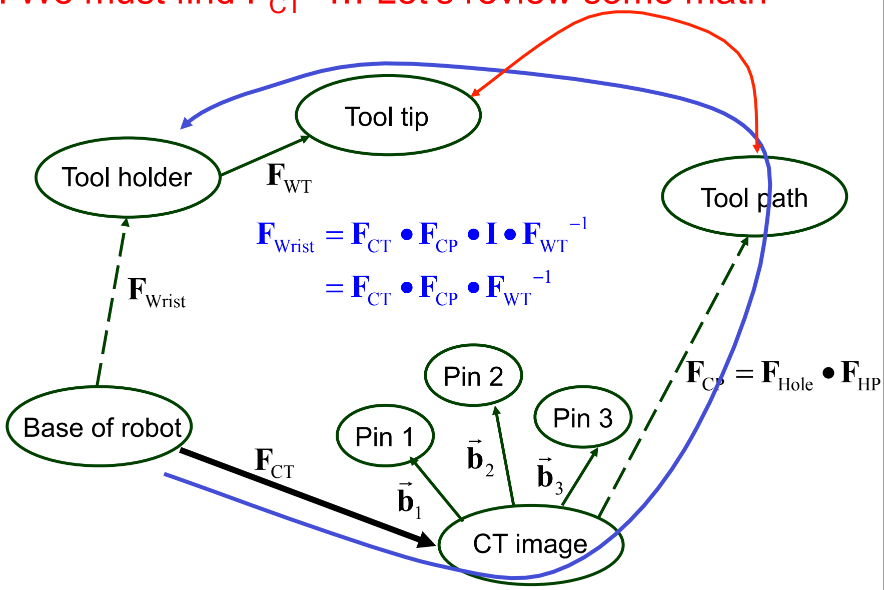


Tool path
$\mathbf{F}_{CP} = \mathbf{F}_{Hole} \bullet \mathbf{F}_{HP}$
$\mathbf{F}_{HP}$
Planned hole
Pin 2
Pin 3
Pin 1
$\vec{\mathbf{b}}_2$
$\vec{\mathbf{b}}_3$
$\vec{\mathbf{b}}_1$
$\mathbf{F}_{Hole}$
Femur
CT image
Assume equal



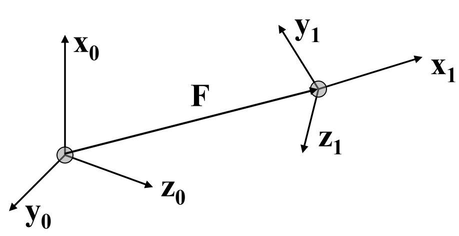
Question: What value of $\mathbf{F}_{Wrist}$ will make these equal?
Tool tip
Tool holder
$\mathbf{F}_{WT}$
Tool path
$\mathbf{F}_{Wrist}$
Pin 2
Pin 3
Pin 1
$\mathbf{F}_{CP} = \mathbf{F}_{Hole} \bullet \mathbf{F}_{HP}$
Base of robot
$\mathbf{F}_{CT}$
$\vec{\mathbf{b}}_2$
$\vec{\mathbf{b}}_3$
$\vec{\mathbf{b}}_1$
CT image

But: We must find F_{CT} … Let's review some math

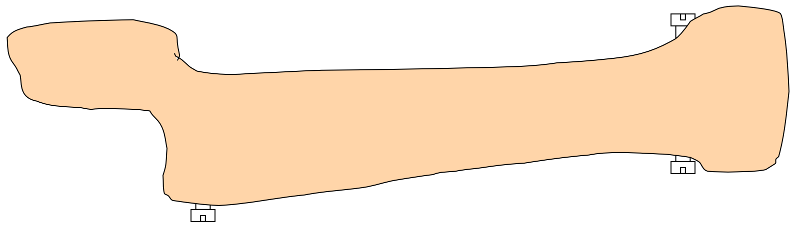


Coordinate Frame Transformation

$$\mathbf{F} = [\mathbf{R}, \mathbf{p}]$$



Slide acknowledgment: Sarah Graham and Andy Bzostek



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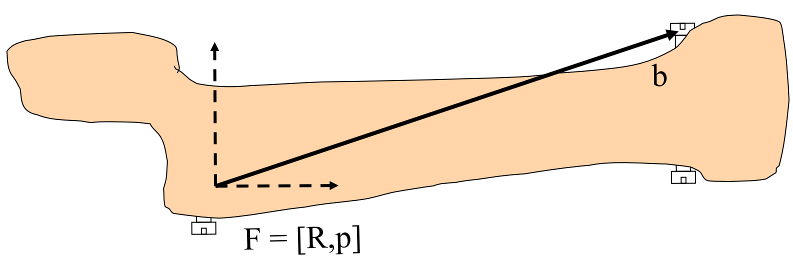

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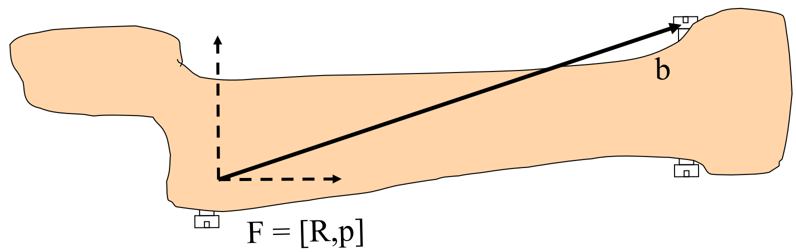
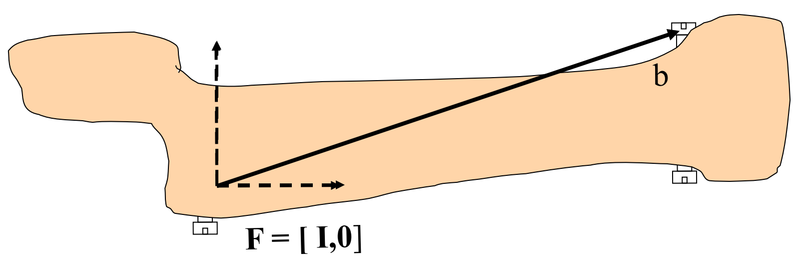
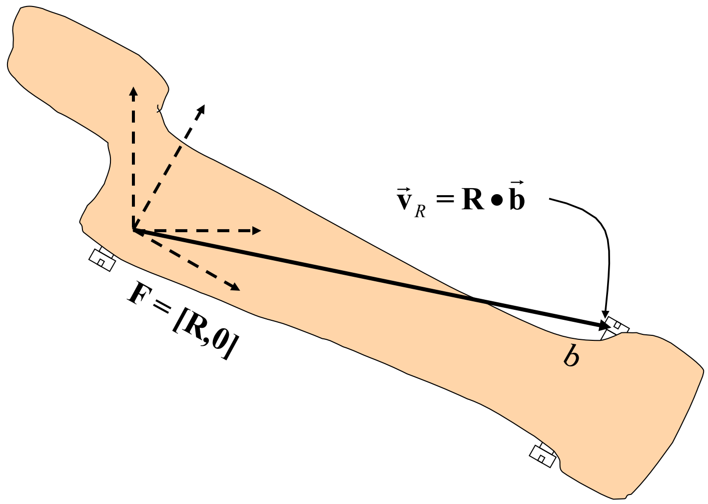
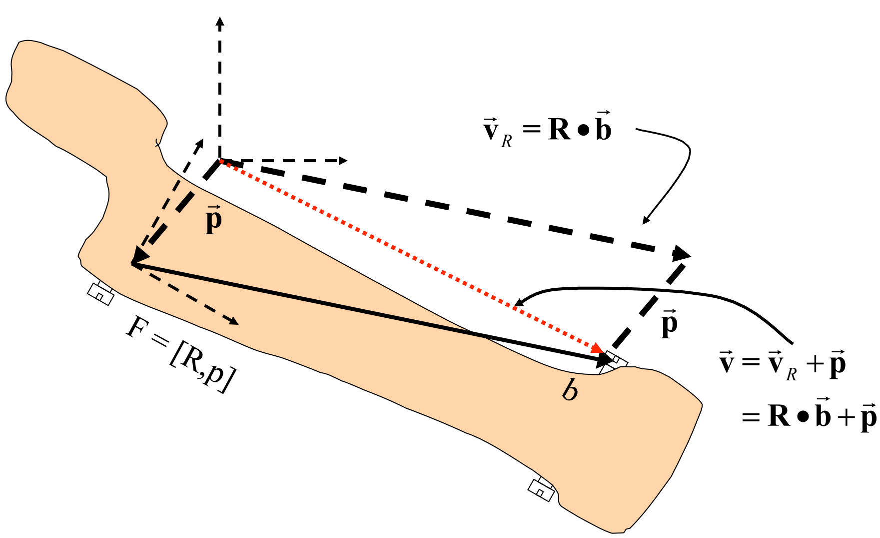

b
F = [R,p]

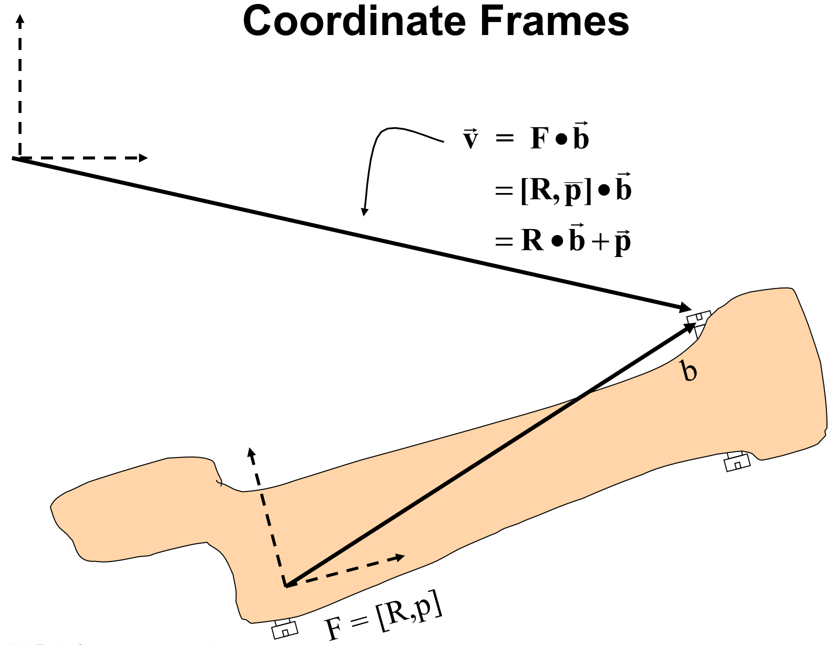

b
F = [I,0]


$\vec{v}_R = \mathbf{R} \bullet \vec{b}$
$F = [R,0]$
b


$\vec{v}_R = \mathbf{R} \bullet \vec{b}$
$\vec{p}$
$\vec{p}$
$F = [R,p]$
b
$\vec{v} = \vec{v}_R + \vec{p}$
$= \mathbf{R} \bullet \vec{b} + \vec{p}$

Coordinate Frames

$$\vec{v} = F \bullet \vec{b}$$
$$= [R, \vec{p}] \bullet \vec{b}$$
$$= R \bullet \vec{b} + \vec{p}$$



Forward and Inverse Frame Transformations

Forward

$$F = [R, p]$$

$$v = F \bullet b$$
$$= [R, p] \bullet b$$
$$= R \bullet b + p$$

Inverse

$$F^{-1} v = b$$
$$b = R^{-1} \bullet (v - p)$$
$$= R^{-1} \bullet v - R^{-1} \bullet p$$

$$F^{-1} = [R^{-1}, -R^{-1} \bullet p]$$

Composition

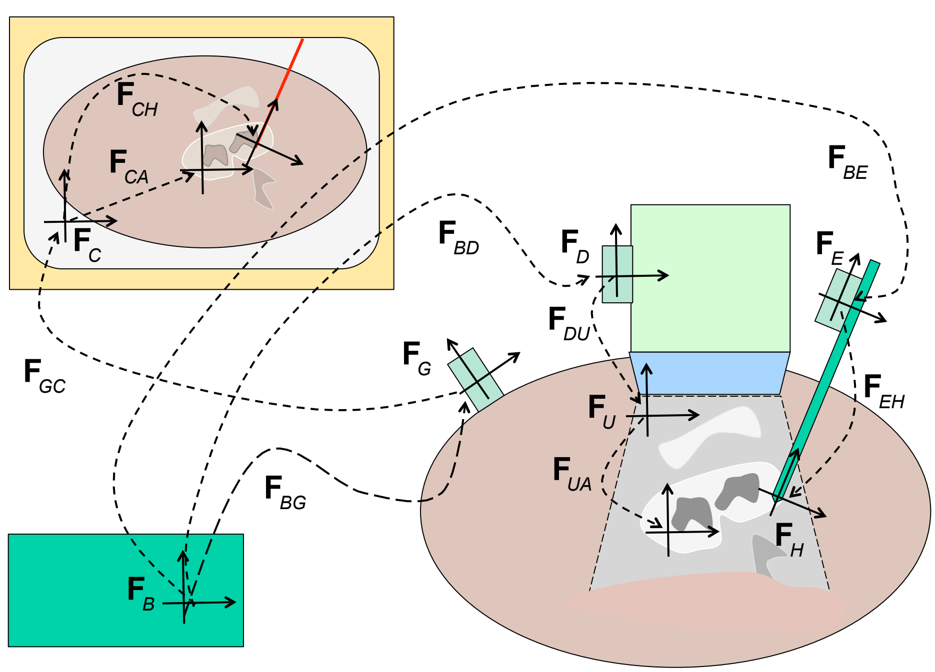
Assume $\mathbf{F}_1 = [\mathbf{R}_1, \vec{\mathbf{p}}_1], \quad \mathbf{F}_2 = [\mathbf{R}_2, \vec{\mathbf{p}}_2]$

