Homework Assignment 2 – 600.455/655 Fall 2021

Instructions and Score Sheet (hand in with answers)

Name	Name
Email	Email
Other contact information (optional)	Other contact information (optional)
I have followed the rules in completing this assignment (signature)	I have followed the rules in completing this assignment (signature)

Question	Points		Question	Points		Total
1A	5		2A	10		
1B	10		2B	10		
1C	5		2C	10		
1D	5		2D	10		
1E	5		2E	10		
1F	5		2F	15		
Subtotal	35		Subtotal	65		

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone or with your partner** and are **not to discuss the problems with anyone** other than the TAs or the instructor. **(NOTE:** You are strongly encouraged to work with a partner).
3. It is otherwise open book, notes, and web. But you should cite any references you consult.
4. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
5. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web. See the course organizational materials.
6. Sign and hand in the score sheet as the first sheet of your assignment.
7. You will submit the assignment in PDF form to Gradescope, as discussed in class.

Scenario: Image Overlay System

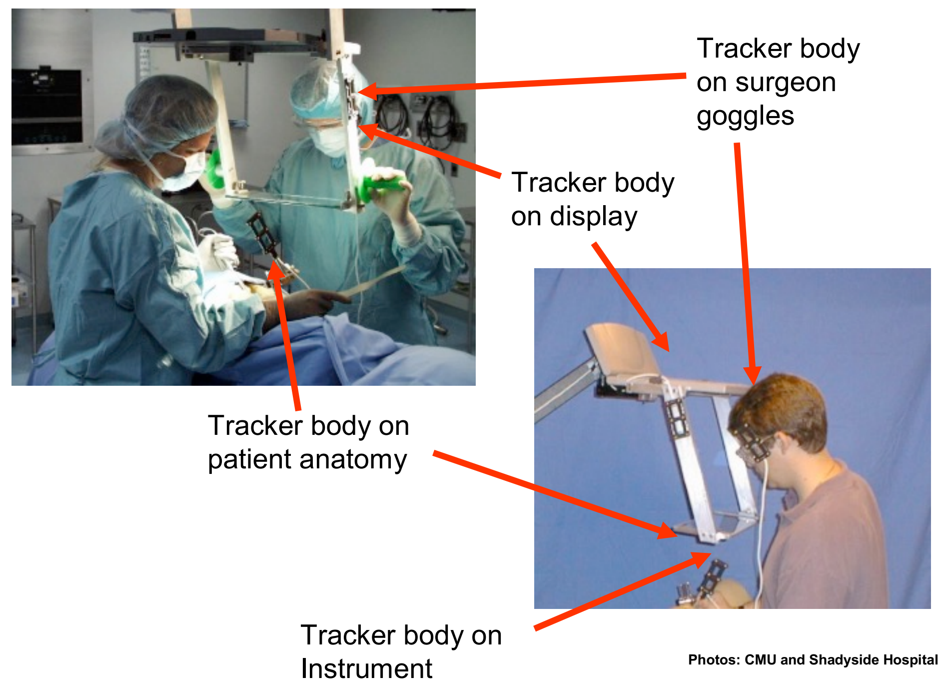

Figure 1: CMU Image Overlay System

This problem is based on an image overlay system similar in some respects to the CMU Image Overlay system shown in Figure 1. The system consists of an optical tracking device with multiple "rigid bodies" attached to tools, to the patient's anatomy, to a pair of eye goggles, and to a see-through display device. For simplicity, we will assume that this see-through display generates graphic images directly on the device, rather than relying on mirrors, as the CMU device does. Figure 2 illustrates these components and provides notation for use in the problems.

