Homework Assignment 3 – 600.455/655 Fall 2022 (Circle One)

Instructions and Score Sheet (hand in with answers)

Name	Name
Email	Email
Other contact information (optional)	Other contact information (optional)
Signature (required) I/We have followed the rules in completing this assignment ____________________________	Signature (required) I/We have followed the rules in completing this assignment ____________________________

Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.

1. You are to work **alone** or **in teams of two** and are **not to discuss the problems with anyone** other than the TAs or the instructor.
2. It is otherwise open book, notes, and web. But you should cite any references you consult.
3. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
4. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
5. Sign and hand in the score sheet as the first sheet of your assignment.

NOTE: This assignment has a total of 110 points. However, at most 100 points will count toward your course grade.

Scenario: Pelvic Osteotomy with Preformed Plates

In a pelvic osteotomy, the acetabular part of a pelvis is cut free from the main part of the pelvis and then reattached to the main part so that the acetabulum is translated and rotated to achieve a desired alignment. The acetabulum will then be fixed in place by plates and screws. In some cases, a surgical navigation system is used to help make the cuts and relocate the bone fragment. Holes are then drilled in the aligned bone fragments using the plate as a drill guide and then screws are inserted through the holes. In this case, we will use the navigation system to predrill the holes and then use the plate to assist in aligning the bone fragments.

Preoperative workflow: The preoperative workflow is illustrated in Figure 1.

Step 1: Several cannulated pins are secured to the pelvis, and a CT scan is made.

Step 2: The CT image is processed to produce a segmented model of the pelvis and to locate the axis of each pin. The line in CT coordinates corresponding to the axis for pin k is characterized by $[\vec{\mathbf{n}}_k, \vec{\mathbf{a}}_k]$, where $\left\| \vec{\mathbf{n}}_k \right\| = 1$. I.e., all points $\vec{\mathbf{p}}$ on the line obey the relation $(\vec{\mathbf{p}} - \vec{\mathbf{a}}_k) \times \vec{\mathbf{n}}_k = \vec{0}$. **Note:** The image processing software only returns a value for $\vec{\mathbf{a}}_k$ that is somewhere on the line, but unfortunately does not constrain the position to be any specific place along the line relative to the pin. We will remedy this defect later in the problem set.

Step 3: The position of each cutting plane is determined in CT coordinates, and the desired transformation $\mathbf{F}_{CA}$ of the freed-up acetabular fragment

Step 4: A metal plate is designed with specified hole positions $\vec{\mathbf{g}}_k$ relative to an arbitrary plate coordinate system and corresponding hole positions $\vec{\mathbf{h}}_k$ in the intact pelvis are defined. The hole positions in the pelvis are defined relative to CT coordinates, and we denote the positions of the holes on the acetabular component as $\vec{\mathbf{h}}_k^{(a)}$ and those on the rest of the pelvis as $\vec{\mathbf{h}}_k^{(p)}$. Similarly, the corresponding hole positions in the plate are denoted as $\vec{\mathbf{g}}_k^{(a)}$ and $\vec{\mathbf{g}}_k^{(p)}$, respectively.

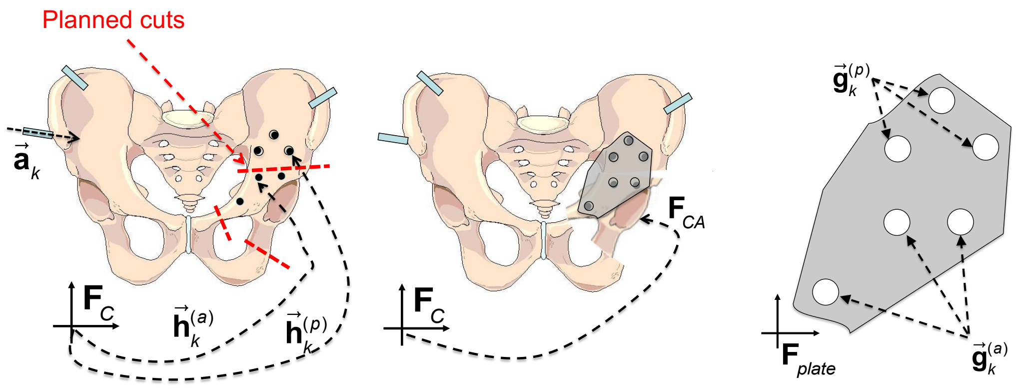

Figure 1: Preoperative scanning and planning

Intraoperative workflow: The initial phases of the procedure are illustrated in Figure 2