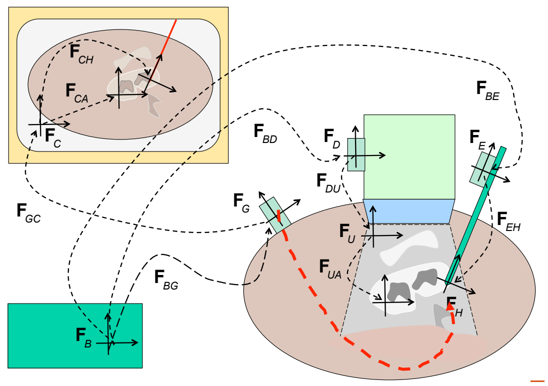
Then

$$\begin{aligned}
\mathbf{F}_1 \bullet \mathbf{F}_2 \bullet \vec{\mathbf{b}} &= \mathbf{F}_1 \bullet (\mathbf{F}_2 \bullet \vec{\mathbf{b}}) \\
&= \mathbf{F}_1 \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\
&= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\
&= \mathbf{R}_1 \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) + \vec{\mathbf{p}}_1 \\
&= \mathbf{R}_1 \bullet \mathbf{R}_2 \bullet \vec{\mathbf{b}} + \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1 \\
&= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1] \bullet \vec{\mathbf{b}}
\end{aligned}$$

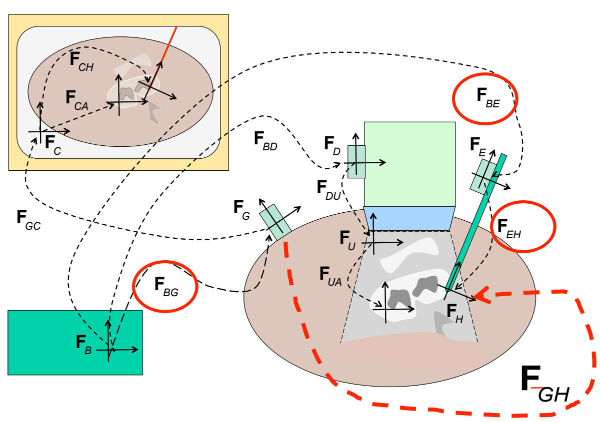
So

$$\begin{aligned}
\mathbf{F}_1 \bullet \mathbf{F}_2 &= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet [\mathbf{R}_2, \vec{\mathbf{p}}_2] \\
&= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1]
\end{aligned}$$





Give a formula for computing the pose $\mathbf{F}_{GH}$
of the surgical tool coordinate system
relative to the patient rigid body coordinate system $\mathbf{F}_G$

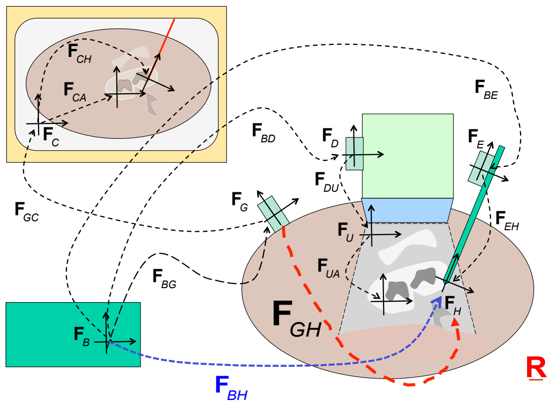


$$\mathbf{F}_{GH} = \mathbf{F}_{BG}^{-1} \cdots$$

$$\mathbf{F}_{GH} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE} \cdots$$

$$\mathbf{F}_{GH} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\mathbf{F}_{EH}$$

Give a formula for computing the pose $\mathbf{F}_{GH}$
of the surgical tool coordinate system
relative to the patient rigid body coordinate system $\mathbf{F}_G$



$$\mathbf{F}_{GH} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\mathbf{F}_{EH} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BH}$$

$$\mathbf{R}_{GH} = \mathbf{R}_{BG}^{-1}\mathbf{R}_{BE}\mathbf{R}_{EH}$$

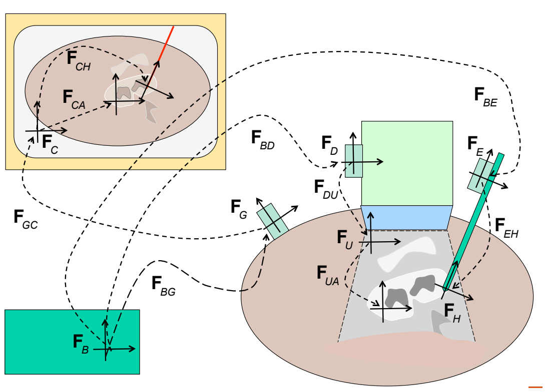
$$\vec{\mathbf{p}}_{GH} = \mathbf{F}_{BG}^{-1}\vec{\mathbf{p}}_{BH} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\vec{\mathbf{p}}_{EH}$$

$$\vec{\mathbf{p}}_{GH} = \mathbf{F}_{BG}^{-1}(\mathbf{R}_{BE}\vec{\mathbf{p}}_{EH} + \vec{\mathbf{p}}_{BE})$$

$$\vec{\mathbf{p}}_{GH} = \mathbf{R}_{BG}^{-1}(\mathbf{R}_{BE}\vec{\mathbf{p}}_{EH} + \vec{\mathbf{p}}_{BE} - \vec{\mathbf{p}}_{BG})$$

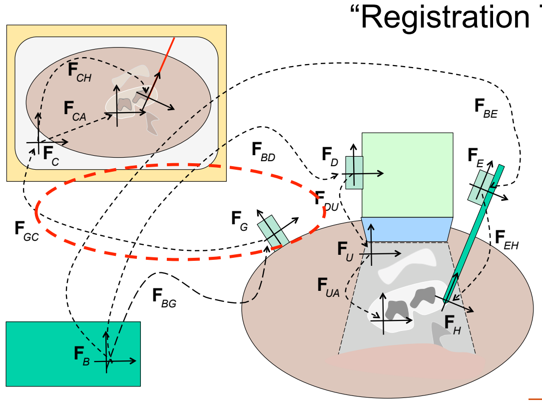
Give a formula for computing the pose $\mathbf{F}_{GH}$
of the surgical tool coordinate system
relative to the patient rigid body coordinate system $\mathbf{F}_{G}$?

What are the components $\mathbf{F}_{GH} = [\mathbf{R}_{GH}, \vec{\mathbf{p}}_{GH}]$?



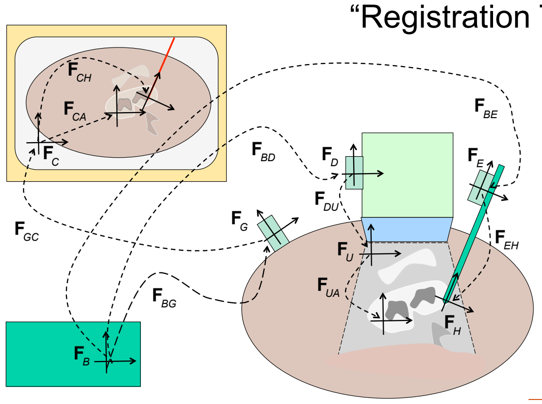
If an anatomic structure is identified at pose $\mathbf{F}_{UA}$
in ultrasound image coordinates
give the formula for computing the corresponding pose
$\mathbf{F}_{CA}$ in CT coordinates

“Registration Transformations”



Given a coordinate system $\mathbf{F}_C$ and another coordinate system $\mathbf{F}_G$ (e.g., a CT scan and a tracked "rigid body" attached to the patient, and points $\vec{\mathbf{c}}_i$ in the coordinate system $\mathbf{F}_C$ and points $\vec{\mathbf{g}}_i$ in the coordinate system $\mathbf{F}_G$, then the "registration transformation" $\mathbf{F}_{GC}$ between $\mathbf{F}_G$ and $\mathbf{F}_C$ one in which for $\mathbf{F}_{GC}\vec{\mathbf{c}}_i = \vec{\mathbf{g}}_i$ if and only if $\vec{\mathbf{c}}_i$ and $\vec{\mathbf{g}}_i$ refer to the same or corresponding points.

“Registration Transformations”



For all points $\vec{\mathbf{a}}_i$ defined relative to the anatomic structure's coordinate system $\mathbf{F}_{CA}\vec{\mathbf{a}}_i$ corresponds to $\mathbf{F}_{UA}\vec{\mathbf{a}}_i$. So ...

$$\mathbf{F}_B\mathbf{F}_{BG}\mathbf{F}_{GC}\mathbf{F}_{CA}\vec{\mathbf{a}}_i = \mathbf{F}_B\mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}\vec{\mathbf{a}}_i$$

$$\begin{aligned}\mathbf{F}_{CA}\vec{\mathbf{a}}_i &= \left(\mathbf{F}_B\mathbf{F}_{BG}\mathbf{F}_{GC}\right)^{-1}\mathbf{F}_B\mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}\vec{\mathbf{a}}_i \\ &= \mathbf{F}_{GC}^{-1}\mathbf{F}_{BG}^{-1}\mathbf{F}_B^{-1}\mathbf{F}_B\mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}\vec{\mathbf{a}}_i \\ &= \mathbf{F}_{GC}^{-1}\mathbf{F}_{BG}^{-1}\mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}\vec{\mathbf{a}}_i \\ &= \left(\mathbf{F}_{BG}\mathbf{F}_{GC}\right)^{-1}\mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}\vec{\mathbf{a}}_i\end{aligned}$$

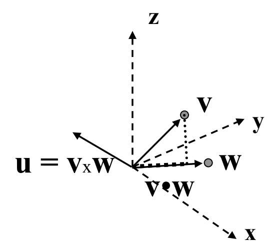
If an anatomic structure is identified at pose $\mathbf{F}_{UA}$ in ultrasound image coordinates, compute the corresponding pose $\mathbf{F}_{CA}$ in CT coordinates.

$$\begin{aligned}\mathbf{F}_{CA} &= \left(\mathbf{F}_{BG}\mathbf{F}_{GC}\right)^{-1}\mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA} \\ &= \mathbf{F}_{GC}^{-1}\mathbf{F}_{BG}^{-1}\mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}\end{aligned}$$

Vectors

$$\mathbf{v}_{col} = \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ \mathbf{v}_z \end{bmatrix}$$

$$\mathbf{v}_{row} = \begin{bmatrix} \mathbf{v}_x & \mathbf{v}_y & \mathbf{v}_z \end{bmatrix}$$



$$\text{length}: \|\mathbf{v}\| = \sqrt{\mathbf{v}_x{}^2 + \mathbf{v}_y{}^2 + \mathbf{v}_z{}^2}$$

$$\text{dot product}: a = \mathbf{v} \cdot \mathbf{w} = \left(\mathbf{v}_x\mathbf{w}_x + \mathbf{v}_y\mathbf{w}_y + \mathbf{v}_z\mathbf{w}_z\right) = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

$$\text{cross product}: \mathbf{u} = \mathbf{v} \times \mathbf{w} = \begin{bmatrix} \mathbf{v}_y\mathbf{w}_z - \mathbf{v}_z\mathbf{w}_y \\ \mathbf{v}_z\mathbf{w}_x - \mathbf{v}_x\mathbf{w}_z \\ \mathbf{v}_x\mathbf{w}_y - \mathbf{v}_y\mathbf{w}_x \end{bmatrix}, \|\mathbf{u}\| = \|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$$

Slide acknowledgment: Sarah Graham and Andy Bzostek

Matrix representation of cross product operator

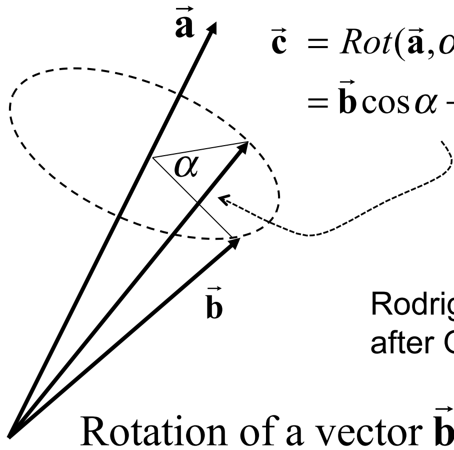
Define

$$\hat{\vec{\mathbf{a}}} \overset{\Delta}{=} skew(\vec{\mathbf{a}}) \overset{\Delta}{=} \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Then

$$\vec{\mathbf{a}} \times \vec{\mathbf{v}} = skew(\vec{\mathbf{a}}) \bullet \vec{\mathbf{v}}$$

Axis-angle Representations of Rotations



$$\vec{\mathbf{c}} = Rot(\vec{\mathbf{a}},\alpha)\bullet\vec{\mathbf{b}}$$
$$= \vec{\mathbf{b}}\cos\alpha + \vec{\mathbf{a}}\times\vec{\mathbf{b}}\sin\alpha + \vec{\mathbf{a}}(\vec{\mathbf{a}}\bullet\vec{\mathbf{b}})\left(1-\cos\alpha\right)$$

Rodrigues' rotation formula (named after Olinde Rodrigues)

Rotation of a vector $\vec{\mathbf{b}}$ by angle α about axis $\vec{\mathbf{a}}$

(Assumes that $\vec{\mathbf{a}}$ is a unit vector, $\left\|\vec{\mathbf{a}}\right\| = 1$)

Exponential representation

Consider a rotation about axis $\vec{\mathbf{n}}$ by angle θ. Then

$$e^{skew(\vec{\mathbf{n}})\theta} = \mathbf{I} + \theta skew(\vec{\mathbf{n}}) + \frac{\theta^2}{2!}skew(\vec{\mathbf{n}})^2 + \ldots$$

