Homework Assignment 3 – 600.455/655 Fall 2021

Instructions and Score Sheet (hand in with answers)

Name	Name
Email	Email
Other contact information (optional)	Other contact information (optional)
I have followed the rules in completing this assignment (signature)	I have followed the rules in completing this assignment (signature)

Question	Points		Question	Points		Total
1A	20		2A	10		
1B	20		2B	20		
			2C	20		
			3	10		
Subtotal	40		Subtotal	60		

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone or with your partner** and are **not to discuss the problems with anyone** other than the TAs or the instructor. **(NOTE:** You are strongly encouraged to work with a partner).
3. It is otherwise open book, notes, and web. But you should cite any references you consult.
4. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
5. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web. See the course organizational materials.
6. Sign and hand in the score sheet as the first sheet of your assignment.
7. You will submit the assignment in PDF form to Gradescope, as discussed in class.

Scenario: Image Overlay System

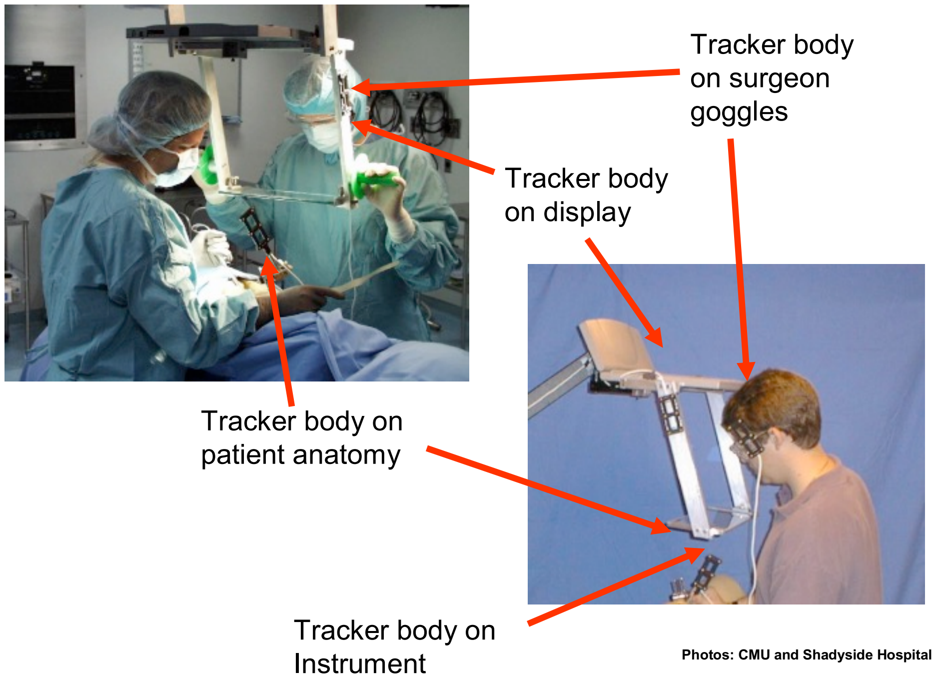
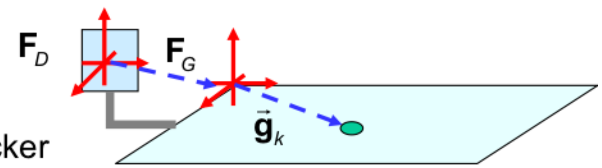


Figure 1: CMU Image Overlay System

This problem is based on an image overlay system similar in some respects to the CMU Image Overlay system shown in Figure 1. The system consists of an optical tracking device with multiple "rigid bodies" attached to tools, to the patient's anatomy, to a pair of eye goggles, and to a see-through display device. For simplicity, we will assume that this see-through display generates graphic images directly on the device, rather than relying on mirrors, as the CMU device does. **Error! Reference source not found.** i llustrates these components and provides notation for use in the problems.