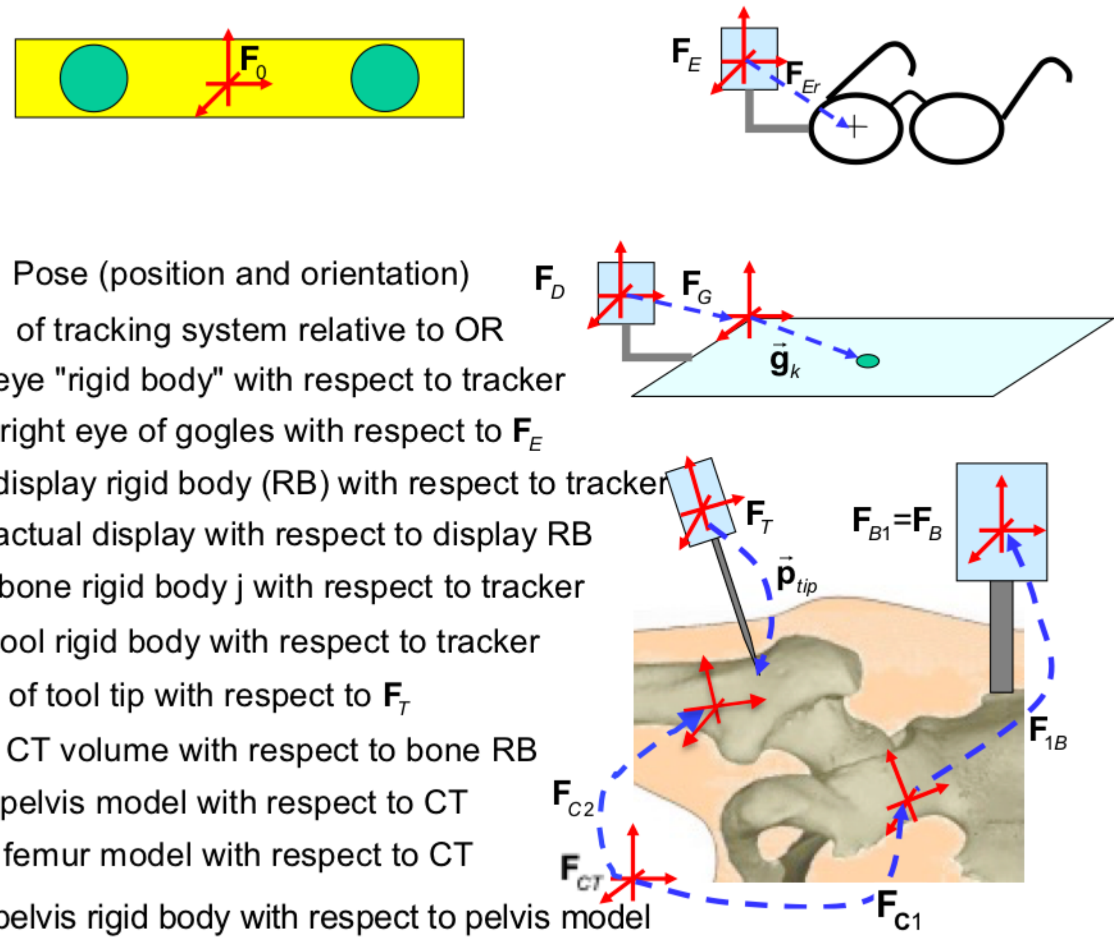

Figure 2

The procedural flow associated with hip surgery example illustrated is as follows:

1. Before surgery, CT images of the patient are made and bone models are made. Coordinate frames (also called "poses") associated with bone models derived from the CT images are denoted $\mathbf{F}_{Ck}$ and defined with respect to the coordinate system of the CT volume (i.e., CT coordinates). Other poses important to the plan may be defined with respect to CT coordinates and denoted $\mathbf{F}_{Ck}$ or defined with respect to bone models and denoted $\mathbf{F}_{ik}$ where i is 1 for the pelvis and 2 for the femur. Where only positions are needed, and not full poses, they are denoted $\vec{\mathbf{p}}_{Cj}$ or $\vec{\mathbf{p}}_{ij}$ depending on what coordinate system they are defined with respect to.
2. In surgery, rigid bodies 1 and 2 are attached to the patient's pelvis and femur respectively, and the tracking system tracks their poses $\mathbf{F}_{B1}$ and $\mathbf{F}_{B2}$ with respect to the tracker. For simplicity, we will also refer to the pose of the rigid body attached to the

pelvis as $\mathbf{F}_B$. I.e., $\mathbf{F}_B = \mathbf{F}_{B1}$. The tracking system also tracks the poses of the display, eye goggle, and tool rigid bodies ($\mathbf{F}_D$, $\mathbf{F}_E$, and $\mathbf{F}_T$).

