Homework Assignment 2

600.455/655 Fall 2022

Instructions and Score Sheet (hand in with answers)

Name	Name
Email	Email
Other contact information (optional)	Other contact information (optional)
Signature (required) I/We have followed the rules in completing this assignment ________________________________	Signature (required) I/We have followed the rules in completing this assignment ________________________________

Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.

1. You are to work **alone** or **in teams of two** and are **not to discuss the problems with anyone** other than the TAs or the instructor.
2. It is otherwise open book, notes, and web. But you should cite any references you consult.
3. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
4. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
5. Sign and hand in the score sheet as the first sheet of your assignment.

NOTE: This assignment has a total of 110 points. However, at most 100 points will count toward your course grade.

Scenario

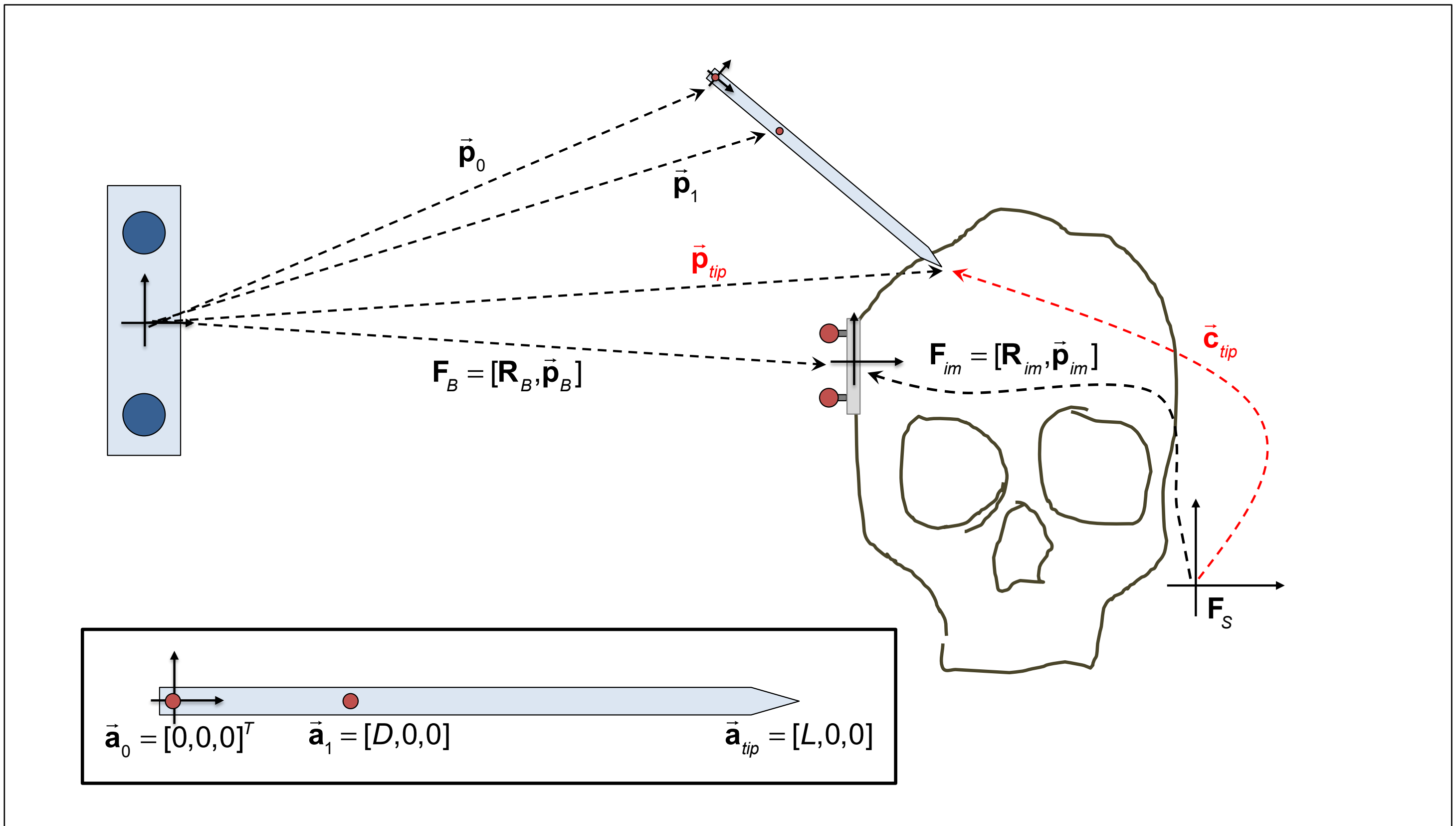

Figure 1: Tracked pointer and skull

Now consider the tracked pointer scenario shown in Figure 1. The pointer has a sharp tip located at position $\vec{\mathbf{a}}_{tip} = [L,0,0]$ in the pointer body coordinate system. There is a stereo camera (which you can think of as the same one as that held in the right arm of the $\Delta\mathbf{F}_{RC}$ two-armed robot) capable of tracking point markers on the pointer body located at positions $\vec{\mathbf{a}}_0 = [0,0,0]$ and $\vec{\mathbf{a}}_1 = [D,0,0]$ in the pointer body coordinate system. The tracking software reports that these two markers are located at positions $\vec{\mathbf{p}}_0$ and $\vec{\mathbf{p}}_1$ relative to the camera coordinate system. An anatomic object (shown here as a skull) is located in the field of view of the camera. A "marker reference body" is attached to the skull. The system has software that can calculate the pose $\mathbf{F}_B = [\mathbf{R}_B, \vec{\mathbf{p}}_B]$ relative to the camera. Also, a CT scan of the skull is available, together with image processing processing software that is able to determine the position and orientation $\mathbf{F}_{im} = [\mathbf{R}_{im}, \vec{\mathbf{p}}_{im}]$ of the marker body in CT coordinates. There are some inaccuracies in all these measurements, so that the actual (unknown) values of all these quantities are given by

$$\mathbf{F}_{im}^* = \Delta\mathbf{F}_{im}\mathbf{F}_{im} \quad \text{where } \Delta\mathbf{F}_{im} = [\Delta\mathbf{R}_{im}, \Delta\vec{\mathbf{p}}_{im}]$$
$$\mathbf{F}_B^* = \Delta\mathbf{F}_B\mathbf{F}_B \quad \text{where } \Delta\mathbf{F}_B = [\Delta\mathbf{R}_B, \Delta\vec{\mathbf{p}}_B]$$