$\mathbf{F}_0 = [\mathbf{R}_0, \vec{\mathbf{p}}_0] =$ Pose (position and orientation) of tracking system relative to OR



$\mathbf{F}_E =$ Pose of eye "rigid body" with respect to tracker

$\mathbf{F}_{Er} =$ Pose of right eye of gogles with respect to $\mathbf{F}_E$

$\mathbf{F}_D =$ Pose of display rigid body (RB) with respect to tracker

$\mathbf{F}_G =$ Pose of actual display with respect to display RB

$\mathbf{F}_{Bj} =$ Pose of bone rigid body j with respect to tracker

$\mathbf{F}_T =$ Pose of tool rigid body with respect to tracker

$\vec{\mathbf{p}}_{tip} =$ Position of tool tip with respect to $\mathbf{F}_T$

$\mathbf{F}_{BC} =$ Pose of CT volume with respect to bone RB

$\mathbf{F}_{C1} =$ Pose of pelvis model with respect to CT

$\mathbf{F}_{C2} =$ Pose of femur model with respect to CT

$\mathbf{F}_{1B} =$ Pose of pelvis rigid body with respect to pelvis model

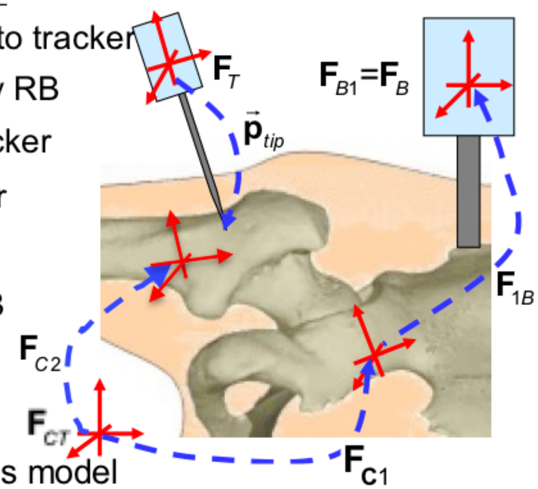


Figure 2

The procedural flow associated with hip surgery example illustrated is as follows:

1. Before surgery, CT images of the patient are made and bone models are made. Coordinate frames (also called "poses") associated with bone models derived from the CT images are denoted $\mathbf{F}_{Ck}$ and defined with respect to the coordinate system of the CT volume (i.e., CT coordinates). Other poses important to the plan may be defined with respect to CT coordinates and denoted $\mathbf{F}_{Ck}$ or defined with respect to bone models and denoted $\mathbf{F}_{ik}$ where i is 1 for the pelvis and 2 for the femur. Where only positions are needed, and not full poses, they are denoted $\vec{\mathbf{p}}_{Cj}$ or $\vec{\mathbf{p}}_{ij}$ depending on what coordinate system they are defined with respect to.
2. In surgery, rigid bodies 1 and 2 are attached to the patient's pelvis and femur respectively, and the tracking system tracks their poses $\mathbf{F}_{B1}$ and $\mathbf{F}_{B2}$ with respect to the tracker. For simplicity, we will also refer to the pose of the rigid body attached to the pelvis as $\mathbf{F}_B$. I.e., $\mathbf{F}_B = \mathbf{F}_{B1}$. The tracking system also tracks the

poses of the display, eye goggle, and tool rigid bodies ($\mathbf{F}_D$, $\mathbf{F}_E$, and $\mathbf{F}_T$).

3. A pivot calibration is performed (if necessary) to determine the displacement $\vec{\mathbf{p}}_{tip}$ with respect to the tool rigid body pose. Thus, the position of the tip with respect to the tracker system is $\mathbf{F}_T \bullet \vec{\mathbf{p}}_{tip}$.
4. A registration process is performed using the calibrated tool to determine the poses $\mathbf{F}_{1B}$and $\mathbf{F}_{2B}$ of the pelvis and femur rigid bodies with respect to the coordinate systems associated with the pelvis and femur models.
5. After this registration process is done, the system displays graphic objects (e.g., bone models, plan advice) on the display so that they appear to be superimposed on the surgeon's view of the patient. In reality, this would be done with a stereo display, but for simplicity, we will only consider the view from the surgeon's right eye. The relevant position and orientation of the right eye relative to the tracking system is $\mathbf{F}_r = \mathbf{F}_E \bullet \mathbf{F}_{Er}$. A

typical point on the graphic display will be given by coordinates $\vec{\mathbf{g}}_k = (u_k, v_k, 0)$ with respect to the coordinate system of the display device.

