NSF Engineering Research Center
for Computer Integrated Surgical
Systems and Technology

LABORATORY FOR Computational Sensing + Robotics
THE JOHNS HOPKINS UNIVERSITY

Coherent Point Drift Registration

601.455/655 Lecture

100 YEARS JOHNS HOPKINS ENGINEERING

WHITING SCHOOL OF ENGINEERING
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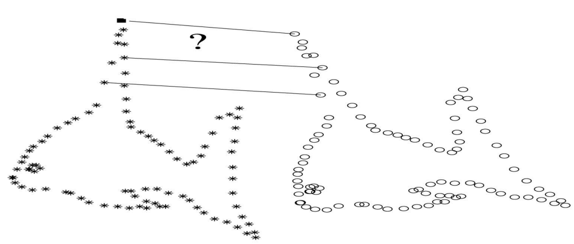
Coherent Point Drift

- A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", *IEEE Trans. on Pattern Analysis and Machine Intelligence,* vol. 32- 12, pp. 2262-2275, 2010.
- Alignment of point clouds
  - Fast method follows "EM" paradigm
  - Tolerates outliers and noise
  - Transformations: Rigid, affine, general deformable



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Buridan’s Ass: Stuck Between Two Haystacks

Image: www.collegetransitions.com

Article: https://en.wikipedia.org/wiki/Buridan%27s_ass

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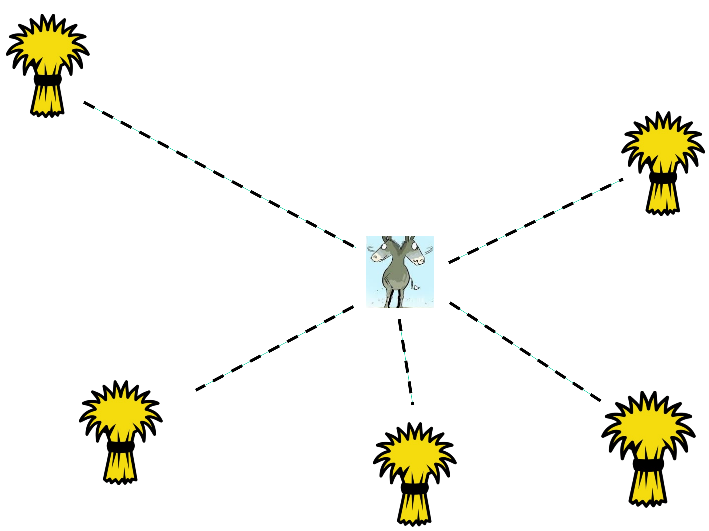
Many Haystacks



Images: www.collegetransitions.com
clickr.com

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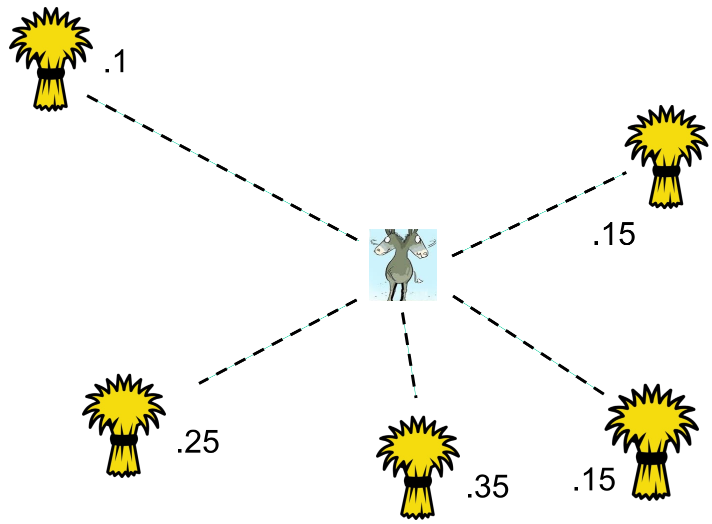
Many Haystacks