$$\vec{\mathbf{p}}_0^* = \vec{\mathbf{p}}_0 + \Delta\vec{\mathbf{p}}_0 = \vec{\mathbf{p}}_0 + \vec{\varepsilon}_0$$
$$\vec{\mathbf{p}}_1^* = \vec{\mathbf{p}}_1 + \Delta\vec{\mathbf{p}}_1 = \vec{\mathbf{p}}_1 + \vec{\varepsilon}_1$$

We can assume that the various errors are sufficiently small so that we can use linearized approximations. I.e.,

$$\Delta \mathbf{F}_{im} \approx [\mathbf{I} + sk(\vec{\alpha}_{im}), \vec{\varepsilon}_{im}]$$
$$\Delta \mathbf{F}_{B} \approx [\mathbf{I} + sk(\vec{\alpha}_{B}), \vec{\varepsilon}_{B}]$$
$$\Delta \vec{\mathbf{p}}_0 = \vec{\varepsilon}_0$$
$$\Delta \vec{\mathbf{p}}_1 = \vec{\varepsilon}_1$$

where the $\vec{\alpha}_x$ and $\vec{\varepsilon}_x$ are small values. Many of the questions below ask you to produce or use linearized error estimates for various quantities. You will be expected to show your work, and the final answers should be expressed in a normalized matrix-vector format:

$$\vec{\eta}_{xxx} = \sum_k \mathbf{M}_k \vec{\eta}_k$$

where the $\mathbf{M}_k$ are matrices which will typically depend on various $\mathbf{R}$ and $\vec{\mathbf{p}}$ variables and the $\vec{\eta}_k^{\,T} = [\vec{\alpha}_k^T, \vec{\varepsilon}_k^T]$. When only an $\vec{\alpha}$ or $\vec{\varepsilon}$ is involved, you can also have terms that look like $\mathbf{M}_k \vec{\alpha}_k$ or $\mathbf{M}_k \vec{\varepsilon}_k$ and (of course) you can leave off the $\mathbf{M}_k$ if it is an identity matrix. However, we do <u>not</u> want to see final answers with terms involving things like $sk(\vec{\alpha}_k)$, although you may have these in intermediate steps showing how you got to the answers. Also, it is fine to have things like $sk(\vec{\mathbf{p}})$, where $\vec{\mathbf{p}}$ is a vector expression not involving any of the small linearized error variables.

Questions

Question 1: (5 Points) Assuming negligible errors, give an expression for computing the unit direction vector $\vec{\mathbf{n}}_{shaft} = (\vec{\mathbf{p}}_{tip} - \vec{\mathbf{p}}_0)/L$ of the pointer shaft from measured values of $\vec{\mathbf{p}}_0$ and $\vec{\mathbf{p}}_1$. Also, give an expression for computing the position $\vec{\mathbf{p}}_{tip}$ of the pointer shaft tip based on $\vec{\mathbf{p}}_1$ and $\vec{\mathbf{n}}_{shaft}$.

Question 2: (5 Points) Assuming negligible errors, give an expression for computing $\vec{\mathbf{c}}_{tip}$ in terms of $\mathbf{F}_B$, $\mathbf{F}_{im}$, and $\vec{\mathbf{p}}_{tip}$.

Question 3: (5 Points) Expand your answer to question 2 in terms of $\mathbf{R}_B, \vec{\mathbf{p}}_B, \mathbf{R}_{im}, \vec{\mathbf{p}}_{im}$.