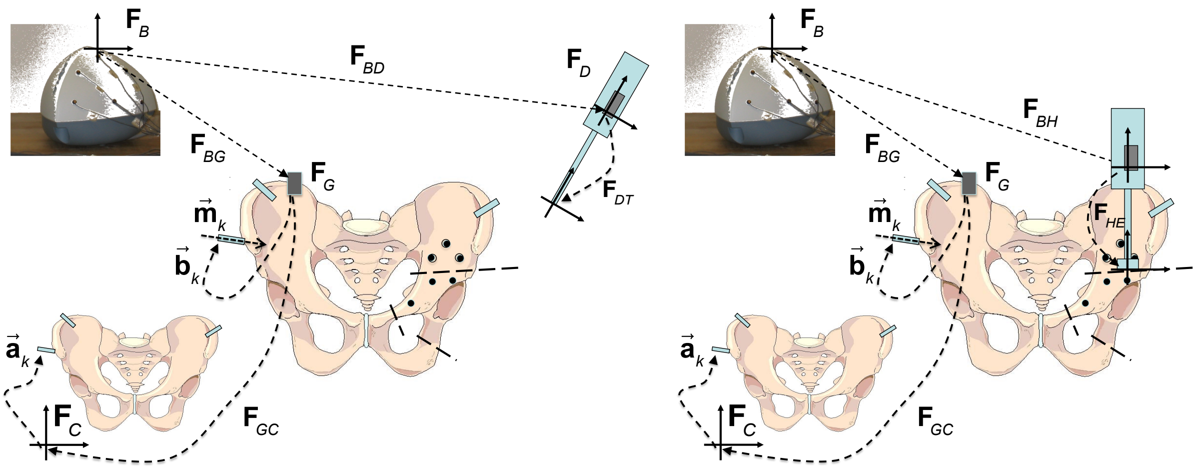

Figure 2: (left) Registration and hole drilling; **(right)** cuts with a tracked osteotome

Three surgical tools are available:

- An awl-like probe/pointer device that also can be used to make holes in the bone. The shaft of this device will fit snugly into the holes in the cannulated pins inserted in the pelvis. (**NOTE:** This is a slightly artificial situation introduced to simplify the actual workflow and problem description).
- An osteotome (essentially a fancy chisel) that will be used to cut the pelvis.

Also, a tracking system (here shown as a Northern Digital Aurora) is present in the room, and a 6 degree-of-freedom tracking devices 6 DOF tracking sensors have been attached to the handle of each tool.

Step 1: An additional 6 DOF sensor is affixed rigidly to the pelvis to serve as a reference body. The tracking system is capable of reading the transformation $\mathbf{F}_G$ of this reference body relative to the tracking system base unit.

Step 2: The shaft of the tracked probe/pointer is inserted into the cannulated pins to a sufficient depth to constrain its direction relative to the reference body. The tracking system then determines computes the line in space corresponding to this axis for each pin k. Relative to the reference body $\mathbf{F}_G$ the line corresponding to this line is characterized by $[\vec{\mathbf{m}}_k, \vec{\mathbf{b}}_k]$, where $\left\|\vec{\mathbf{m}}_k\right\| = 1$. I.e., all points $\vec{\mathbf{p}}$ on the line obey the relation $(\vec{\mathbf{p}} - \vec{\mathbf{b}}_k) \times \vec{\mathbf{m}}_k = \vec{0}$.

Step 3: A registration transformation $\mathbf{F}_{CG}$ is computed such that any point $\vec{\mathbf{c}}$ in CT coordinates corresponds to a point on the pelvis at location $\mathbf{F}_{GC}^{-1}\vec{\mathbf{c}}$ on the physical pelvis relative to the reference body.

Step 4: The navigation system assists the surgeon to use the tracked pointer/probe device to drill the desired holes $\vec{\mathbf{h}}_k$ in the intact pelvis.

Step 5: The navigation system assists the surgeon to use the tracked osteotome to make the desired cuts to separate the acetabular fragment from the rest of the pelvis.

Step 6: The plate is attached to the acetabular fragment with screws through the $\vec{\mathbf{g}}_k^{(a)}$ holes into the corresponding $\vec{\mathbf{h}}_k^{(a)}$

Step 7: The acetabular fragment is then moved to line up the $\vec{\mathbf{g}}_k^{(p)}$ holes with the corresponding $\vec{\mathbf{h}}_k^{(p)}$ holes and screws are inserted to hold the plate and fragment firmly in place relative to the rest of the pelvis, as shown in Figure 1(right).