Images: www.collegetransitions.com
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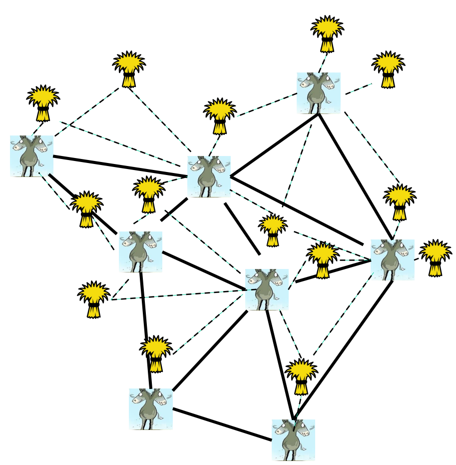
Many Haystacks, Many Donkeys Tied Together



Images: www.collegetransitions.com
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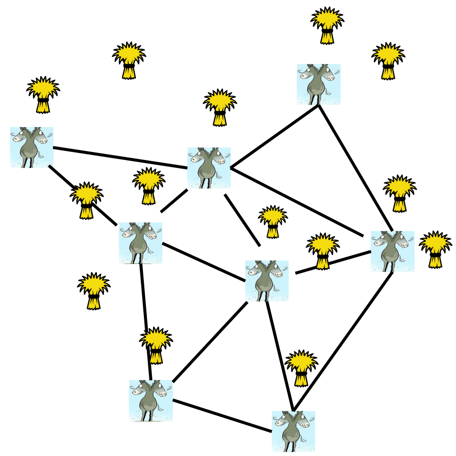
Many Haystacks, Many Donkeys Tied Together



Images: www.collegetransitions.com
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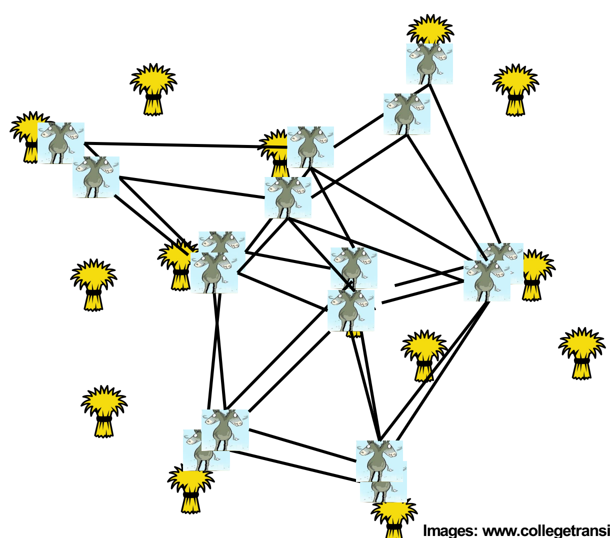
Many Donkeys With Stretchy Ropes



Images: www.collegetransitions.com
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CPD – Basic EM Paradigm

- Initialization
  - Given initial guess of registration, compute the variance of distances between all possible point pairs
  - Assumes independent isotropic Gaussian distribution for matches, uniform distribution for outliers

$$p_{mn} = \frac{\exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (s\mathbf{R}\mathbf{y}_m + \mathbf{t})\|^2}}{\sum_{k=1}^{M} \exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (s\mathbf{R}\mathbf{y}_k + \mathbf{t})\|^2} + (2\pi\sigma^2)^{D/2}\frac{w}{1-w}\frac{M}{N}}$$

- "E Step"
  - Based on current variance, compute probability of matches of all possible point pairs, decide what are outliers
- "M Step"
  - Compute new transformation that increases probability
  - Update probabilities based on registration

A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", ***IEEE Trans. on Pattern Analysis and Machine Intelligence,*** **vol. 32- 12, pp. 2262-2275, 2010.**

CPD Inputs

- D—dimension of the point sets,
- N, M—number of points in the point sets,
- $\mathbf{X}_{N\times D} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)^T$—the first point set (the data points),
- $\mathbf{Y}_{M\times D} = (\mathbf{y}_1, \ldots, \mathbf{y}_M)^T$—the second point set (the GMM centroids),
- $\mathcal{T}(\mathbf{Y}, \theta)$—Transformation $\mathcal{T}$ applied to $\mathbf{Y}$, where θ is a set of the transformation parameters,
- $\mathbf{I}$—identity matrix,
- $\mathbf{1}$—column vector of all ones,
- $\mathrm{d}(\mathbf{a})$—diagonal matrix formed from the vector $\mathbf{a}$.