By doing some manipulation, you can show

$$\begin{aligned} Rot(\vec{\mathbf{n}},\theta) &= e^{skew(\vec{\mathbf{n}})\theta} \\ &= \mathbf{I} + skew(\vec{\mathbf{n}})\sin\theta + skew(\vec{\mathbf{n}})^2(1-\cos\theta) \\ &= \mathbf{I} + skew(\vec{\mathbf{n}})\sin\theta + (\vec{\mathbf{n}}\bullet\vec{\mathbf{n}}^T - \mathbf{I})(1-\cos\theta) \\ &= \mathbf{I}\cos\theta + skew(\vec{\mathbf{n}})\sin\theta + \vec{\mathbf{n}}\bullet\vec{\mathbf{n}}^T(1-\cos\theta) \end{aligned}$$

Note that for small θ, this reduces to

$$Rot(\vec{\mathbf{n}},\theta) \approx \mathbf{I} + skew(\theta\vec{\mathbf{n}})$$

Rotations: Some Notation

$$Rot(\vec{\mathbf{a}},\alpha) = \text{ Rotation by angle } \alpha \text{ about axis } \vec{\mathbf{a}}$$

$$\mathbf{R}_{\vec{\mathbf{a}}}(\alpha) = \text{ Rotation by angle } \alpha \text{ about axis } \vec{\mathbf{a}}$$

$$\mathbf{R}(\vec{\mathbf{a}}) = Rot(\vec{\mathbf{a}},\|\vec{\mathbf{a}}\|)$$

$$\mathbf{R}_{\mathbf{xyz}}(\alpha,\beta,\gamma) = \mathbf{R}(\vec{\mathbf{x}},\alpha) \bullet \mathbf{R}(\vec{\mathbf{y}},\beta) \bullet \mathbf{R}(\vec{\mathbf{z}},\gamma)$$

$$\mathbf{R}_{\mathbf{zyz}}(\alpha,\beta,\gamma) = \mathbf{R}(\vec{\mathbf{z}},\alpha) \bullet \mathbf{R}(\vec{\mathbf{y}},\beta) \bullet \mathbf{R}(\vec{\mathbf{z}},\gamma)$$

Rotations: A few useful facts

$$Rot(s\vec{\mathbf{a}},\alpha) \bullet \vec{\mathbf{a}} = \vec{\mathbf{a}} \quad \text{and} \quad \left\|Rot(\vec{\mathbf{a}},\alpha) \bullet \vec{\mathbf{b}}\right\| = \left\|\vec{\mathbf{b}}\right\|$$

$$Rot(\vec{\mathbf{a}},\alpha) = Rot(\hat{\mathbf{a}},\alpha) \text{ where } \hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|}$$

$$Rot(\hat{\mathbf{a}},\alpha) \bullet Rot(\hat{\mathbf{a}},\beta) = Rot(\hat{\mathbf{a}},\alpha+\beta)$$

$$Rot(\hat{\mathbf{a}},\alpha)^{-1} = Rot(\hat{\mathbf{a}},-\alpha)$$

$$Rot(\vec{\mathbf{a}},0) \bullet \vec{\mathbf{b}} = \vec{\mathbf{b}} \quad \text{i.e., } Rot(\vec{\mathbf{a}},0) = \mathbf{I}_{Rot} = \text{ the identity rotation}$$

$$Rot(\hat{\mathbf{a}},\alpha) \bullet \vec{\mathbf{b}} = \left(\hat{\mathbf{a}} \bullet \vec{\mathbf{b}}\right)\hat{\mathbf{a}} + Rot(\hat{\mathbf{a}},\alpha) \bullet \left(\vec{\mathbf{b}} - \left(\hat{\mathbf{a}} \bullet \vec{\mathbf{b}}\right)\hat{\mathbf{a}}\right)$$

$$Rot(\hat{\mathbf{a}},\alpha) \bullet Rot(\hat{\mathbf{b}},\beta) = Rot(\hat{\mathbf{b}},\beta) \bullet Rot(Rot(\hat{\mathbf{b}},-\beta) \bullet \hat{\mathbf{a}},\alpha)$$

$$Rot(\hat{\mathbf{a}},\alpha) \bullet \mathbf{R}_{\beta} = \mathbf{R}_{\beta} \bullet Rot(\mathbf{R}_{\beta}^{-1} \bullet \hat{\mathbf{a}},\alpha)$$

$$\mathbf{R}_{\alpha} \bullet Rot(\hat{\mathbf{b}},\beta) = Rot(\mathbf{R}_{\alpha} \bullet \hat{\mathbf{b}},\beta) \bullet \mathbf{R}_{\alpha}$$

Rotations: more facts

If $\vec{\mathbf{v}} = [v_x, v_y, v_z]^T$ then a rotation $\mathbf{R} \bullet \vec{\mathbf{v}}$ may be described in terms of the effects of $\mathbf{R}$ on orthogonal unit vectors, $\vec{\mathbf{e}}_x = [1,0,0]^T$, $\vec{\mathbf{e}}_y = [0,1,0]^T$, $\vec{\mathbf{e}}_z = [0,0,1]^T$

$$\mathbf{R} \bullet \vec{\mathbf{v}} = v_x \vec{\mathbf{r}}_x + v_y \vec{\mathbf{r}}_y + v_z \vec{\mathbf{r}}_z$$

where

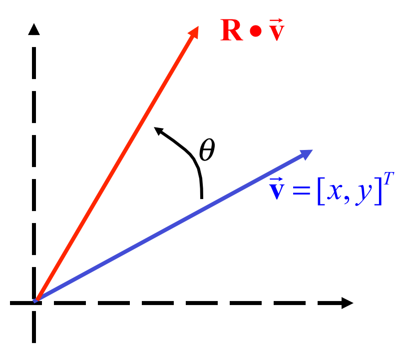
$$\vec{\mathbf{r}}_x = \mathbf{R} \bullet \vec{\mathbf{e}}_x$$
$$\vec{\mathbf{r}}_y = \mathbf{R} \bullet \vec{\mathbf{e}}_y$$
$$\vec{\mathbf{r}}_z = \mathbf{R} \bullet \vec{\mathbf{e}}_z$$

Note that rotation doesn't affect inner products

$$\left(\mathbf{R} \bullet \vec{\mathbf{b}}\right) \bullet \left(\mathbf{R} \bullet \vec{\mathbf{c}}\right) = \vec{\mathbf{b}} \bullet \vec{\mathbf{c}}$$

Rotations in the plane

$$\mathbf{R} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix}$$

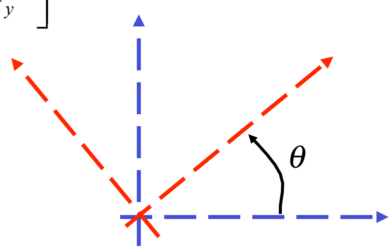


Rotations in the plane

$$\mathbf{R} \bullet \begin{bmatrix} \vec{\mathbf{e}}_x & \vec{\mathbf{e}}_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R} \bullet \vec{\mathbf{e}}_x & \mathbf{R} \bullet \vec{\mathbf{e}}_y \end{bmatrix}$$

$$= \begin{bmatrix} \vec{\mathbf{r}}_x & \vec{\mathbf{r}}_y \end{bmatrix}$$



3D Rotation Matrices

$$\mathbf{R} \bullet \begin{bmatrix} \vec{\mathbf{e}}_x & \vec{\mathbf{e}}_y & \vec{\mathbf{e}}_z \end{bmatrix} = \begin{bmatrix} \mathbf{R} \bullet \vec{\mathbf{e}}_x & \mathbf{R} \bullet \vec{\mathbf{e}}_y & \mathbf{R} \bullet \vec{\mathbf{e}}_z \end{bmatrix}$$

$$= \begin{bmatrix} \vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_z \end{bmatrix}$$

$$\mathbf{R}^T \bullet \mathbf{R} = \begin{bmatrix} \hat{\mathbf{r}}_{\mathbf{x}}^{\;T} \\ \hat{\mathbf{r}}_y^{\;T} \\ \hat{\mathbf{r}}_z \end{bmatrix} \bullet \begin{bmatrix} \vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_z \end{bmatrix}$$

$$= \begin{bmatrix} \vec{\mathbf{r}}_{\mathbf{x}}^{\;T} \bullet \vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_{\mathbf{x}}^{\;T} \bullet \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_{\mathbf{x}}^{\;T} \bullet \vec{\mathbf{r}}_z \\ \vec{\mathbf{r}}_y^{\;T} \bullet \vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_y^{\;T} \bullet \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_y^{\;T} \bullet \vec{\mathbf{r}}_z \\ \vec{\mathbf{r}}_z^{\;T} \bullet \vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_z^{\;T} \bullet \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_z^{\;T} \bullet \vec{\mathbf{r}}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of Rotation Matrices

Inverse of a Rotation Matrix equals its transpose:

$$\mathbf{R}^{-1} = \mathbf{R}^{T}$$

$$\mathbf{R}^{T}\mathbf{R} = \mathbf{R}\,\mathbf{R}^{T} = \mathbf{I}$$

The Determinant of a Rotation matrix is equal to +1:

$$\mathbf{det(R) = +1}$$

Any Rotation can be described by consecutive rotations about the three primary axes, x, y, and z:

$$\mathbf{R} = \mathbf{R}_{z,\theta}\,\mathbf{R}_{y,\varphi}\,\mathbf{R}_{x,\psi}$$

Canonical 3D Rotation Matrices

Note: **Right-Handed Coordinate System**

$$\mathbf{R}_{\vec{\mathbf{x}}}(\theta) = Rot(\vec{\mathbf{x}},\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\vec{\mathbf{y}}}(\theta) = Rot(\vec{\mathbf{y}},\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\vec{\mathbf{z}}}(\theta) = Rot(\vec{\mathbf{z}},\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Coordinates

- Widely used in graphics, geometric calculations
- Represent 3D vector as 4D quantity
- For our current purposes, we will keep the "scale" *s = 1*

$$\vec{\mathbf{v}} \equiv \begin{bmatrix} xs \\ ys \\ zs \\ s \end{bmatrix} \cong \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

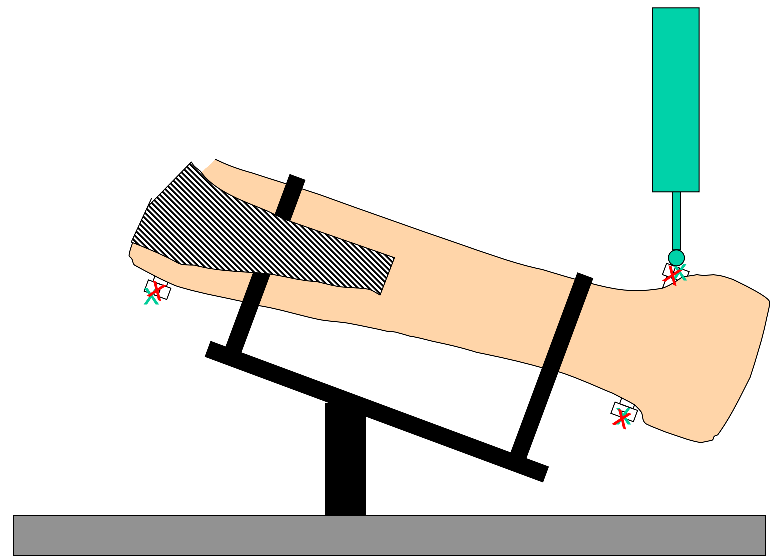
Representing Frame Transformations as Matrices

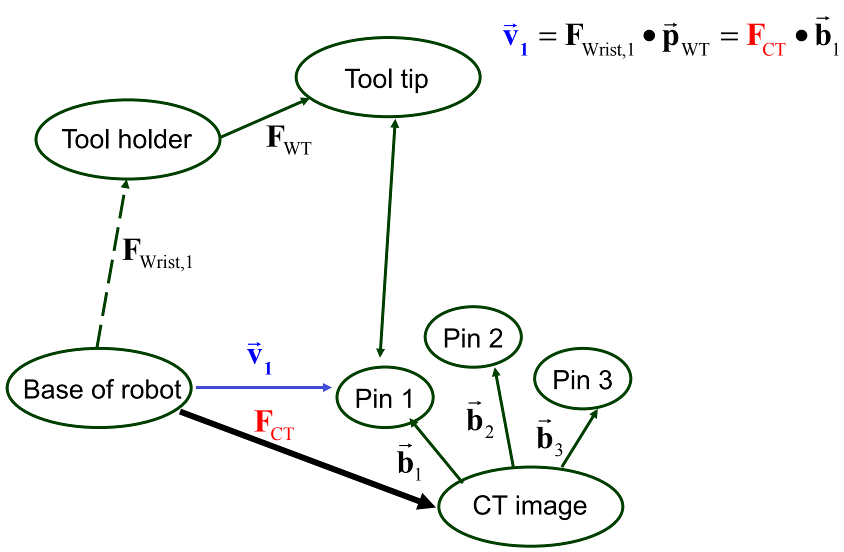
$$\mathbf{v}+\mathbf{p} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \mathbf{p}_x \\ 0 & 1 & 0 & \mathbf{p}_y \\ 0 & 0 & 1 & \mathbf{p}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ \mathbf{v}_z \\ 1 \end{bmatrix} = \mathbf{P} \bullet \mathbf{v}$$

$$\mathbf{R} \bullet \mathbf{v} \rightarrow \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}$$

$$\mathbf{P} \bullet \mathbf{R} \rightarrow \begin{bmatrix} \mathbf{I} & \mathbf{p} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \bullet \begin{bmatrix} \mathbf{R} & 0 \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = [\mathbf{R}, \mathbf{p}] = \mathbf{F}$$

$$\mathbf{F} \bullet \mathbf{v} \rightarrow \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = \begin{bmatrix} (\mathbf{R} \bullet \mathbf{v}) + \mathbf{p} \\ 1 \end{bmatrix}$$