Question 4: (10 Points) Assume that the values of $\vec{\mathbf{p}}_0$ and $\vec{\mathbf{p}}_1$ are now subject to small measurement errors $\Delta\vec{\mathbf{p}}_0 = \vec{\varepsilon}_0$ and $\Delta\vec{\mathbf{p}}_1 = \vec{\varepsilon}_1$, give a formulas estimating the approximate error $\Delta\vec{\mathbf{p}}_{tip} = \vec{\varepsilon}_{tip}$ in the tip position and the angle γ between the computed direction $\vec{\mathbf{n}}_{shaft}$ and the actual direction $\vec{\mathbf{n}}^*_{shaft}$, based on your answers to Question 1.

Question 5: (10 Points) Suppose that the true values for $\mathbf{F}_B$, $\mathbf{F}_{im}$, and $\vec{\mathbf{p}}_{tip}$ have some error, so that the actual values are given by $\mathbf{F}^*_B = \Delta\mathbf{F}_B\mathbf{F}_B$, $\mathbf{F}^*_{im} = \Delta\mathbf{F}_{im}\mathbf{F}_{im}$, and $\vec{\mathbf{p}}^*_{tip} = \vec{\mathbf{p}}_{tip} + \Delta\vec{\mathbf{p}}_{tip}$. Produce a formula for the error $\Delta\vec{\mathbf{c}}_{tip}$ in terms of the $\Delta\mathbf{F}$'s and other variables such that the actual value is $\vec{\mathbf{c}}^*_{tip} = \vec{\mathbf{c}}_{tio} + \Delta\vec{\mathbf{c}}_{tip}$.

Question 6: (5 Points) Express your answer to Question 5 in terms of $\mathbf{R}_B, \vec{\mathbf{p}}_B, \mathbf{R}_{im}, \vec{\mathbf{p}}_{im}$ and $\Delta\mathbf{R}_B, \Delta\vec{\mathbf{p}}_B, \Delta\mathbf{R}_{im}, \Delta\vec{\mathbf{p}}_{im}$.

Question 7: (10 Points) Give a linearized expression for estimating the error $\Delta\vec{\mathbf{c}}_{tip}$, assuming that the measurement errors described in the scenario are non-negligible.

Question 8: (10 Points) Now, assume that the you know a bound on the total measurement error for the optical tracking system is given by $\left\|\Delta\vec{\mathbf{p}}_j\right\| \le \delta_{tracker}$ and that the measurement error in the optical tracking system has "random" and "smoothly varying components", such that $\Delta\vec{\mathbf{p}}_j = \vec{\varepsilon}_j = \vec{\varepsilon}_{j,r} + \vec{\varepsilon}_{j,s}$ where $\left\|\vec{\varepsilon}_{j,s} - \vec{\varepsilon}_{k,s}\right\| \le \rho\left\|\vec{\mathbf{p}}_{j,s} - \vec{\mathbf{p}}_{k,s}\right\|$ and $\left\|\vec{\varepsilon}_{j,r}\right\| \le \delta_r$. Give an expression for estimating a bound on the maximum error $\left\|\vec{\varepsilon}_{tip}\right\|$. **Hint:** Consider the worst-case error in the tool shaft direction under these assumptions.

Question 9: (15 Points) In addition to the assumptions in Question 8 and the problem scenario, assume that $\vec{\varepsilon}_B = \vec{\varepsilon}_{B,r} + \vec{\varepsilon}_{B,s}$, where $\vec{\varepsilon}_{B,r}$ and $\vec{\varepsilon}_{B,s}$ are the "random" and "smoothly varying" error components. Also assume that $\left\|\vec{\varepsilon}_{j,s} - \vec{\varepsilon}_{B,s}\right\| \le \rho\left\|\vec{\mathbf{p}}_{j,s} - \vec{\mathbf{p}}_{B,s}\right\|$, $\left\|\vec{\varepsilon}_{B,r}\right\| \le \delta_r$, and $\left\|\vec{\alpha}_B\right\| \le \beta$, but that superior image processing has made $\Delta\mathbf{F}_{im}$ negligible. Give an expression for estimating a bound on the maximum error $\left\|\Delta\mathbf{c}_{tip}\right\|$. **Hint:** Consider errors relative to the marker body $\mathbf{F}_B$ and consider the position and orientation of the pointer relative to $\mathbf{F}_B$.

Question 10: (5 Points) Suppose that the position and orientation of the marker body $\mathbf{F}_B$ is computed by measuring the positions of 4 spheres located at positions $\vec{\mathbf{b}}_{ij} = [\pm d, \pm d, 0]$ in the marker body coordinate system. Compute an approximate bound on the value of $\|\vec{\alpha}_B\|$. I.e., provide an estimate for the value of β in Question 9. **Hint:** A geometric argument for a worst-case scenario can help you establish an acceptable bound.