Problem 1

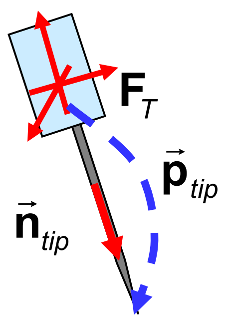


Figure 2

Suppose that the pointing tool is an awl-like instrument. Suppose also that planning system has defined two positions $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ relative to the pelvis model, where point $\vec{\mathbf{a}}$ is on the surface of the pelvis and $\vec{\mathbf{b}}$ is somewhere inside the pelvis. The points $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ define a hole to be drilled into the bone using the tool, so that the tool enters the bone at point $\vec{\mathbf{a}}$ and finishes with the tip at point $\vec{\mathbf{b}}$. Suppose that the direction of the shaft of the tool is given by a vector $\vec{\mathbf{n}}_{tip}$, as shown in Figure 2. When the tool is inserted in the hole, it will be free to rotate about its shaft. I.e., if the tool is inserted into the hole, its possible poses relative to the tracking system will be given by an expression of the form

$$\mathbf{F}_{ins} = \left[\mathbf{R}_{ins}(\theta), \vec{\mathbf{p}}_{ins} \right]$$

where

$$\mathbf{R}_{ins}(\theta) = \mathbf{R}^0_{ins} \bullet Rot(\vec{\mathbf{n}}_{tip}, \theta)$$
$$= Rot(\vec{\mathbf{c}}_{tip}, \theta) \bullet \mathbf{R}^0_{ins}$$

I.e., $\mathbf{F}_T$ will have a value of this form whenever the tool is in the hole. Note that $\vec{\mathbf{c}}_{tip}$ gives the axis of rotation **<u>in tracking system coordinates</u>** and $\vec{\mathbf{n}}_{tip}$ gives the axis **<u>in tool coordinates</u>**.

A. Give expressions for $\vec{\mathbf{p}}_{ins}, \mathbf{R}^0_{ins}$, and $\vec{\mathbf{c}}_{tip}$ in terms of $\mathbf{F}_p$ and other entities described in the scenario. Note that $\vec{\mathbf{c}}_{tip}$ should be a unit vector. For the purpose of this question, you can assume that $\mathbf{F}_p = \mathbf{F}_B \mathbf{F}_{1B}^{-1}$ is the known transformation from pelvis model to tracker coordinates.

B. Suppose that the pointer tool is designed so that $\vec{\mathbf{p}}_{tip} = \vec{\mathbf{p}}_{mount} + L_{tip}\vec{\mathbf{n}}_{tip}$ but that it has been dropped so that its shaft mount has been twisted so that you do not have accurate values for $\vec{\mathbf{n}}_{tip}$ and $\vec{\mathbf{p}}_{tip}$. I.e.,

$\vec{\mathbf{n}}^*_{tip} = \Delta\mathbf{R}_{twist}\vec{\mathbf{n}}_{tip}$. However, there is a convenient hole drilled into the frame of the display. The position and orientation $\mathbf{F}_{GH} = [\mathbf{R}_{GH}, \vec{\mathbf{p}}_{GH}]$ of this hole relative to $\mathbf{F}_G$ is not known, but its diameter is a snug fit for the shaft of the pointer tool and its depth is an unknown value $d_{hole} > 2cm$. Describe a method for determining new values for $\vec{\mathbf{n}}^*_{tip}$ and $\vec{\mathbf{p}}^*_{tip}$. Give the workflow, explain what sensor values you will measure, and give the mathematical formulas and methods that you will use to get your answer.