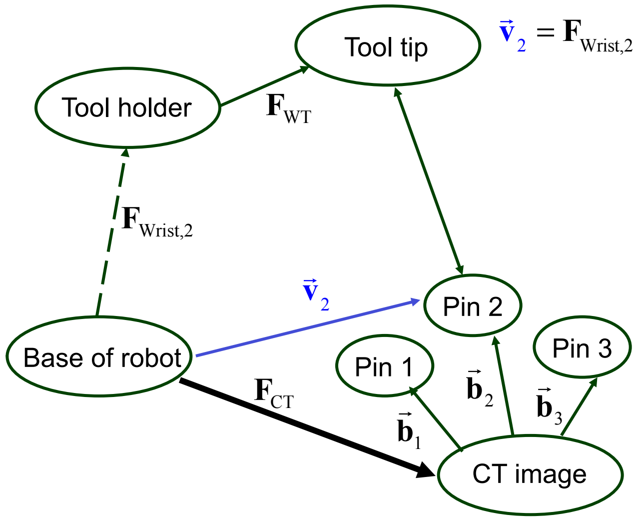




$\vec{v}_1 = F_{Wrist,1} \bullet \vec{p}_{WT} = F_{CT} \bullet \vec{b}_1$
Tool tip
Tool holder
F_{WT}
$F_{Wrist,1}$
Pin 2
Pin 3
$\vec{v}_1$
Base of robot
Pin 1
F_{CT}
$\vec{b}_2$
$\vec{b}_3$
$\vec{b}_1$
CT image

$$\vec{v}_1 = \mathbf{F}_{Wrist,1} \bullet \vec{p}_{WT} = \mathbf{F}_{CT} \bullet \vec{b}_1$$

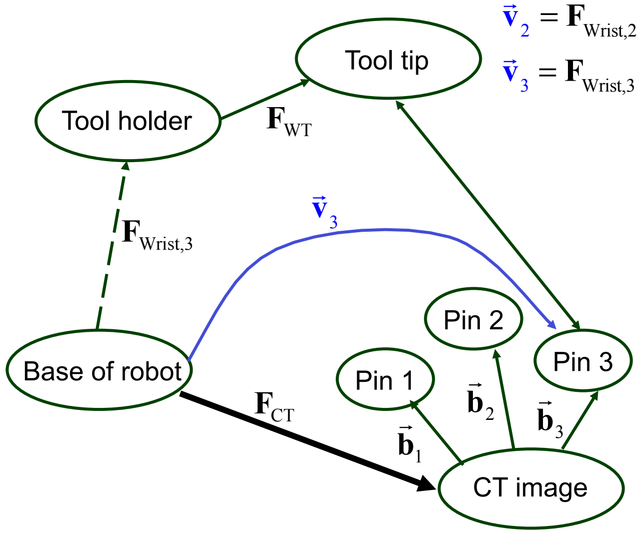
$$\vec{v}_2 = \mathbf{F}_{Wrist,2} \bullet \vec{p}_{WT} = \mathbf{F}_{CT} \bullet \vec{b}_2$$



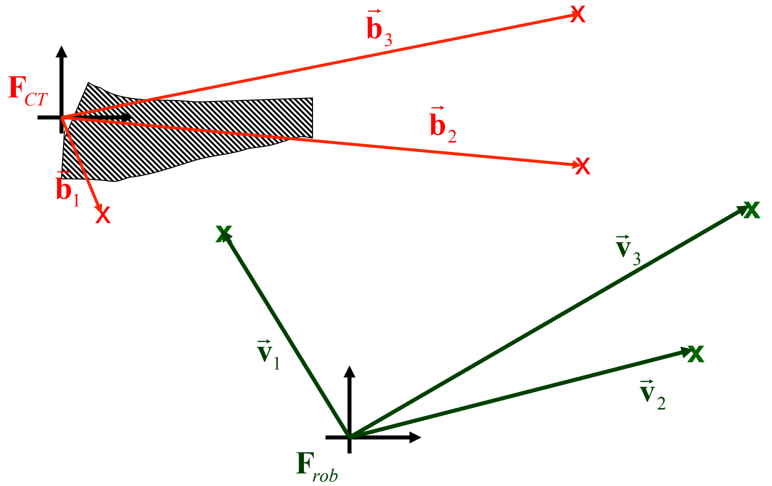
$$\vec{v}_1 = \mathbf{F}_{Wrist,1} \bullet \vec{p}_{WT} = \mathbf{F}_{CT} \bullet \vec{b}_1$$

$$\vec{v}_2 = \mathbf{F}_{Wrist,2} \bullet \vec{p}_{WT} = \mathbf{F}_{CT} \bullet \vec{b}_2$$

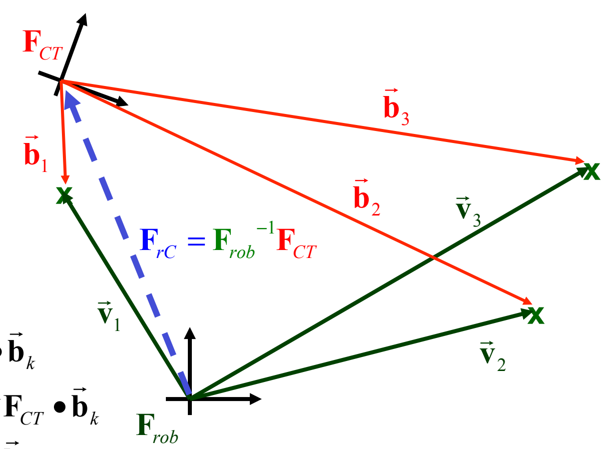
$$\vec{v}_3 = \mathbf{F}_{Wrist,3} \bullet \vec{p}_{WT} = \mathbf{F}_{CT} \bullet \vec{b}_3$$



Frame transformation from 3 point pairs



Frame transformation from 3 point pairs



$$F_{rob} \bullet \vec{v}_k = F_{CT} \bullet \vec{b}_k$$

$$\vec{v}_k = F_{rob}^{-1} F_{CT} \bullet \vec{b}_k$$

$$\vec{v}_k = F_{rC} \bullet \vec{b}_k$$

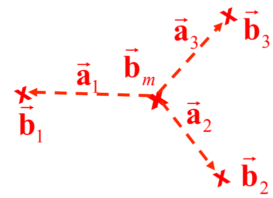
Frame transformation from 3 point pairs

$$\vec{\mathbf{v}}_k = \mathbf{F}_{rC}\vec{\mathbf{b}}_k = \mathbf{R}_{rC}\vec{\mathbf{b}}_k + \vec{\mathbf{p}}_{rC}$$

Define

$$\vec{\mathbf{v}}_m = \frac{1}{3}\sum_1^3 \vec{\mathbf{v}}_k \quad \vec{\mathbf{b}}_m = \frac{1}{3}\sum_1^3 \vec{\mathbf{b}}_k$$

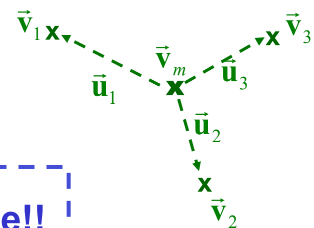
$$\vec{\mathbf{u}}_k = \vec{\mathbf{v}}_k - \vec{\mathbf{v}}_m \quad \vec{\mathbf{a}}_k = \vec{\mathbf{b}}_k - \vec{\mathbf{b}}_m$$



$$\mathbf{F}_{rC}\vec{\mathbf{a}}_k = \mathbf{R}_{rC}\vec{\mathbf{a}}_k + \vec{\mathbf{p}}_{rC}$$

$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k + \vec{\mathbf{p}}_{rC} = \mathbf{R}_{rC}\left(\vec{\mathbf{b}}_k - \vec{\mathbf{b}}_m\right) + \vec{\mathbf{p}}_{rC}$$

$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k = \mathbf{R}_{rC}\vec{\mathbf{b}}_k + \vec{\mathbf{p}}_{rC} - \mathbf{R}_{rC}\vec{\mathbf{b}}_m - \vec{\mathbf{p}}_{rC}$$



$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k = \vec{\mathbf{v}}_k - \vec{\mathbf{v}}_m = \vec{\mathbf{u}}_k$$

$$\vec{\mathbf{p}}_{rC} = \vec{\mathbf{v}}_m - \mathbf{R}_{rC}\vec{\mathbf{b}}_m$$

Solve These!!

Rotation from multiple vector pairs

Given a system $\mathbf{R}\vec{\mathbf{a}}_k = \vec{\mathbf{u}}_k$ for $k = 1,\cdots,n$ the problem is to estimate $\mathbf{R}$. This will require at least three such point pairs. Later in the course we will cover some good ways to solve this system. Here is a not-so-good way that will produce roughly correct answers:

Step 1: Form matrices $\mathbf{U} = [\vec{\mathbf{u}}_1 \ \cdots \ \vec{\mathbf{u}}_n]$ and $\mathbf{A} = [\vec{\mathbf{a}}_1 \ \cdots \ \vec{\mathbf{a}}_n]$

Step 2: Solve the system $\mathbf{RA} = \mathbf{U}$ for $\mathbf{R}$. E.g., by $\mathbf{R} = \mathbf{UA}^{-1}$

Step 3: Renormalize $\mathbf{R}$ to guarantee $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

Renormalizing Rotation Matrix

Given "rotation" matrix $\mathbf{R} = \left[\vec{\mathbf{r}}_x \mid \vec{\mathbf{r}}_y \mid \vec{\mathbf{r}}_z \right]$, modify it so $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

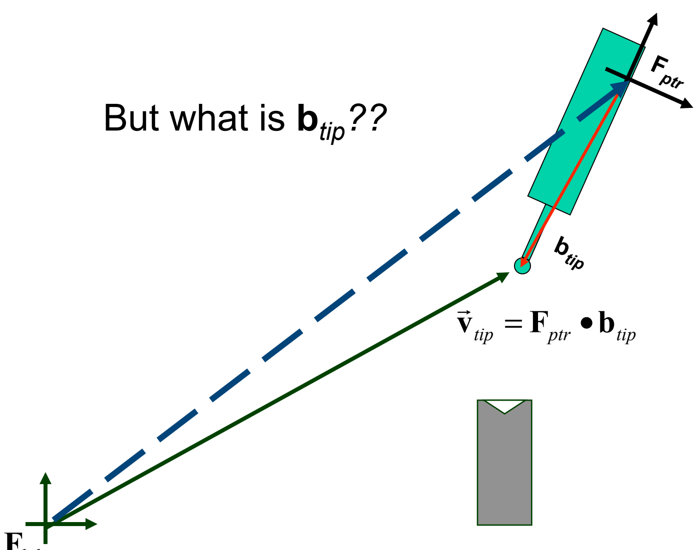
Step 1: $\vec{\mathbf{a}} = \vec{\mathbf{r}}_y \times \vec{\mathbf{r}}_z$

Step 2: $\vec{\mathbf{b}} = \vec{\mathbf{r}}_z \times \vec{\mathbf{a}}$

Step 3: $\mathbf{R}_{normalized} = \left[\frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|} \middle| \frac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|} \middle| \frac{\vec{\mathbf{r}}_z}{\|\vec{\mathbf{r}}_z\|} \right]$

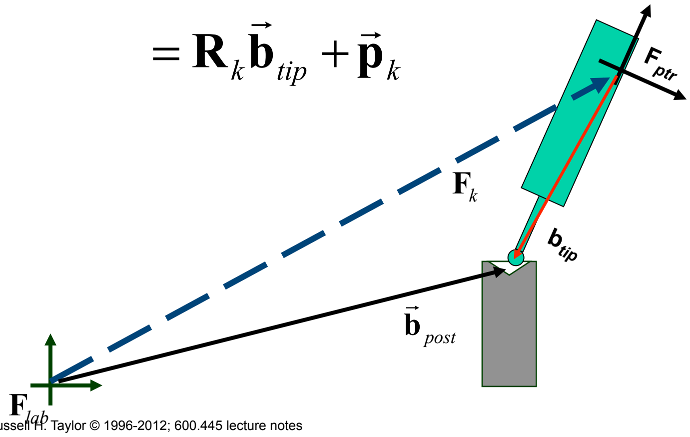
Calibrating a pointer

But what is $\mathbf{b}_{tip}$??