To recapitulate the key coordinate systems:

$\mathbf{F}_B =$ Coordinate system of tracking system base unit
$\mathbf{F}_D =$ Coordinate system of tracking device on probe/pointer handle
$\mathbf{F}_H =$ Coordinate system of tracking device on osteotome handle
$\mathbf{F}_G =$ Coordinate system of tracking device attached to pelvis
$\mathbf{F}_C =$ Coordinate system of CT image
$\mathbf{F}_{plate} =$ Coordinate system of the plate

We also have the following relationships

$\mathbf{F}_{Bx} =$ Measured 6 DOF pose of tracking device x relative to base unit
$\mathbf{F}_{HE} =$ 6 DOF pose of osteotome blade relative to osteotome handle tracking device
$\mathbf{F}_{DT} =$ 6 DOF pose of probe/pointer tip relative to probe/pointer handle tracking device
$[\vec{\mathbf{n}}_k, \vec{\mathbf{a}}_k] =$ axis parameters for pin k in CT coordinates
$[\vec{\mathbf{m}}_k, \vec{\mathbf{b}}_k] =$ axis parameters for pin k relative to tracking device G
$[\mathbf{R}_{DT}\vec{\mathbf{z}}, \vec{\mathbf{p}}_{DT}] =$ axis parameters of the shaft of the probe relative to probe handle tracking device

We will follow our usual conventions where frame position and orientation components are represented by $\mathbf{F} = [\mathbf{R}, \vec{\mathbf{p}}]$, and errors are represented by $\Delta\mathbf{F} = [\Delta\mathbf{R}, \Delta\mathbf{p}]$. We will also use the approximation convention $\Delta\mathbf{R} \approx \mathbf{I} + sk(\vec{\alpha})$ and may interchangeably use $\Delta\vec{\mathbf{p}} = \vec{\varepsilon}$.

Questions

Question 1: (5 points) Give a formula for determining the coordinate transformation $\mathbf{F}_{GT}$ of the pointer/probe end relative to the reference body $\mathbf{F}_G$, assuming that the tracking system has negligible error.

Question 2: (5 points) Use your result to compute the line parameters $[\vec{\mathbf{m}}_k, \vec{\mathbf{b}}_k]$ for pin *k*, assuming negligible tracking system error. Here, you should compute $\vec{\mathbf{b}}_k$ from the position of the probe tip when the tool is inserted in the cannulated pin, since you know that the probe tip is along the pin axis.

Question 3: (10 points) Assume now that the tracking system does, indeed, have some error, so that $\mathbf{F}^*_{Bx} = \mathbf{F}_{Bx}\Delta\mathbf{F}_{Bx}$, what would be the new values for $[\vec{\mathbf{m}}^*_k, \vec{\mathbf{b}}^*_k]$, expressed in terms of the various $\mathbf{R}_{Bx}, \Delta\mathbf{R}_{Bx}, \vec{\mathbf{p}}_{Bx}, \Delta\vec{\mathbf{p}}_{Bx}$ terms and the other quantities in the problem scenario.

Question 4: (10 points) Give approximated values for $[\vec{\mathbf{m}}^*_k, \vec{\mathbf{b}}^*_k]$ using $\Delta\mathbf{R}_{Bx} \approx \mathbf{I} + sk(\vec{\alpha}_{Bx})$ and $\Delta\vec{\mathbf{p}}_{Bx} = \vec{\varepsilon}_{Bx}$. Express your answer by giving normalized linearized expressions for $\Delta\mathbf{m}_k$ and $\Delta\vec{\mathbf{b}}_k$ where $\vec{\mathbf{m}}^*_k \approx \vec{\mathbf{m}}_k + \Delta\vec{\mathbf{m}}_k$ and $\vec{\mathbf{b}}^*_k = \vec{\mathbf{b}}_k + \Delta\vec{\mathbf{b}}_k$.

Question 5: (15 points) Assuming again that the tracking system has negligible error, describe a method using the workflow described above for the computer to determine $\mathbf{F}_{GC}$. Include all relevant formulas.

Question 6: (5 points) Give a formula for determining the coordinate transformation $\mathbf{F}_{EC}$ of the pose of the osteotome cutter blade relative to CT coordinates, assuming that you have successfully computed $\mathbf{F}_{GC}$ (I.e., you should use $\mathbf{F}_{GC} = [\mathbf{R}_{GC}, \vec{\mathbf{p}}_{GC}]$ in your answer) but there is no error.

