Question 1

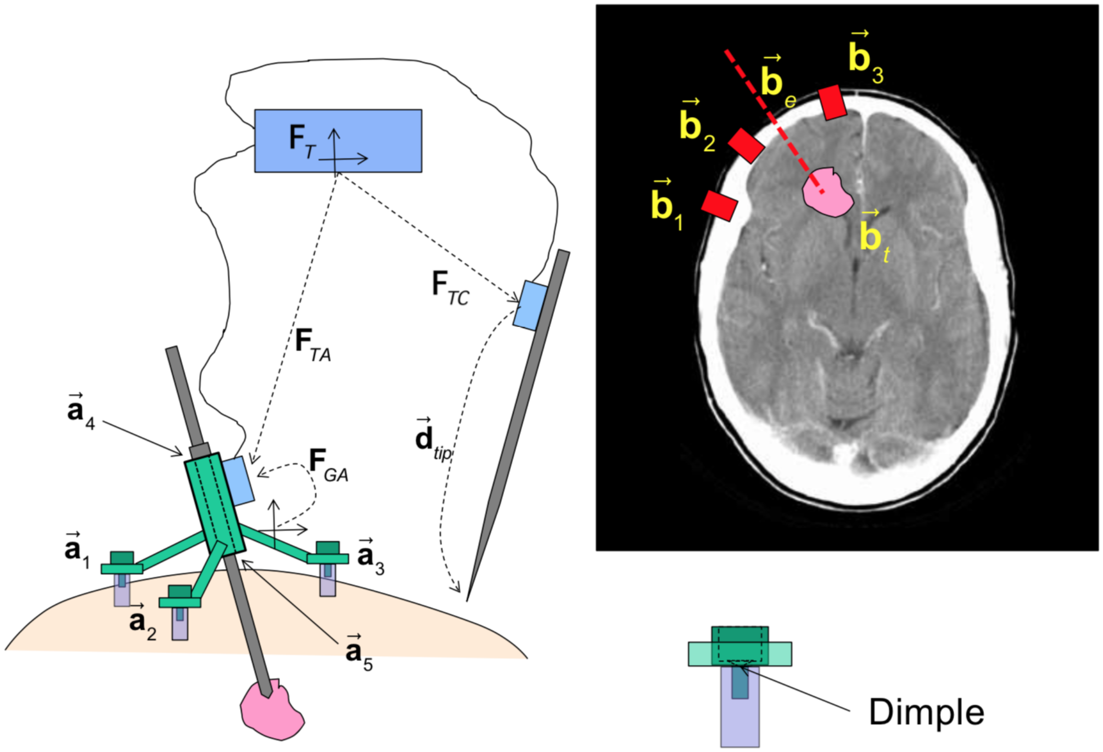
Suppose that we have $\mathbf{F}^* = \mathbf{F}\Delta\mathbf{F}_R = \Delta\mathbf{F}_L\mathbf{F}$, where

$$\mathbf{F} = [\mathbf{R},\vec{\mathbf{p}}]$$
$$\Delta\mathbf{F}_L = [\Delta\mathbf{R}_L,\Delta\vec{\mathbf{p}}_L] \approx [\mathbf{I} + sk(\vec{\alpha}_L),\vec{\varepsilon}_L]$$
$$\Delta\mathbf{F}_R = [\Delta\mathbf{R}_R,\Delta\vec{\mathbf{p}}_R] \approx [\mathbf{I} + sk(\vec{\alpha}_R),\vec{\varepsilon}_R]$$

A. Give expressions for $\Delta\mathbf{F}_L$ and $\Delta\mathbf{F}_R$ in terms of $\mathbf{F}$, $\Delta\mathbf{F}_R$, and $\Delta\mathbf{F}_L$, avoiding tautologies like $\Delta\mathbf{F}_L = \Delta\mathbf{F}_L$.

B. Give expressions for $\Delta\mathbf{R}_L$, $\Delta\mathbf{R}_R$, $\Delta\vec{\mathbf{p}}_L$, and $\Delta\vec{\mathbf{p}}_R$ in terms of the other quantities while avoiding tautologies.

C. Give simplified expressions for the linearized error variables $\vec{\alpha}_L$, $\vec{\alpha}_R$, $\vec{\varepsilon}_L$, and $\vec{\varepsilon}_R$ in terms of the other quantities while avoiding tautologies. Also, express your answer in “standard” form, in which small error variables are shown as sums of terms with the general form $\mathbf{M}_k\vec{\eta}_k$ where $\vec{\eta}_k$ are small error variables and $\mathbf{M}_k$ is an expression containing quantities known to the computer. For example, one might imagine an answer $\vec{\gamma} = \mathbf{R}sk(\vec{\mathbf{v}})\vec{\alpha} + \vec{\beta}$.