Calibrating a pointer

$$\vec{\mathbf{b}}_{post} = \mathbf{F}_k \vec{\mathbf{b}}_{tip}$$
$$= \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k$$



Calibrating a pointer

For each measurement k, we have

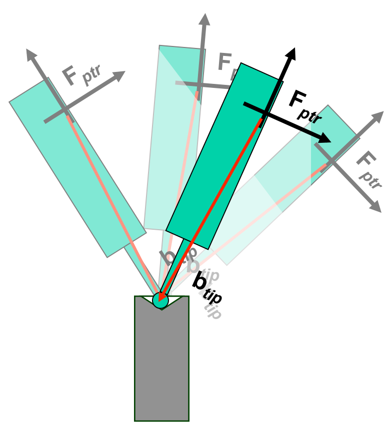
$$\vec{\mathbf{b}}_{post} = \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k$$

I. e.,

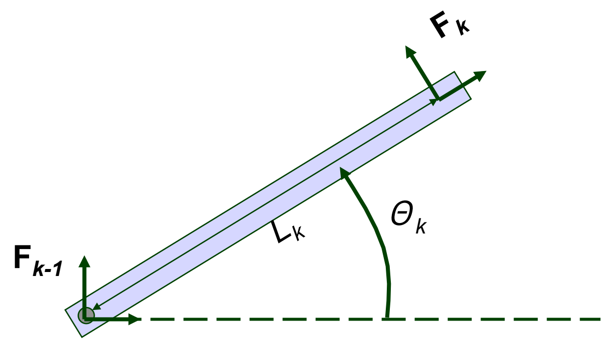
$$\mathbf{R}_k \vec{\mathbf{b}}_{tip} - \vec{\mathbf{b}}_{post} = -\vec{\mathbf{p}}_k$$

Set up a least squares problem

$$\begin{bmatrix} \vdots & \vdots \\ \mathbf{R}_k & -\mathbf{I} \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vec{\mathbf{b}}_{tip} \\ \vec{\mathbf{b}}_{post} \end{bmatrix} \cong \begin{bmatrix} \vdots \\ -\vec{\mathbf{p}}_k \\ \vdots \end{bmatrix}$$



Kinematic Links

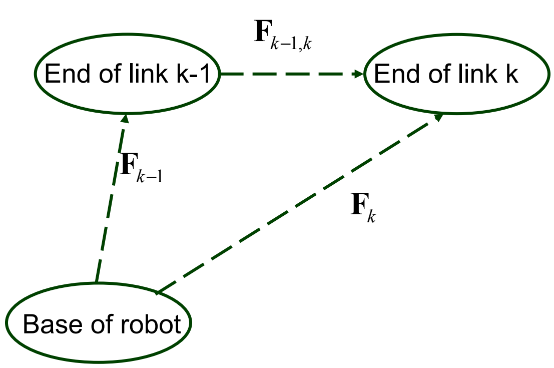


$$\mathbf{F}_k = \mathbf{F}_{k-1} \bullet \mathbf{F}_{k-1,k}$$

$$[\mathbf{R}_k, \vec{\mathbf{p}}_k] = [\mathbf{R}_{k-1}, \mathbf{p}_{k-1}] \bullet [\mathbf{R}_{k-1,k}, \mathbf{p}_{k-1,k}]$$

$$= [\mathbf{R}_{k-1}, \mathbf{p}_{k-1}] \bullet [Rot(\vec{\mathbf{r}}_k, \theta_k), L_k Rot(\vec{\mathbf{r}}_k, \theta_k) \bullet \vec{\mathbf{x}}]$$

Kinematic Links



$$\mathbf{F}_k = \mathbf{F}_{k-1} \bullet \mathbf{F}_{k-1,k}$$

$$[\mathbf{R}_k, \vec{\mathbf{p}}_k] = [\mathbf{R}_{k-1}, \mathbf{p}_{k-1}] \bullet [\mathbf{R}_{k-1,k}, \mathbf{p}_{k-1,k}]$$

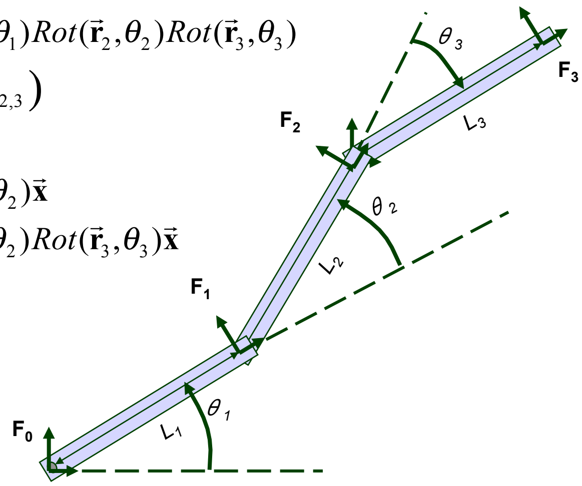
$$= [\mathbf{R}_{k-1}, \mathbf{p}_{k-1}] \bullet [Rot(\vec{\mathbf{r}}_k, \theta_k), L_k Rot(\vec{\mathbf{r}}_k, \theta_k) \bullet \vec{\mathbf{x}}]$$

Kinematic Chains

$$\mathbf{F}_0 = [\mathbf{I}, \vec{\mathbf{0}}]$$

$$\mathbf{R}_3 = \mathbf{R}_{0,1}\mathbf{R}_{1,2}\mathbf{R}_{2,3} = Rot(\vec{\mathbf{r}}_1,\theta_1)Rot(\vec{\mathbf{r}}_2,\theta_2)Rot(\vec{\mathbf{r}}_3,\theta_3)$$

$$\begin{aligned}\vec{\mathbf{p}}_3 &= \vec{\mathbf{p}}_{0,1} + \mathbf{R}_{0,1}\left(\vec{\mathbf{p}}_{1,2} + \mathbf{R}_{1,2}\vec{\mathbf{p}}_{2,3}\right) \\ &= L_1 Rot(\vec{\mathbf{r}}_1,\theta_1)\vec{\mathbf{x}} \\ &\quad + L_2 Rot(\vec{\mathbf{r}}_1,\theta_1)Rot(\vec{\mathbf{r}}_2,\theta_2)\vec{\mathbf{x}} \\ &\quad + L_3 Rot(\vec{\mathbf{r}}_1,\theta_1)Rot(\vec{\mathbf{r}}_2,\theta_2)Rot(\vec{\mathbf{r}}_3,\theta_3)\vec{\mathbf{x}}\end{aligned}$$

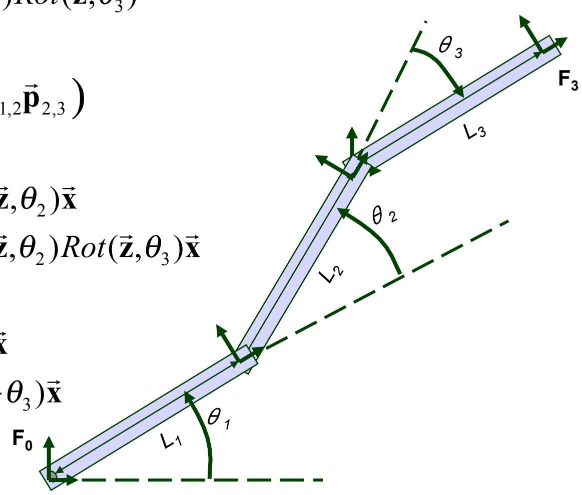


Kinematic Chains

If $\vec{\mathbf{r}}_1 = \vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_3 = \vec{\mathbf{z}}$,

$$\begin{aligned}\mathbf{R}_3 &= Rot(\vec{\mathbf{z}},\theta_1)Rot(\vec{\mathbf{z}},\theta_2)Rot(\vec{\mathbf{z}},\theta_3) \\ &= Rot(\vec{\mathbf{z}},\theta_1+\theta_2+\theta_3)\end{aligned}$$

$$\begin{aligned}\vec{\mathbf{p}}_3 &= \vec{\mathbf{p}}_{0,1} + \mathbf{R}_{0,1}\left(\vec{\mathbf{p}}_{1,2} + \mathbf{R}_{1,2}\vec{\mathbf{p}}_{2,3}\right) \\ &= L_1 Rot(\vec{\mathbf{z}},\theta_1)\vec{\mathbf{x}} \\ &\quad + L_2 Rot(\vec{\mathbf{z}},\theta_1)Rot(\vec{\mathbf{z}},\theta_2)\vec{\mathbf{x}} \\ &\quad + L_3 Rot(\vec{\mathbf{z}},\theta_1)Rot(\vec{\mathbf{z}},\theta_2)Rot(\vec{\mathbf{z}},\theta_3)\vec{\mathbf{x}} \\ &= L_1 Rot(\vec{\mathbf{z}},\theta_1)\vec{\mathbf{x}} \\ &\quad + L_2 Rot(\vec{\mathbf{z}},\theta_1+\theta_2)\vec{\mathbf{x}} \\ &\quad + L_3 Rot(\vec{\mathbf{z}},\theta_1+\theta_2+\theta_3)\vec{\mathbf{x}}\end{aligned}$$

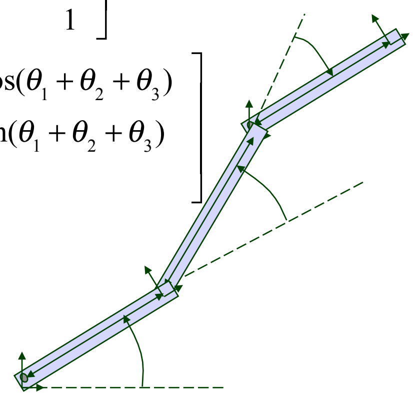


Kinematic Chains

If $\vec{\mathbf{r}}_1 = \vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_3 = \vec{\mathbf{z}},$

$$\mathbf{R}_3 = \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{\mathbf{p}}_3 = \begin{bmatrix} L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) \\ L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) + L_3\sin(\theta_1+\theta_2+\theta_3) \\ 0 \end{bmatrix}$$



"Small" Transformations

- A great deal of CIS is concerned with computing and using geometric information based on imprecise knowledge
- Similarly, one is often concerned with the effects of relatively small rotations and displacements
- Essentially, we will be using fairly straightforward linearizations to model these situations, but a specialized notation is often useful

“Small” Frame Transformations

Represent a "small" pose shift consisting of a small rotation $\Delta\mathbf{R}$ followed by a small displacement $\Delta\vec{\mathbf{p}}$ as

$$\Delta\mathbf{F} = [\Delta\mathbf{R}, \Delta\vec{\mathbf{p}}]$$

Then

$$\Delta\mathbf{F} \bullet \vec{\mathbf{v}} = \Delta\mathbf{R} \bullet \vec{\mathbf{v}} + \Delta\vec{\mathbf{p}}$$

Small Rotations

$\Delta\mathbf{R} =$ a small rotation

$\mathbf{R}_{\vec{\mathbf{a}}}(\Delta\alpha) =$ a rotation by a small angle $\Delta\alpha$ about axis $\vec{\mathbf{a}}$

$Rot(\vec{\mathbf{a}}, \|\vec{\mathbf{a}}\|) \bullet \vec{\mathbf{b}} \approx \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{b}}$ for $\|\vec{\mathbf{a}}\|$ sufficiently small

$\Delta\mathbf{R}(\vec{\mathbf{a}}) =$ a rotation that is small enough so that any error introduced by this approximation is negligible

$$\Delta\mathbf{R}(\lambda\vec{\mathbf{a}}) \bullet \Delta\mathbf{R}(\mu\vec{\mathbf{b}}) \cong \Delta\mathbf{R}(\lambda\vec{\mathbf{a}} + \mu\vec{\mathbf{b}}) \quad \text{(Linearity for small rotations)}$$

Exercise: Work out the linearity proposition by substitution

Approximations to "Small" Frames

$$
\begin{aligned}
\Delta\mathbf{F}(\vec{\mathbf{a}},\Delta\vec{\mathbf{p}}) &\approx [\Delta\mathbf{R}(\vec{\mathbf{a}}),\Delta\vec{\mathbf{p}}] \\
\Delta\mathbf{F}(\vec{\mathbf{a}},\Delta\vec{\mathbf{p}})\bullet\vec{\mathbf{v}} &= \Delta\mathbf{R}(\vec{\mathbf{a}})\bullet\vec{\mathbf{v}}+\Delta\vec{\mathbf{p}} \\
&\approx \vec{\mathbf{v}}+\vec{\mathbf{a}}\times\vec{\mathbf{v}}+\Delta\vec{\mathbf{p}}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\mathbf{a}}\times\vec{\mathbf{v}} &= skew(\vec{\mathbf{a}})\bullet\vec{\mathbf{v}} \\
&= \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \bullet \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}
\end{aligned}
$$

$$skew(\vec{\mathbf{a}})\bullet\vec{\mathbf{a}} = \vec{0}$$

$$
\begin{aligned}
\Delta\mathbf{R}(\vec{\mathbf{a}}) &\approx \mathbf{I}+skew(\vec{\mathbf{a}}) \\
\Delta\mathbf{R}(\vec{\mathbf{a}})^{-1} &\approx \mathbf{I}-skew(\vec{\mathbf{a}}) = \mathbf{I}+skew(-\vec{\mathbf{a}})
\end{aligned}
$$

Approximations to "Small" Frames

Notational NOTE:

We often use $\vec{\alpha}$ to represent a vector of small angles
and $\vec{\varepsilon}$ to represent a vector of small displacements

In using these approximations, we typically ignore second order terms. I.e.,

$\vec{\alpha}_A\vec{\alpha}_B \approx \vec{0},\ \vec{\alpha}_A\vec{\varepsilon}_B \approx \vec{0},\ \vec{\varepsilon}_A\vec{\varepsilon}_B \approx \vec{0}$, etc.