Rigid and Similarity Transform CPD

Rigid point set registration algorithm:

- Initialization: $\mathbf{R} = \mathbf{I}, \mathbf{t} = 0, s = 1, 0 \leq w \leq 1$
  $\sigma^2 = \frac{1}{DNM} \sum_{n=1}^{N} \sum_{m=1}^{M} \|\mathbf{x}_n - \mathbf{y}_m\|^2$
- EM optimization, repeat until convergence:
  - E-step: Compute $\mathbf{P}$,
    $$p_{mn} = \frac{\exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (s\mathbf{R}\mathbf{y}_m + \mathbf{t})\|^2}}{\sum_{k=1}^{M} \exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (s\mathbf{R}\mathbf{y}_k + \mathbf{t})\|^2} + (2\pi\sigma^2)^{D/2} \frac{w}{1-w} \frac{M}{N}}$$
  - M-step: Solve for $\mathbf{R}, s, \mathbf{t}, \sigma^2$:
    - $N_{\mathbf{P}} = \mathbf{1}^T\mathbf{P}\mathbf{1}, \mu_{\mathbf{x}} = \frac{1}{N_{\mathbf{P}}}\mathbf{X}^T\mathbf{P}^T\mathbf{1}, \mu_{\mathbf{y}} = \frac{1}{N_{\mathbf{P}}}\mathbf{Y}^T\mathbf{P}\mathbf{1},$
    - $\hat{\mathbf{X}} = \mathbf{X} - \mathbf{1}\mu_{\mathbf{x}}^T, \ \hat{\mathbf{Y}} = \mathbf{Y} - \mathbf{1}\mu_{\mathbf{y}}^T,$
    - $\mathbf{A} = \hat{\mathbf{X}}^T\mathbf{P}^T\hat{\mathbf{Y}}$, compute SVD of $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T$,
    - $\mathbf{R} = \mathbf{U}\mathbf{C}\mathbf{V}^T$, where $\mathbf{C} = \mathrm{d}(1, .., 1, \det(\mathbf{U}\mathbf{V}^T))$,
    - $s = \frac{\mathrm{tr}(\mathbf{A}^T\mathbf{R})}{\mathrm{tr}(\hat{\mathbf{Y}}^T \mathrm{d}(\mathbf{P}\mathbf{1})\hat{\mathbf{Y}})},$
    - $\mathbf{t} = \mu_{\mathbf{x}} - s\mathbf{R}\mu_{\mathbf{y}},$
    - $\sigma^2 = \frac{1}{N_{\mathbf{P}}D}(\mathrm{tr}(\hat{\mathbf{X}}^T \mathrm{d}(\mathbf{P}^T\mathbf{1})\hat{\mathbf{X}}) - s\,\mathrm{tr}(\mathbf{A}^T\mathbf{R})).$
- The aligned point set is $\mathcal{T}(\mathbf{Y}) = s\mathbf{Y}\mathbf{R}^T + 1\mathbf{t}^T$,
- The probability of correspondence is given by $\mathbf{P}$.