Consider the stereotactic system shown in Fig. 1, which is based loosely on work from Vanderbilt University (e.g., Henderson JM, Holloway KL, Gaede SE, Rosenow JM: "The application accuracy of a skull-mounted trajectory guide for image-guided functional neurosurgery". *Computer Aided Surgery* 2004;9:155–160). The work flow for this system is as follows:

1. Three small fiducial devices are screwed into the patient's skull. Each fiducial device has a threaded hole in its top.
2. A 3D volumetric scan (CT or MRI) is taken of the patient's head. For simplicity, we will assume CT. For the purpose of this problem, you can assume that all CT coordinates are expressed in mm.
3. The CT coordinates $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2$, and $\vec{\mathbf{b}}_3$ of the tops (outer ends) the threaded holes in the fiducials are determined by image processing.
4. A CT coordinates of desired entry point $\vec{\mathbf{b}}_e$ and target point $\vec{\mathbf{b}}_t$ for a stereotactic needle insertion are determined by the surgeon, using planning software.
5. A custom needle guide is fabricated. This guide has three holes located at points $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2$, and $\vec{\mathbf{a}}_3$ in a local coordinate system

6. In surgery, the guide is placed so that the holes in the guide are lined up with the corresponding fiducials and secured with three small screws. The tube is used to guide a drill for creating a burr hole at the entry point. Then, a needle with a stop at distance from the tip is passed through the tube to hit the planned target.

[The next part is a bit artificial, but is here for the assignment]: The guide is also constructed so that a navigational marker is placed at a pose $\mathbf{F}_{GA} = [\mathbf{R}_{GA}, \vec{\mathbf{p}}_{GA}]$ relative to the guide coordinate system. A navigational tracking system is able to track the position and orientation $\mathbf{F}_{TA} = [\mathbf{R}_{TA}, \vec{\mathbf{p}}_{TA}]$ of this marker relative to the base unit of the tracking system. It is also able to track the position and orientation $\mathbf{F}_{TC} = [\mathbf{R}_{TC}, \vec{\mathbf{p}}_{TC}]$ of a second marker attached to a pointer device. The tip of this pointer is located at a point $\vec{\mathbf{d}}_{tip}$ relative to the tracker marker on the pointer, so that the tip of the pointer is at position $\mathbf{F}_{TC} \bullet \vec{\mathbf{d}}_{tip}$ relative to the tracker base unit.

4. Give expression for computing the positions $\vec{\mathbf{p}}_{ptr}$ of the pointer tip relative to the coordinate systems of the tracking marker attached to the guide. Also, give an expression for the position $\vec{\mathbf{p}}_{Gp}$ of the pointer tip relative to the coordinate system of the guide. Give these first in terms of the **F** values and then expand it out into an expression involving the corresponding **R** and $\vec{\mathbf{p}}$ values.

5. Suppose that we have computed a transformation $\mathbf{F}_{reg}$ such that any point $\vec{\mathbf{a}}$ in guide coordinates corresponds to a point $\vec{\mathbf{b}} = \mathbf{F}_{reg} \bullet \vec{\mathbf{a}}$ in CT coordinates for computing the CT coordinates $\vec{\mathbf{b}}_{ptr}$ corresponding to $\vec{\mathbf{p}}_{ptr}$, assuming that the navigational tracker is perfectly accurate. Give an expression for $\vec{\mathbf{b}}_{ptr}$ in terms of the **R**'s and $\vec{\mathbf{p}}$'s known to the system.

6. Suppose now that the navigational tracker is not perfectly accurate, so that the actual values for $\mathbf{F}_{TA}$ and $\mathbf{F}_{TC}$ are $\mathbf{F}_{TA}^{*} = \mathbf{F}_{TA} \bullet \Delta\mathbf{F}_{TA}$ and $\mathbf{F}_{TC}^{*} = \mathbf{F}_{TC} \bullet \Delta\mathbf{F}_{TC}$. What is the error $\Delta\vec{\mathbf{p}}_{ptr}$ between the computed and actual values of $\vec{\mathbf{p}}_{ptr}$. Express this in terms first of the $\Delta\mathbf{F}$'s and then in terms of the $\Delta\mathbf{R}$'s and $\Delta\vec{\mathbf{p}}$'s. **Hint:** first compute expressions for $\Delta\mathbf{F}_{AC}$, where $\mathbf{F}_{AC}{}^{*} = \mathbf{F}_{AC}\Delta\mathbf{F}_{AC}$. You may use $\mathbf{F}_{AC} = \left[\mathbf{R}_{AC}, \vec{\mathbf{p}}_{AC}\right]$ in your answer provided that you first show the formula for computing it in terms of the **F**'s.

7. Assume now that we know that the $\Delta\mathbf{R}$'s and $\Delta\vec{\mathbf{p}}$'s are "small", so that $\Delta\mathbf{F}_i \approx [\mathbf{I} + sk(\vec{\alpha}_i), \vec{\varepsilon}_i]$ for each of the measured frames $\mathbf{F}_i$. Compute a simplified expression for $\Delta\mathbf{p}_{ptr}$. Please express your answer as a sum of linearized products $\mathbf{M}_{TC\alpha}\vec{\alpha}_{TC} + \mathbf{M}_{TC\varepsilon}\vec{\varepsilon}_{TC} + \mathbf{M}_{TA\alpha}\vec{\alpha}_{TA} + \mathbf{M}_{TA\varepsilon}\vec{\varepsilon}_{TA}$ where each of the **M**'s can be expressed as a 3x3 matrix of quantities known to the computer. **Hint:** Some of these M's may involve skew matrices.