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Errors & sensitivity

Often, we do not have an accurate value for a transformation, so we need to model the error. We model this as a composition of a "nominal" frame and a small displacement

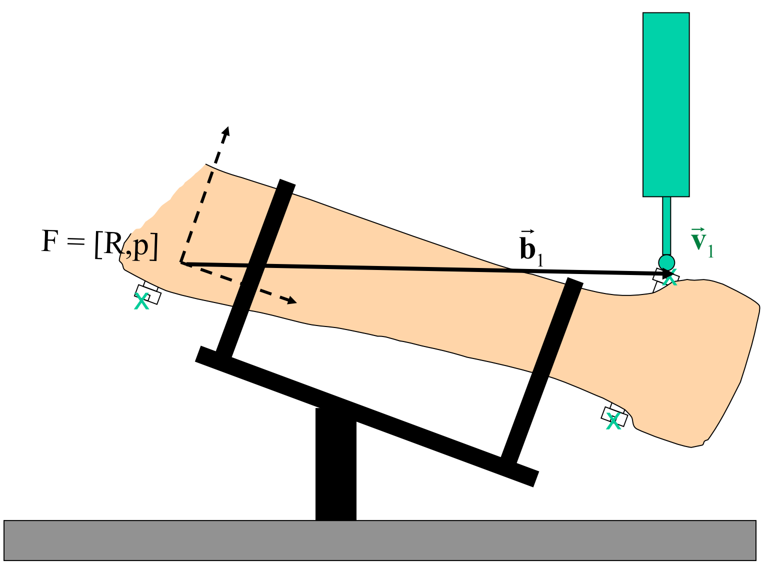
$$\mathbf{F}_{actual} = \mathbf{F}_{nominal} \bullet \Delta\mathbf{F}$$

Often, we will use the notation $\mathbf{F}^*$ for $\mathbf{F}_{actual}$ and will just use $\mathbf{F}$ for $\mathbf{F}_{nominal}$. Thus we may write something like

$$\mathbf{F}^* = \mathbf{F} \bullet \Delta\mathbf{F}$$

or (less often) $\mathbf{F}^* = \Delta\mathbf{F} \bullet \mathbf{F}$. We also use $\vec{\mathbf{v}}^* = \vec{\mathbf{v}} + \Delta\vec{\mathbf{v}}, etc.$
Thus, if we use the former form (error on the right), and have nominal relationship $\vec{\mathbf{v}} = \mathbf{F} \bullet \vec{\mathbf{b}}$, we get

$$\begin{aligned}\vec{\mathbf{v}}^* &= \mathbf{F}^* \bullet \vec{\mathbf{b}}^* \\ &= \mathbf{F} \bullet \Delta\mathbf{F} \bullet (\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}})\end{aligned}$$



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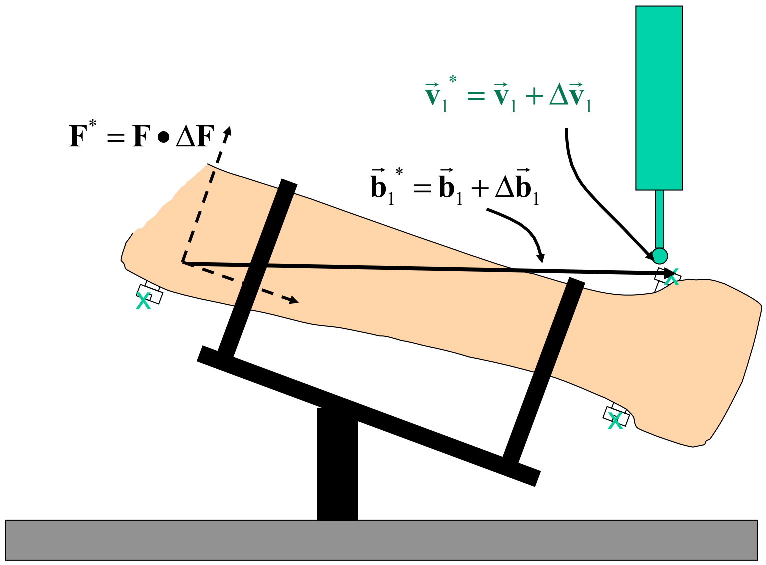
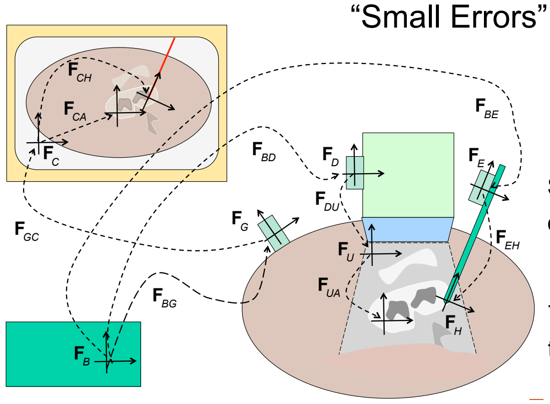


"Small Errors"



Suppose that there is a small systematic error in the tracking system so that

$$\mathbf{F}^*_{Bx} = \Delta\mathbf{F}_B\mathbf{F}_{Bx}$$

for $\mathbf{F}_{BG}, \mathbf{F}_{BD}, \mathbf{F}_{BE}$. How does this affect the calculation of $\mathbf{F}_{GH}$?

$$\mathbf{F}^*_{GH} = (\mathbf{F}^*_{BG})^{-1}\mathbf{F}^*_{BE}\mathbf{F}_{EH}$$

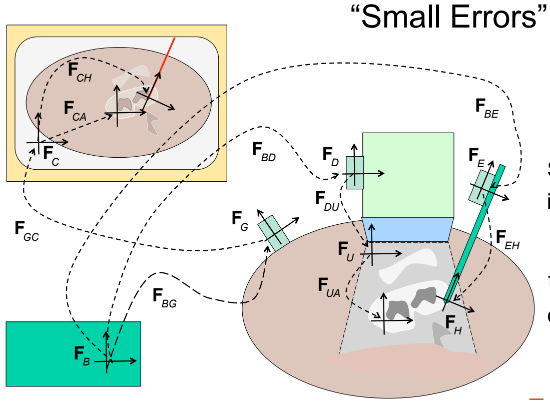
$$\mathbf{F}_{GH}\Delta\mathbf{F}_{GH} = \left(\Delta\mathbf{F}_B\mathbf{F}_{BG}\right)^{-1}\Delta\mathbf{F}_B\mathbf{F}_{BE}\mathbf{F}_{EH}$$

$$\Delta\mathbf{F}_{GH} = \mathbf{F}_{BG}^{-1}\Delta\mathbf{F}_B^{-1}\Delta\mathbf{F}_B\mathbf{F}_{BE}\mathbf{F}_{EH}\mathbf{F}_{GH}^{-1}$$

$$= \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\mathbf{F}_{EH}\mathbf{F}_{GH}^{-1}$$

$$= \mathbf{F}_{GH}\mathbf{F}_{GH}^{-1} = \mathbf{I}$$

“Small Errors”



Suppose that there are additional errors in the tracking of each tracker body so that

$$\mathbf{F}^*_{Bx} = \Delta\mathbf{F}_B \mathbf{F}_{Bx} \Delta\mathbf{F}_{Bx}$$

for $\mathbf{F}_{BG}, \mathbf{F}_{BD}, \mathbf{F}_{BE}$. How does this affect the calculation of $\mathbf{F}_{GH}$?

$$\mathbf{F}^*_{GH} = \mathbf{F}_{GH}\Delta\mathbf{F}_{GH} = (\mathbf{F}^*_{BG})^{-1}\mathbf{F}^*_{BE}\,\mathbf{F}_{EH}$$

$$\Delta\mathbf{F}_{GH} = \mathbf{F}_{GH}^{-1}\left(\Delta\mathbf{F}_B\mathbf{F}_{BG}\Delta\mathbf{F}_{BG}\right)^{-1}\left(\Delta\mathbf{F}_B\mathbf{F}_{BE}\Delta\mathbf{F}_{BE}\right)\mathbf{F}_{EH}$$

$$\Delta\mathbf{F}_{GH} = \left(\mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\mathbf{F}_{EH}\right)^{-1}\Delta\mathbf{F}_{BG}^{-1}\mathbf{F}_{BG}^{-1}\Delta\mathbf{F}_B^{-1}\Delta\mathbf{F}_B\mathbf{F}_{BE}\Delta\mathbf{F}_{BG}\mathbf{F}_{EH}$$

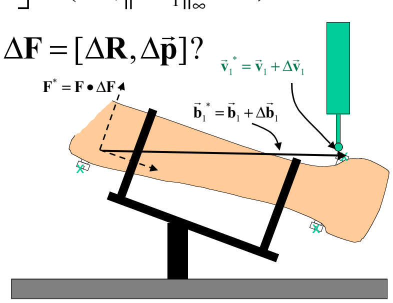
$$= \mathbf{F}_{EH}^{-1}\mathbf{F}_{BE}^{-1}\mathbf{F}_{BG}\Delta\mathbf{F}_{BG}^{-1}\mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\Delta\mathbf{F}_{BG}\mathbf{F}_{EH}$$

Errors & Sensitivity

Suppose that we know nominal values for $\mathbf{F}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{v}}$ and that

$$\left[-\varepsilon,-\varepsilon,-\varepsilon\right]^T \le \Delta\vec{\mathbf{v}}_1 \le \left[\varepsilon,\varepsilon,\varepsilon\right]^T \quad (i.e., \left\|\Delta\vec{\mathbf{v}}_1\right\|_\infty \le \varepsilon)$$

What does this tell us about $\Delta\mathbf{F} = [\Delta\mathbf{R}, \Delta\vec{\mathbf{p}}]$?



Errors & Sensitivity

$$
\begin{aligned}
\vec{\mathbf{v}}^* &= \mathbf{F}^* \bullet \vec{\mathbf{b}}^* \\
&= \mathbf{F} \bullet \Delta\mathbf{F} \bullet (\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}}) \\
&= \mathbf{R} \bullet \left(\Delta\mathbf{R}(\vec{\alpha}) \bullet \left(\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} \right) + \Delta\vec{\mathbf{p}} \right) + \vec{\mathbf{p}} \\
&\cong \mathbf{R} \bullet \left(\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \vec{\alpha} \times \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}} \right) + \vec{\mathbf{p}} \\
&= \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}} + \mathbf{R} \bullet \left(\Delta\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \vec{\alpha} \times \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}} \right) \\
&\cong \vec{\mathbf{v}} + \mathbf{R} \bullet \left(\Delta\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{p}} \right)
\end{aligned}
$$

if $\left\| \vec{\alpha} \times \Delta\vec{\mathbf{b}} \right\| \le \left\| \vec{\alpha} \right\| \left\| \Delta\vec{\mathbf{b}} \right\|$ is negligible (it usually is)

so

$$
\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}^* - \vec{\mathbf{v}} \cong \mathbf{R} \bullet \left(\Delta\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{p}} \right) = \mathbf{R} \bullet \Delta\vec{\mathbf{b}} + \mathbf{R} \bullet \vec{\alpha} \times \vec{\mathbf{b}} + \mathbf{R} \bullet \Delta\vec{\mathbf{p}}
$$

Digression: "rotation triple product"

Expressions like $\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}}$ are linear in $\vec{\mathbf{a}}$, but are not always convenient to work with. Often we would prefer something like $\mathbf{M}(\mathbf{R}, \vec{\mathbf{b}}) \bullet \vec{\mathbf{a}}$.

$$
\begin{aligned}
\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}} &= -\mathbf{R} \bullet \vec{\mathbf{b}} \times \vec{\mathbf{a}} \\
&= \mathbf{R} \bullet skew(-\vec{\mathbf{b}}) \bullet \vec{\mathbf{a}} \\
&= \left[\mathbf{R} \bullet skew(\vec{\mathbf{b}})^T \right] \bullet \vec{\mathbf{a}}
\end{aligned}
$$

Digression: “rotation triple product”

Here are a few more useful facts:

$$\mathbf{R}\bullet\left(\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right)=\left(\mathbf{R}\bullet\vec{\mathbf{a}}\right)\times\left(\mathbf{R}\bullet\vec{\mathbf{b}}\right)$$

$$\vec{\mathbf{a}}\times\left(\mathbf{R}\bullet\vec{\mathbf{b}}\right)=\mathbf{R}\bullet\left(\left(\mathbf{R}^{-1}\bullet\vec{\mathbf{a}}\right)\times\vec{\mathbf{b}}\right)$$

Consequently

$$skew(\vec{\mathbf{a}})\bullet\mathbf{R}=\mathbf{R}\bullet skew(\mathbf{R}^{-1}\bullet\vec{\mathbf{a}})$$

$$\mathbf{R}^{-1}skew(\vec{\mathbf{a}})\bullet\mathbf{R}=skew(\mathbf{R}^{-1}\bullet\vec{\mathbf{a}})$$

Errors & Sensitivity

Previous expression was

$$\Delta\vec{\mathbf{v}}_1\cong\mathbf{R}\bullet\left(\Delta\vec{\mathbf{b}}_1+\vec{\mathbf{a}}\times\vec{\mathbf{b}}+\Delta\vec{\mathbf{p}}_1\right)$$

Substituting triple product and rearranging gives

$$\Delta\vec{\mathbf{v}}_1\cong\left[\mathbf{R}\;\middle|\;\mathbf{R}\;\middle|\;\mathbf{R}\bullet skew(-\vec{\mathbf{b}})\right]\bullet\begin{bmatrix}\Delta\vec{\mathbf{b}}_1\\ \Delta\vec{\mathbf{p}}\\ \vec{\mathbf{a}}\end{bmatrix}$$

So

$$\begin{bmatrix}-\varepsilon\\ -\varepsilon\\ -\varepsilon\end{bmatrix}\le\left[\mathbf{R}\;\middle|\;\mathbf{R}\;\middle|\;\mathbf{R}\bullet skew(-\vec{\mathbf{b}})\right]\begin{bmatrix}\Delta\vec{\mathbf{b}}_1\\ \Delta\vec{\mathbf{p}}\\ \vec{\mathbf{a}}\end{bmatrix}\le\begin{bmatrix}\varepsilon\\ \varepsilon\\ \varepsilon\end{bmatrix}$$

Errors & Sensitivity

Now, suppose we know that $\left|\Delta\vec{\mathbf{b}}_1\right| \le \beta$, this will give us a system of linear constraints

$$\begin{bmatrix} -\varepsilon \\ -\varepsilon \\ -\varepsilon \\ \hline -\beta \\ -\beta \\ -\beta \end{bmatrix} \le \left[\begin{array}{c|c|c} \mathbf{R} & \mathbf{R} & \mathbf{R}\bullet skew(-\vec{\mathbf{b}}) \\ \hline \mathbf{I} & \mathbf{0} & \mathbf{0} \end{array}\right] \begin{bmatrix} \Delta\vec{\mathbf{b}}_1 \\ \Delta\vec{\mathbf{p}}_1 \\ \vec{\mathbf{a}} \end{bmatrix} \le \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \\ \hline \beta \\ \beta \\ \beta \end{bmatrix}$$