3. A pivot calibration is performed (if necessary) to determine the displacement $\vec{\mathbf{p}}_{tip}$ with respect to the tool rigid body pose. Thus, the position of the tip with respect to the tracker system is $\mathbf{F}_T \bullet \vec{\mathbf{p}}_{tip}$.
4. A registration process is performed using the calibrated tool to determine the poses $\mathbf{F}_{1B}$and $\mathbf{F}_{2B}$ of the pelvis and femur rigid bodies with respect to the coordinate systems associated with the pelvis and femur models.
5. After this registration process is done, the system displays graphic objects (e.g., bone models, plan advice) on the display so that they appear to be superimposed on the surgeon's view of the patient. In reality, this would be done with a stereo display, but for simplicity, we will only consider the view from the surgeon's right eye. The relevant position and orientation of the

right eye relative to the tracking system is $\mathbf{F}_r = \mathbf{F}_E \bullet \mathbf{F}_{Er}$. A typical point on the graphic display will be given by coordinates $\vec{\mathbf{g}}_k = (u_k, v_k, 0)$ with respect to the coordinate system of the display device.

Problem 1

The questions below ask you to give expressions for positions or poses relative to various coordinate systems. In answering them, give answers first in terms of the **F**'s and in terms of **R**'s and $\vec{\mathbf{p}}$'s. For example, if a question asks for the pose of the right eye relative to the tracking system, you should give the following

$$\mathbf{F}_r = \mathbf{F}_E \bullet \mathbf{F}_{Er}$$
$$\mathbf{R}_r = \mathbf{R}_E \bullet \mathbf{R}_{Er}$$
$$\vec{\mathbf{p}}_r = \mathbf{R}_E \bullet \vec{\mathbf{p}}_{Er} + \vec{\mathbf{p}}_E$$

A. Write expressions for pose $\mathbf{F}_p$ of the pelvis model relative to the tracker in terms of $\mathbf{F}_{B1}$ and $\mathbf{F}_{1B}$.

B. Write an expression giving the position $\vec{\mathbf{p}}_{Gtip}$ of the tool tip relative to the pelvis model using the components of $\mathbf{F}_p$ and the other quantities given in the scenario.

C. Write an expression giving the intraoperative pose $\mathbf{F}_{pf}$ of the femur model relative to the pelvis model (i.e., do not assume that the femur is in the same place relative to the pelvis as it was during the CT scan). Here, you may assume that there is a rigid tracker body (not shown on the figure) affixed to the femur, whose pose relative to the tracking system is $\mathbf{F}_{B2}$, and whose pose relative to the femur model is assumed to be $\mathbf{F}_{2B}$. Express you answer in terms of $\mathbf{F}_{p}$ and other quantities defined in the scenario. Expand into components. **Hint:** Define a frame $\mathbf{F}_{f}$ for the pose of the femur model relative to the tracking system. Provide an answer in terms of the components of $\mathbf{F}_{p}$ and $\mathbf{F}_{f}$.

D. Write an expression giving the pose $\mathbf{F}_{G1} = \left[\mathbf{R}_{G1}, \vec{\mathbf{p}}_{G1}\right]$ of the pelvis model relative to the display screen, using $\mathbf{F}_{p}$. Expand into components. **Hint:** Define a frame $\mathbf{F}_{dpy}$ for the pose of the display relative to the tracking system. Express your answer in terms of the components of $\mathbf{F}_{D}$, $\mathbf{F}_{G}$, and $\mathbf{F}_{p}$.

E. Write an expressions for the components of the pose $\mathbf{F}_{eye} = [\mathbf{R}_{eye}, \vec{\mathbf{p}}_{eye}]$ in terms of other variables defined in the scenario. Then write an expression giving the pose $\mathbf{F}_{Gr} = \left[\mathbf{R}_{Gr}, \vec{\mathbf{p}}_{Gr}\right]$ of the right eye relative to the display screen in terms of the components of $\mathbf{F}_{dpy}$ and $\mathbf{F}_{eye}$.

F. Suppose that an anatomic feature with position $\vec{\mathbf{p}}_{Cj}$ defined relative to the CT image is known to move with the pelvis. Give an expression for the position $\vec{\mathbf{p}}_{Gj}$ of this feature relative to the display screen. You may express this answer in terms of $\mathbf{F}_{dpy}$ and $\mathbf{F}_{p}$. (**Hint:** your answer should include the computation of the position $\vec{\mathbf{p}}_{1j}$ of the feature relative to the pelvis model).