Problem 2

A. For the moment, suppose that we know the pose $\mathbf{F}_G$ of the actual display device relative to the display rigid body (whose pose relative to the tracker is given by $\mathbf{F}_D$). Then $\mathbf{F}_{dpy} = \mathbf{F}_D\mathbf{F}_G$ is the coordinate system for the actual display device relative to the tracking system. Suppose that there is some tracking error so that the actual values are $\mathbf{F}^*_{dpy} = \Delta\mathbf{F}_{dpy}\mathbf{F}_{dpy} = \Delta\mathbf{F}_D\mathbf{F}_D\Delta\mathbf{F}_G\mathbf{F}_G$ where $\Delta\mathbf{F}_D \approx [\mathbf{I} + sk(\vec{\alpha}_D), \vec{\varepsilon}_D]$ and $\Delta\mathbf{F}_G \approx [\mathbf{I} + sk(\vec{\alpha}_G), \vec{\varepsilon}_G]$. Give formulas providing values for $\Delta\mathbf{F}_{dpy} \approx [\mathbf{I} + sk(\vec{\alpha}_{dpy}), \vec{\varepsilon}]$ in terms of the components of $\mathbf{F}_D$ and $\mathbf{F}_G$ and $\vec{\alpha}_D, \vec{\varepsilon}_D, \vec{\alpha}_G, \vec{\varepsilon}_G$. Express your answer in normalized form,

$$\vec{\alpha}_{dpy} = A_1\vec{\alpha}_D + A_2\vec{\varepsilon}_D + A_3\vec{\alpha}_G + A_4\vec{\varepsilon}_G$$

$$\vec{\varepsilon}_{dpy} = A_5\vec{\alpha}_D + A_6\vec{\varepsilon}_D + A_7\vec{\alpha}_G + A_8\vec{\varepsilon}_G$$

in which the A_k are matrices.

B. Now suppose that we do not have a prior value for the pose $\mathbf{F}_G$ of the actual display device relative to the display rigid body (whose pose relative to the tracker is given by $\mathbf{F}_D$). Describe a simple technique for determining $\mathbf{F}_G$ using only the equipment described in the problem scenario. Include step-by-step instructions and all relevant formulas. In this problem, you can assume that the computer is capable of displaying arbitrary content on the screen. In answering this question, you do not need to recite the steps of algorithms described in class, but you need to explain the formulation and make clear what the inputs and outputs are and give suitable formulas for those. If your answer requires taking multiple measurements over time, use the notation $\mathbf{F}_X[t]$ to indicate a measurement taken at time t.

C. Give an expression for the coordinates $\vec{\mathbf{g}}_j = [u_j, v_j, 0]$ relative to the display coordinate system $\mathbf{F}_{dpy}$ of the intersection of the line between the right eye and a point located at $\vec{\mathbf{p}}_{dj}$ relative to $\mathbf{F}_{dpy}$. You may express your answer in terms of $\vec{\mathbf{p}}_{dj} = [x_{dj}, y_{dj}, z_{dj}]$ and $\vec{\mathbf{p}}_r = [x_r, y_r, z_r]$, where $\vec{\mathbf{p}}_r$ is the position of the right eye relative to $\mathbf{F}_{dpy}$.

Problem 3

We can associate a rotation by angle θ about an axis $\vec{\mathbf{n}}$ with the unit quaternion:

$$Rot(\vec{\mathbf{n}},\theta) \Leftrightarrow \left[\cos\frac{\theta}{2},\ \sin\frac{\theta}{2}\vec{\mathbf{n}}\right]$$

Demonstrate this relationship. I.e., show that

$$Rot((\vec{\mathbf{n}},\theta)\bullet\vec{\mathbf{p}} = \left[\cos\frac{\theta}{2},\ \sin\frac{\theta}{2}\vec{\mathbf{n}}\right]\circ\left[0,\vec{\mathbf{p}}\right]\circ\left[\cos\frac{\theta}{2},\ -\sin\frac{\theta}{2}\vec{\mathbf{n}}\right]$$

To reduce the amount of typing, please use the notation

$$s = \sin\frac{\theta}{2} \qquad c = \cos\frac{\theta}{2}$$

HINT: In addition to trigonometric identities, you may find it useful to recall the identity $\vec{\mathbf{a}}\times(\vec{\mathbf{b}}\times\vec{\mathbf{a}}) = (\vec{\mathbf{a}}\bullet\vec{\mathbf{a}})\vec{\mathbf{b}} - (\vec{\mathbf{a}}\bullet\vec{\mathbf{b}})\vec{\mathbf{a}}$