Error from frame composition

Consider $\mathbf{R}_1^*\mathbf{R}_2^* = \mathbf{R}_3^*$ where $\mathbf{R}_1^* = \mathbf{R}_1\Delta\mathbf{R}_1, \mathbf{R}_2^* = \mathbf{R}_2\Delta\mathbf{R}_2,\ \mathbf{R}_3^* = \mathbf{R}_3\Delta\mathbf{R}_3$ and $\Delta\mathbf{R}_1 \approx \mathbf{I} + sk(\vec{\alpha}_1),\ \Delta\mathbf{R}_2 \approx \mathbf{I} + sk(\vec{\alpha}_2)$, estimate $\Delta\mathbf{R}_3 \approx \mathbf{I} + sk(\vec{\alpha}_3)$

$$\mathbf{R}_1\Delta\mathbf{R}_1\mathbf{R}_2\Delta\mathbf{R}_2 = \mathbf{R}_1\mathbf{R}_2\Delta\mathbf{R}_3$$

$$\mathbf{R}_1(\mathbf{I} + sk(\vec{\alpha}_1))\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_2)) \approx \mathbf{R}_1\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_3))$$

$$(\mathbf{R}_1\mathbf{R}_2)^{-1}\mathbf{R}_1(\mathbf{I} + sk(\vec{\alpha}_1))\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_2)) \approx \mathbf{I} + sk(\vec{\alpha}_3)$$

$$\mathbf{R}_2^{-1}\mathbf{R}_1^{-1}\mathbf{R}_1(\mathbf{I} + sk(\vec{\alpha}_1))\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_2)) \approx \mathbf{I} + sk(\vec{\alpha}_3)$$

$$\mathbf{I} + \mathbf{R}_2^{-1}sk(\vec{\alpha}_1)\mathbf{R}_2 + sk(\vec{\alpha}_2) + \mathbf{R}_2^{-1}sk(\vec{\alpha}_1)\mathbf{R}_2 sk(\vec{\alpha}_2) \approx \mathbf{I} + sk(\vec{\alpha}_3)$$

$$\mathbf{R}_2^{-1}sk(\vec{\alpha}_1)\mathbf{R}_2 + sk(\vec{\alpha}_2) \approx sk(\vec{\alpha}_3)$$

Since $\mathbf{R}^{-1}\bullet(\vec{\mathbf{a}}\times\mathbf{R}\vec{\mathbf{b}}) = (\mathbf{R}^{-1}\vec{\mathbf{a}})\times\vec{\mathbf{b}}$ for all $\mathbf{R}, \vec{\mathbf{a}}, \vec{\mathbf{b}}$ we get $\mathbf{R}_2^{-1}sk(\vec{\alpha}_1)\mathbf{R}_2 = sk(\mathbf{R}_2^{-1}\vec{\alpha}_1)$

$$sk(\vec{\alpha}_3) \approx sk(\mathbf{R}_2^{-1}\vec{\alpha}_1) + sk(\vec{\alpha}_2) = sk(\mathbf{R}_2^{-1}\vec{\alpha}_1 + \vec{\alpha}_2)$$

$$\vec{\alpha}_3 \approx \mathbf{R}_2^{-1}\vec{\alpha}_1 + \vec{\alpha}_2$$

Error from frame composition

Consider $\mathbf{F}_1^*\mathbf{F}_2^* = \mathbf{F}_3^*$ where $\mathbf{F}_1^* = \mathbf{F}_1\Delta\mathbf{F}_1, \mathbf{F}_2^* = \mathbf{F}_2\Delta\mathbf{F}_2,\ \mathbf{F}_3^* = \mathbf{F}_3\Delta\mathbf{F}_3$
and $\Delta\mathbf{F}_1 \approx \left[\mathbf{I} + sk\left(\vec{\alpha}_1\right), \vec{\varepsilon}_1\right],\ \Delta\mathbf{F}_2 \approx \left[\mathbf{I} + sk\left(\vec{\alpha}_2\right), \vec{\varepsilon}_3\right],$
estimate $\Delta\mathbf{F}_3 \approx \left[\mathbf{I} + sk\left(\vec{\alpha}_3\right), \vec{\varepsilon}_3\right]$

From before, we have $\vec{\alpha}_3 \approx \mathbf{R}_2^{-1}\vec{\alpha}_1 + \vec{\alpha}_2$. So now we just need $\vec{\varepsilon}_3$.

$$
\begin{aligned}
\vec{\mathbf{p}}_3^* &= \mathbf{R}_1^*\vec{\mathbf{p}}_2^* + \vec{\mathbf{p}}_1^* \\
\vec{\mathbf{p}}_3 + \vec{\varepsilon}_3 &\approx \mathbf{R}_1\left(\mathbf{I} + sk(\vec{\alpha}_1)\right)\left(\vec{\mathbf{p}}_2 + \vec{\varepsilon}_2\right) + \left(\vec{\mathbf{p}}_1 + \vec{\varepsilon}_1\right) \\
&= \mathbf{R}_1\vec{\mathbf{p}}_2 + \mathbf{R}_1\vec{\varepsilon}_2 + \mathbf{R}_1\bullet\left(\vec{\alpha}_1 \times \vec{\mathbf{p}}_2 + \vec{\alpha}_1 \times \vec{\varepsilon}_2\right) + \vec{\mathbf{p}}_1 + \vec{\varepsilon}_1 \\
&= \vec{\mathbf{p}}_3 + \mathbf{R}_1\vec{\varepsilon}_2 + \mathbf{R}_1\bullet\left(\vec{\alpha}_1 \times \vec{\mathbf{p}}_2 + \vec{\alpha}_1 \times \vec{\varepsilon}_2\right) + \vec{\varepsilon}_1 \\
\vec{\varepsilon}_3 &\approx \mathbf{R}_1\vec{\varepsilon}_2 + \mathbf{R}_1\bullet\vec{\alpha}_1 \times \vec{\mathbf{p}}_2 + \vec{\varepsilon}_1 \\
&= \mathbf{R}_1\vec{\varepsilon}_2 - \mathbf{R}_1\bullet\vec{\mathbf{p}}_2 \times \vec{\alpha}_1 + \vec{\varepsilon}_1 \\
&= \vec{\varepsilon}_1 - \mathbf{R}_1 sk(\vec{\mathbf{p}}_2)\vec{\alpha}_1 + \mathbf{R}_1\vec{\varepsilon}_2
\end{aligned}
$$

Inverse of frame transformation with errors

$$
\begin{aligned}
\mathbf{F}_i &= \mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1}\vec{\mathbf{p}}] \\
\mathbf{F}_i^* &= \left(\mathbf{F}\Delta\mathbf{F}\right)^{-1} \\
\mathbf{F}_i\Delta\mathbf{F}_i &= \left[\mathbf{R}\Delta\mathbf{R}, \mathbf{R}\Delta\vec{\mathbf{p}} + \vec{\mathbf{p}}\right]^{-1} \\
&= [(\mathbf{R}\Delta\mathbf{R})^{-1}, -(\mathbf{R}\Delta\mathbf{R})^{-1}\left(\mathbf{R}\Delta\mathbf{p} + \vec{\mathbf{p}}\right)] \\
&= [\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\left(\mathbf{R}\Delta\mathbf{p} + \vec{\mathbf{p}}\right)] \\
&= [\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}}] \\
\Delta\mathbf{F}_i &= \left(\mathbf{F}^{-1}\right)^{-1}[\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}}] \\
&= [\mathbf{R}, \vec{\mathbf{p}}]\bullet[\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}}] \\
&= [\mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\mathbf{R}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}} + \vec{\mathbf{p}}]
\end{aligned}
$$

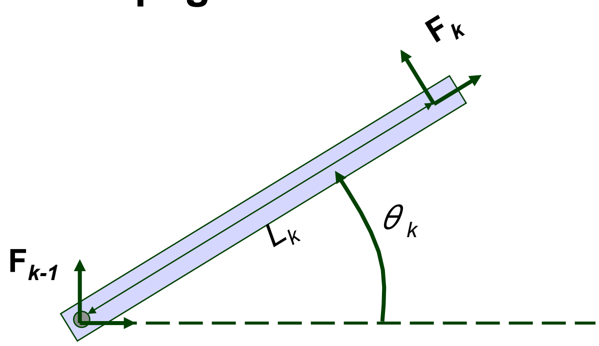
Inverse of frame transformation with errors

Suppose we know that $\Delta\mathbf{R}$ is "small", i.e.,$\Delta\mathbf{R} \approx \mathbf{I} + sk(\vec{\alpha})$, and for notational convenience we write $\Delta\vec{\mathbf{p}} = \vec{\varepsilon}$, we get

$$
\begin{aligned}
\Delta\mathbf{R}_i &= \mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1} \approx \mathbf{R}\left(\mathbf{I} + sk(\vec{\alpha})\right)^{-1}\mathbf{R}^{-1} \\
&\approx \mathbf{R}\left(\mathbf{I} - sk(\vec{\alpha})\right)\mathbf{R}^{-1} \\
&= \mathbf{R}\mathbf{R}^{-1} - \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1} \\
&= \mathbf{I} - \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1} \\
&= \mathbf{I} - sk\left(\left(\mathbf{R}^{-1}\right)^{-1}\vec{\alpha}\right) = \mathbf{I} - sk(\mathbf{R}\vec{\alpha})
\end{aligned}
$$

$$
\begin{aligned}
\Delta\vec{\mathbf{p}}_i &= -\mathbf{R}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}} + \vec{\mathbf{p}} \\
&\approx -\mathbf{R}\left(\mathbf{I} - sk(\vec{\alpha})\right)\vec{\varepsilon} - \left(\mathbf{I} - sk(\mathbf{R}\vec{\alpha})\right)\vec{\mathbf{p}} + \vec{\mathbf{p}} \\
&= -\mathbf{R}\vec{\varepsilon} + \mathbf{R}\left(\vec{\alpha} \times \vec{\varepsilon}\right) - \vec{\mathbf{p}} + (\mathbf{R}\vec{\alpha}) \times \vec{\mathbf{p}} + \vec{\mathbf{p}} \\
&\approx -\mathbf{R}\vec{\varepsilon} + (\mathbf{R}\vec{\alpha}) \times \vec{\mathbf{p}} = -\mathbf{R}\vec{\varepsilon} - \vec{\mathbf{p}} \times (\mathbf{R}\vec{\alpha}) = -\mathbf{R}\vec{\varepsilon} - \mathbf{R}\left(\vec{\mathbf{p}} \times (\mathbf{R}^{-1}\mathbf{R}\vec{\alpha})\right) \\
&= -\mathbf{R}\bullet\left(\vec{\varepsilon} + \vec{\mathbf{p}} \times \vec{\alpha}\right) = -\mathbf{R}\vec{\varepsilon} + \mathbf{R}sk(\vec{\mathbf{p}})\vec{\alpha}
\end{aligned}
$$

Error Propagation in Chains



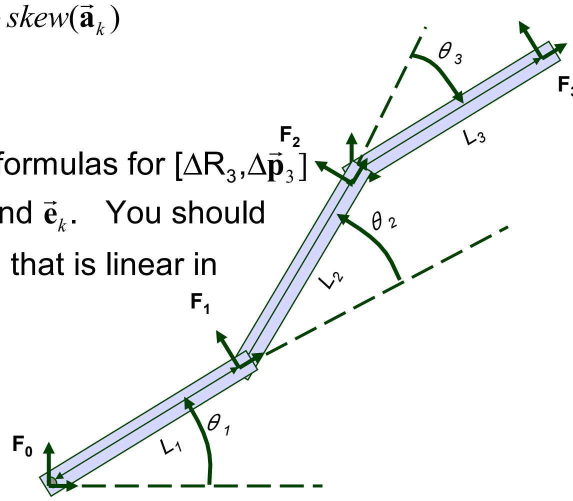
$$
\begin{aligned}
\mathbf{F}^*_k &= \mathbf{F}^*_{k-1} \bullet \mathbf{F}^*_{k-1,k} \\
\mathbf{F}_k\Delta\mathbf{F}_k &= \mathbf{F}_{k-1}\Delta\mathbf{F}_{k-1}\mathbf{F}_{k-1,k}\Delta\mathbf{F}_{k-1,k} \\
\Delta\mathbf{F}_k &= \left(\mathbf{F}_k^{-1}\mathbf{F}_{k-1}\right)\Delta\mathbf{F}_{k-1}\mathbf{F}_{k-1,k}\Delta\mathbf{F}_{k-1,k} \\
&= \left(\mathbf{F}_{k-1,k}{}^{-1}\Delta\mathbf{F}_{k-1}\mathbf{F}_{k-1,k}\right)\Delta\mathbf{F}_{k-1,k}
\end{aligned}
$$

Exercise

Suppose that you have

$$\Delta \mathbf{R}_{k-1,k} = \Delta \mathbf{R}(\vec{\mathbf{a}}_k) \cong \mathbf{I} + skew(\vec{\mathbf{a}}_k)$$
$$\Delta \vec{\mathbf{p}}_{k-1,k} = \vec{\mathbf{e}}_k$$

Work out approximate formulas for $[\Delta \mathrm{R}_3, \Delta \vec{\mathbf{p}}_3]$ in terms of $L_k, \vec{\mathbf{r}}_k, \theta_k, \vec{\mathbf{a}}_k$ and $\vec{\mathbf{e}}_k$. You should come up with a formula that is linear in L_k, $\vec{\mathbf{a}}_k$, and $\vec{\mathbf{e}}_k$.