Problem 2

Suppose now that the tracking system has some error, so that if the tracking system reports the pose of a rigid body to be $\mathbf{F}_X$, then the actual pose will be $\mathbf{F}_X^* = \mathbf{F}_X \Delta\mathbf{F}_X$, where $\Delta\mathbf{F}_X \approx [\mathbf{I} + sk(\vec{\alpha}_x), \vec{\varepsilon}_X]$.

A. Suppose that that there has been some small manufacturing error in the tool or that the tool has been bent slightly, so that the actual position of the tool tip relative to the tracker body is $\vec{\mathbf{p}}^*_{tip} = \vec{\mathbf{p}}_{tip} + \Delta\vec{\mathbf{p}}_{tip} = \vec{\mathbf{p}}_{tip} + \vec{\varepsilon}_{tip}$. Recall that the nominal position of the tool tip relative to the tracker system is $\vec{\mathbf{p}}_{Tt} = \mathbf{F}_T \bullet \vec{\mathbf{p}}_{tip}$, but that the tracker system has some error. Give an expression for the error $\Delta\vec{\mathbf{p}}_{Tt}$ in the position of the tip relative to the tracking system in terms of $\Delta\mathbf{R}_T, \Delta\vec{\mathbf{p}}_T, \vec{\mathbf{p}}_{tip}, \Delta\vec{\mathbf{p}}_{tip}$ and the components of $\mathbf{F}_T$.

B. Now give a linearized formula approximating $\vec{\varepsilon}_{Tt}$ in terms of $\vec{\alpha}_T, \vec{\varepsilon}_T, \vec{\varepsilon}_{tip}$. Express your answer in "normalized" matrix form as

$$\vec{\varepsilon}_{Tt} \approx \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} \bullet \begin{bmatrix} \vec{\alpha}_T \\ \vec{\varepsilon}_T \\ \vec{\varepsilon}_{tip} \end{bmatrix}$$

where you will provide formulas for the A_k. Equivalently, you can write this as $\vec{\varepsilon}_{Tt} \approx A_1\vec{\alpha}_T + A_2\vec{\varepsilon}_T + A_3\vec{\varepsilon}_{tip}$

The key thing is that all the small error variables should be on the right hand sides.

C. Similarly give expressions giving the error in the system's estimate of $\mathbf{F}_p$ where the actual value is $\mathbf{F}_p^* = \mathbf{F}_p \Delta\mathbf{F}_p$. I.e., give expressions for the errors $\Delta\mathbf{R}_p$ and $\Delta\vec{\mathbf{p}}_p$ in terms of the $\Delta\mathbf{R}$'s and $\Delta\vec{\mathbf{p}}$'s and other variables in the system.

D. Now give linearized approximations in "normalized" form for these error values. I.e., give normalized form expressions for $\vec{\alpha}_p$ and $\vec{\varepsilon}_p$ in terms of the various $\vec{\alpha}$'s, $\vec{\varepsilon}$'s and other variables in the system.

E. Similarly give linearized expressions in "normalized" form to approximate the error $\Delta\vec{\mathbf{p}}_{Gtip} \approx \vec{\varepsilon}_{Gtip}$ in the system's estimate of the position of the tip of the pointer relative to the pelvis, under the assumptions of Questions 2A-2D. Give your answer in terms of the various $\vec{\alpha}$'s, $\vec{\varepsilon}$'s and other variables in the system. Here you may use the variables $\vec{\alpha}_p$ and $\vec{\varepsilon}_p$ (computed in Question 2D) without expanding them.

F. Similarly give linearized expressions in "normalized" form to approximate the error in the system's estimate of $\mathbf{F}_{pf}$ where the actual value is $\mathbf{F}^*_{pf} = \mathbf{F}_{pf} \Delta \mathbf{F}_{pf}$ and $\Delta \mathbf{F}_{pf} \approx [\mathbf{I} + sk(\vec{\alpha}_{pf}), \vec{\varepsilon}_{pf}]$. I.e., give expressions for the $\vec{\alpha}_{pf}$ and $\vec{\varepsilon}_{pf}$. **Hint:** First, find expressions for $\vec{\alpha}_f$ and $\vec{\varepsilon}_f$. Then combine with your answer to Question 2D and make suitable linearization approximations. You can express your answer in terms of $\vec{\alpha}_f, \vec{\varepsilon}_f, \vec{\alpha}_p, \vec{\varepsilon}_p$ and other variables defined in the scenario.