Question 7: (10 points) Suppose that $\mathbf{F}_{GC}$ has some error, so that the true value is $\mathbf{F}^*_{GC} = \Delta\mathbf{F}_{GC}\mathbf{F}_{GC}$, what is the resulting error $\Delta\mathbf{F}_{EC}$ in $\mathbf{F}_{EC}$, where $\mathbf{F}^*_{EC} = \mathbf{F}_{EC}\Delta\mathbf{F}_{EC}$. Give your answer both in terms of the "**F**" variables and then in terms of the rotational and translational variables. Then, give linearized expressions for $\vec{\alpha}_{EC}$ and $\vec{\varepsilon}_{EC}$.

Question 8: (15 points) Suppose that the tracking system has negligible error, but that the tracking device on the handle of the probe/pointer has been displaced and reattached in an unknown position, so that the transformation $\mathbf{F}_{DT}$ is not known. However, the design of the cannulated pins has been modified so that the tip of the probe rests at a consistent place when the probe shaft (which is cylindrical) is inserted all the way into the pin holes. Describe a method for finding the registration transformation $\mathbf{F}_{GC}$. Your answer should include the step-by-step workflow, all measurements taken, and the computational details describing how you will determine the desired values from the measurements. **Note:** This problem will necessarily taking multiple measurements of various tracking devices. You should adopt the notation $\mathbf{F}_{Bx}[t]$ to denote a measurement of tracked marker "*x*" at time *t*. **And a hint:** The recommended method is to find a way to find a new set of $[\vec{\mathbf{m}}_k,\vec{\mathbf{b}}_k]$ values and then say that you will use the answer from Question 5.

Question 9: (15 points) Suppose that you have successfully answered Question 8. Suppose, also, that the image processing software has been improved so it can determine the positions $\vec{\mathbf{a}}_k$ within the cannulated pins where the tip probe rests when the probe is inserted all the way into the pin holes. Describe a method for determining $\mathbf{R}_{DT}\vec{\mathbf{z}}$ and $\vec{\mathbf{p}}_{DT}$ using only the cannulated pins and the other equipment in the problem scenario.

Question 10: (10 points) Suppose that the positions of the holes $\vec{\mathbf{h}}_k$ to be drilled into the pelvis have been determined, along with the positions of the holes $\vec{\mathbf{g}}_k^{(a)}$ in the plate and the desired transformation $\mathbf{F}_{CA}$ of the acetabular bone fragment. Assume that the coordinate system $\mathbf{F}_{Plate}$ of the plate is defined such that $\mathbf{F}_{PC}\vec{\mathbf{g}}_k^{(a)} = \vec{\mathbf{h}}_k^{(a)}$, where $\mathbf{F}_{PC} = \mathbf{F}_{CT}^{-1}\mathbf{F}_{Plate}$ is the transformation between Plate and CT coordinate systems. Where should the holes $\vec{\mathbf{g}}_k^{(p)}$ be drilled into the plate?

Question 11: (10 points) In actual practice, there will always be some error in the positioning of the screw holes, so that the plate holes need to be a bit larger than the diameter of the screws, as illustrated in Figure 3. In general, there will be both random errors and smoothly varying distortion errors coming from various sources, so that the actual position of each hole in CT coordinates is given by $\vec{\mathbf{h}}_k^* = \vec{\mathbf{h}}_k + \vec{\varepsilon}_k^{(rand)} + \vec{\varepsilon}^{(distort)}(\vec{\mathbf{h}}_k)$, where $\left\|\vec{\varepsilon}_k^{(rand)}\right\| \le \delta_k^{(rand)}$ and $\left\|\vec{\varepsilon}^{(distort)}(\vec{\mathbf{c}}_i) - \vec{\varepsilon}^{(distort)}(\vec{\mathbf{c}}_j)\right\| \le \rho\left\|\vec{\mathbf{c}}_i - \vec{\mathbf{c}}_j\right\|$ for any two points $\vec{\mathbf{c}}_i$ and $\vec{\mathbf{c}}_j$. In general, it is not necessary that all plate holes have the same diameter, but it must be possible to insert all the screws. Suppose that the diameter of the screws is D_{screw} and the diameter of plate hole k is $D_k = D_{screw} + \delta_k^{(screw)}$. Specify a family of (nonlinear) constraints on the values of the plate hole clearances $\delta_k^{(screw)}$ that will enforce this condition. **Hint:** Do some geometric reasoning to figure out the simplest situation that will make it impossible to do the insertion task, express this mathematically, and then apply this formulation to your entire set of screw holes.

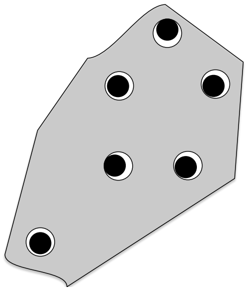

Figure 3