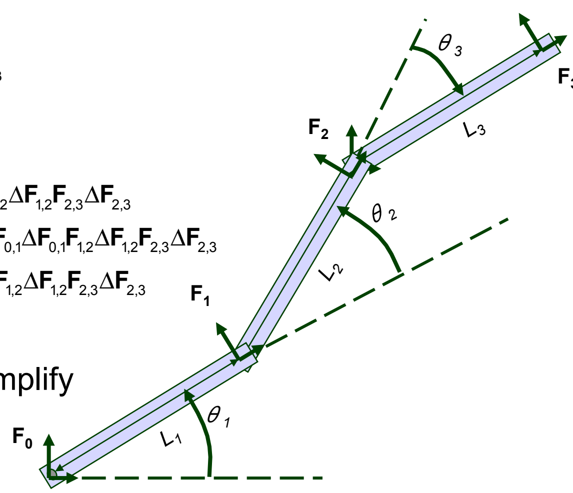


Exercise

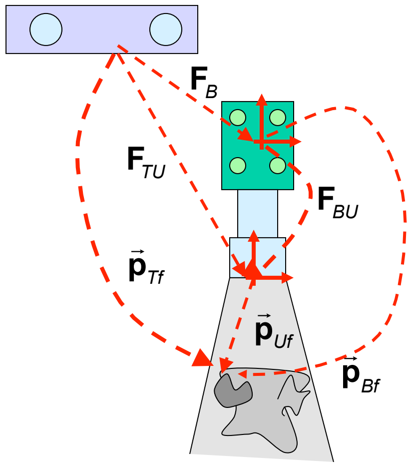
Suppose we want to know error in $\mathbf{F}_{0,3} = \mathbf{F}_0^{-1}\mathbf{F}_3$

$$\mathbf{F}_{0,3} = \mathbf{F}_0^{-1}\mathbf{F}_0\mathbf{F}_{0,1}\mathbf{F}_{1,2}\mathbf{F}_{2,3}$$
$$\mathbf{F}_{0,3}^{*} = \mathbf{F}_{0,1}^{*}\mathbf{F}_{1,2}^{*}\mathbf{F}_{2,3}^{*}$$
$$\mathbf{F}_{0,3}\Delta\mathbf{F}_{0,3} = \mathbf{F}_{0,1}^{*}\mathbf{F}_{1,2}^{*}\mathbf{F}_{2,3}^{*}$$
$$\Delta\mathbf{F}_3 = \mathbf{F}_{0,3}^{-1}\mathbf{F}_{0,1}\Delta\mathbf{F}_{0,1}\mathbf{F}_{1,2}\Delta\mathbf{F}_{1,2}\mathbf{F}_{2,3}\Delta\mathbf{F}_{2,3}$$
$$= \mathbf{F}_{2,3}^{-1}\mathbf{F}_{1,2}^{-1}\mathbf{F}_{0,1}^{-1}\mathbf{F}_{0,1}\Delta\mathbf{F}_{0,1}\mathbf{F}_{1,2}\Delta\mathbf{F}_{1,2}\mathbf{F}_{2,3}\Delta\mathbf{F}_{2,3}$$
$$= \mathbf{F}_{2,3}^{-1}\mathbf{F}_{1,2}^{-1}\Delta\mathbf{F}_{0,1}\mathbf{F}_{1,2}\Delta\mathbf{F}_{1,2}\mathbf{F}_{2,3}\Delta\mathbf{F}_{2,3}$$

Now substitute and simplify



Another Example



Another Example

$$\vec{p}_{Tf} = F_{TU} \bullet \vec{p}_{Uf}$$

$$F_{TU} = F_B \bullet F_{BU}$$

$$= [R_B \bullet R_{BU}, R_B \bullet \vec{p}_{BU} + \vec{p}_B]$$

$$\vec{p}_{Tf} = R_B \bullet R_{BU} \bullet \vec{p}_{Uf} + R_B \bullet \vec{p}_{BU} + \vec{p}_B$$

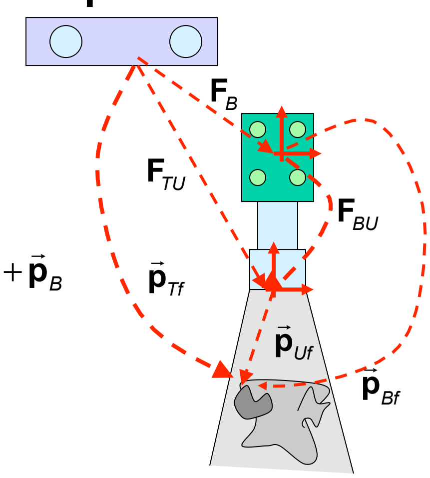
Also

$$\vec{p}_{Tf} = F_B \bullet \vec{p}_{Bf}$$

$$\vec{p}_{Bf} = F_{BU} \bullet \vec{p}_{Uf}$$

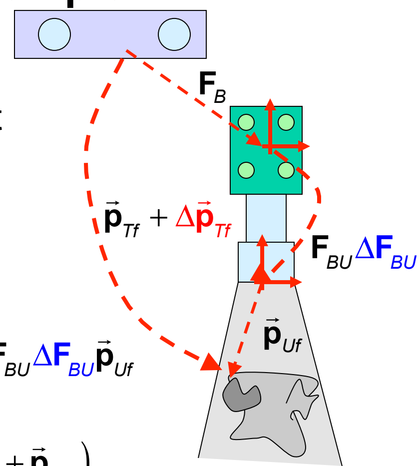
$$= R_{BU} \bullet \vec{p}_{Uf} + \vec{p}_{BU}$$

$$\vec{p}_{Tf} = R_B \bullet R_{BU} \bullet \vec{p}_{Uf} + R_B \bullet \vec{p}_{BU} + \vec{p}_B$$



Another Example

Suppose that the track body to US calibration is not perfect



$$F_{BU}^{*} = F_{BU}\Delta F_{BU}$$
$$= [R_{BU}\Delta R_{BU}, R_{BU}\Delta\vec{p}_{BU} + \vec{p}_{BU}]$$

$$\vec{p}_{Bf}^{*} = F_{BU}^{*} \bullet \vec{p}_{Uf} \quad \text{I.e.,} \quad \vec{p}_{Bf} + \Delta\vec{p}_{BF} = F_{BU}\Delta F_{BU}\vec{p}_{Uf}$$

$$\Delta\vec{p}_{Bf} = F_{BU}\Delta F_{BU}\vec{p}_{Uf} - \vec{p}_{Bf}$$
$$= F_{BU}(\Delta R_{BU}\vec{p}_{Uf} + \Delta\vec{p}_{BU}) - (R_{BU}\vec{p}_{Uf} + \vec{p}_{BU})$$
$$= R_{BU}\Delta R_{BU}\vec{p}_{Uf} + R_{BU}\Delta\vec{p}_{BU} + \vec{p}_{BU} - R_{BU}\vec{p}_{Uf} - \vec{p}_{BU}$$
$$= R_{BU}\Delta R_{BU}\vec{p}_{Uf} + R_{BU}\Delta\vec{p}_{BU} - R_{BU}\vec{p}_{Uf}$$

Another Example

Continuing …

$$\Delta\vec{p}_{Bf} = R_{BU}\Delta R_{BU}\vec{p}_{Uf} + R_{BU}\Delta\vec{p}_{BU} - R_{BU}\vec{p}_{Uf}$$
$$\approx R_{BU}(I + \mathbf{skew}(\vec{\alpha}_{BU}))\vec{p}_{Uf} + R_{BU}\Delta\vec{p}_{BU} - R_{BU}\vec{p}_{Uf}$$
$$= R_{BU}\vec{p}_{Uf} + R_{BU} \bullet \vec{\alpha}_{BU} \times \vec{p}_{Uf} + R_{BU}\Delta\vec{p}_{BU} - R_{BU}\vec{p}_{Uf}$$
$$= R_{BU} \bullet \vec{\alpha}_{BU} \times \vec{p}_{Uf} + R_{BU}\Delta\vec{p}_{BU}$$
$$= -R_{BU} \bullet \vec{p}_{Uf} \times \vec{\alpha}_{BU} + R_{BU}\Delta\vec{p}_{BU}$$
$$= R_{BU}\,skew(-\vec{p}_{Uf})\vec{\alpha}_{BU} + R_{BU}\Delta\vec{p}_{BU}$$

Another Example

$$\vec{\mathbf{p}}_{Tf} + \Delta\vec{\mathbf{p}}_{Tf} = \mathbf{F}_B \Delta\mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right)$$

$$\Delta\vec{\mathbf{p}}_{Tf} = \mathbf{F}_B \Delta\mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) - \mathbf{F}_B \vec{\mathbf{p}}_{Bf}$$

$$\Delta\mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) = \Delta\mathbf{R}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \Delta\vec{\mathbf{p}}_B$$

$$\approx \left(\mathbf{I} + skew(\vec{\alpha}_B)\right)\left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \Delta\vec{\mathbf{p}}_B$$

$$= \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \Delta\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B$$

$$\approx \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B$$

$$\Delta\vec{\mathbf{p}}_{Tf} \approx \mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right) - \mathbf{F}_B \vec{\mathbf{p}}_{Bf}$$

$$= \mathbf{R}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right) + \vec{\mathbf{p}}_B - \left(\mathbf{R}_B \vec{\mathbf{p}}_{Bf} + \vec{\mathbf{p}}_B\right)$$

$$= \mathbf{R}_B \left(\Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right)$$

$$\Delta\vec{\mathbf{p}}_{Bf} \approx \mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}$$

Another Example

$$\Delta\vec{\mathbf{p}}_{Tf} \approx \mathbf{R}_B \left(\Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right)$$

$$\Delta\vec{\mathbf{p}}_{Bf} \approx \mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}$$

$$\Delta\vec{\mathbf{p}}_{Tf} \approx \mathbf{R}_B \left(\mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right)$$

$$= \left(\begin{array}{c} \mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} \\ +\mathbf{R}_B skew(-\vec{\mathbf{p}}_{Bf})\vec{\alpha}_B + \Delta\vec{\mathbf{p}}_B \end{array}\right)$$

$$= \left[\mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU}) \;\middle|\; \mathbf{R}_{BU} \;\middle|\; \mathbf{R}_B skew(-\vec{\mathbf{p}}_{Bf}) \;\middle|\; \mathbf{I}\right] \left[\begin{array}{c} \vec{\alpha}_{BU} \\ \Delta\vec{\mathbf{p}}_{BU} \\ \vec{\alpha}_B \\ \Delta\vec{\mathbf{p}}_B \end{array}\right]$$

Parametric Sensitivity

Suppose you have an explicit formula like

$$\vec{\mathbf{p}}_3 = \begin{bmatrix} L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) \\ L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) + L_3\sin(\theta_1+\theta_2+\theta_3) \\ 0 \end{bmatrix}$$

and know that the only variation is in parameters like L_k and θ_k. Then you can estimate the variation in $\vec{\mathbf{p}}_3$ as a function of variation in L_k and θ_k by remembering your calculus.

$$\Delta\vec{\mathbf{p}}_3 \cong \begin{bmatrix} \frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{L}} & \frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{\theta}} \end{bmatrix} \begin{bmatrix} \Delta\vec{L} \\ \Delta\vec{\theta} \end{bmatrix}$$

Parametric Sensitivity

Grinding this out gives:

$$\Delta\vec{\mathbf{p}}_3 \cong \begin{bmatrix} \frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{L}} & \frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{\theta}} \end{bmatrix} \begin{bmatrix} \Delta\vec{L} \\ \Delta\vec{\theta} \end{bmatrix}$$

where

$$\vec{L} = [L_1, L_2, L_3]^T$$

$$\vec{\theta} = [\theta_1, \theta_2, \theta_3]^T$$

$$\frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{L}} = \begin{bmatrix} \cos(\theta_1) & \cos(\theta_1+\theta_2) & \cos(\theta_1+\theta_2+\theta_3) \\ \sin(\theta_1) & \sin(\theta_1+\theta_2) & \sin(\theta_1+\theta_2+\theta_3) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{\theta}} = \begin{bmatrix} -L_1\sin(\theta_1) - L_2\sin(\theta_1+\theta_2) - L_3\sin(\theta_1+\theta_2+\theta_3) & -L_2\sin(\theta_1+\theta_2) - L_3\sin(\theta_1+\theta_2+\theta_3) & -L_3\sin(\theta_1+\theta_2+\theta_3) \\ L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) & L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) & L_3\cos(\theta_1+\theta_2+\theta_3) \\ 0 & 0 & 0 \end{bmatrix}$$

More generally …

Suppose that we have a vector function

$$\vec{\mathbf{v}} = \vec{\mathbf{g}}(\vec{\mathbf{q}}) \quad = [g_1(\vec{\mathbf{q}}),...,g_m(\vec{\mathbf{q}})]^T$$

of parameters $\vec{\mathbf{q}} = [q_1,...,q_n]$. Then we can estimate the value of

$$\vec{\mathbf{v}} + \Delta\vec{\mathbf{v}} = \vec{\mathbf{g}}(\vec{\mathbf{q}} + \Delta\vec{\mathbf{q}})$$

by

$$\vec{\mathbf{v}} + \Delta\vec{\mathbf{v}} \approx \vec{\mathbf{g}}(\vec{\mathbf{q}}) + \mathbf{J}_{\mathbf{G}}(\vec{\mathbf{q}}) \bullet \Delta\vec{\mathbf{q}}$$

where

$$\mathbf{J}_{\mathbf{G}}(\vec{\mathbf{q}}) = \begin{bmatrix} \frac{\partial g_1}{\partial q_1} & & \frac{\partial g_1}{\partial q_j} & & \frac{\partial g_1}{\partial q_n} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_i}{\partial q_1} & \cdots & \frac{\partial g_i}{\partial q_j} & \cdots & \frac{\partial g_i}{\partial q_n} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_m}{\partial q_1} & & \frac{\partial g_m}{\partial q_j} & & \frac{\partial g_m}{\partial q_n} \end{bmatrix}$$