A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", ***IEEE Trans. on Pattern Analysis and Machine Intelligence*****, vol. 32- 12, pp. 2262-2275, 2010.**

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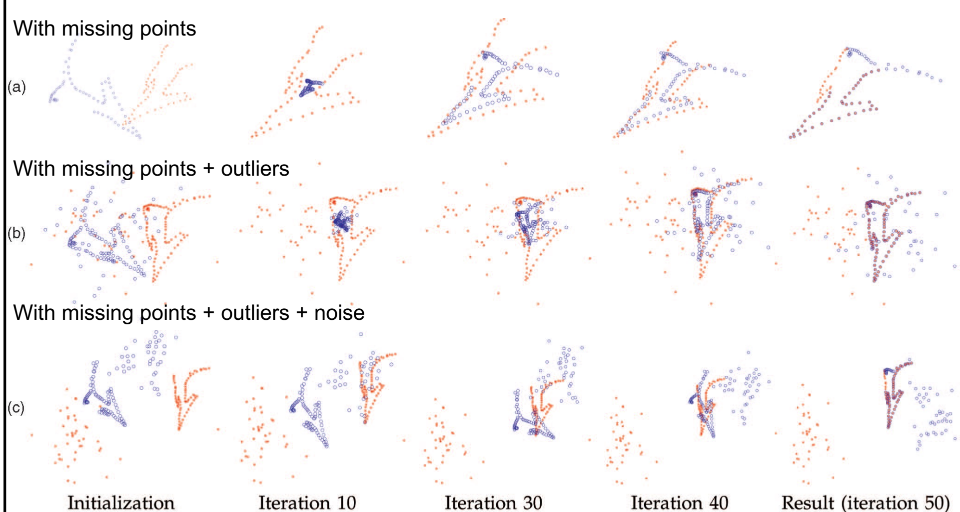
Rigid and Similarity Transform CPD



A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", ***IEEE Trans. on Pattern Analysis and Machine Intelligence*****, vol. 32- 12, pp. 2262-2275, 2010.**

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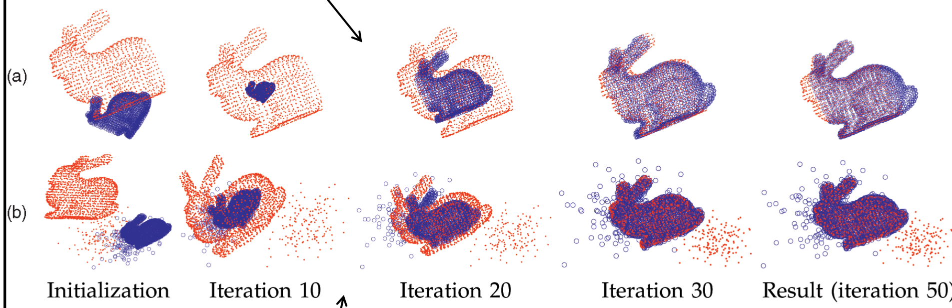
Rigid and Similarity Transform CPD

With missing points

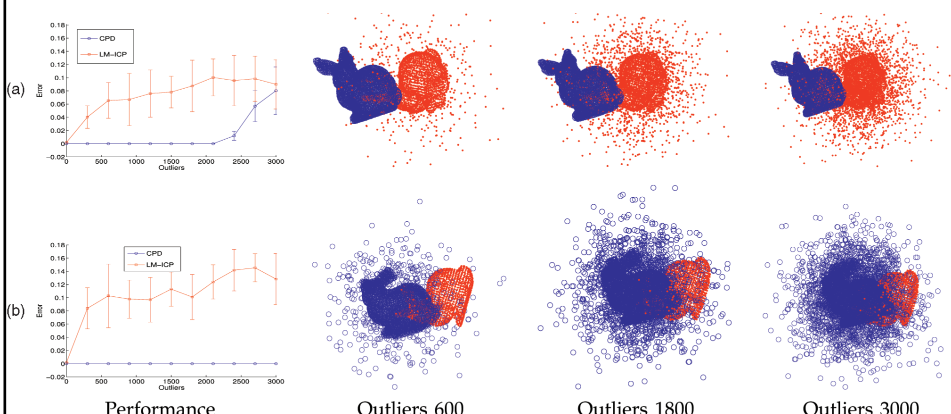


With missing points, outliers and noise

A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 32- 12, pp. 2262-2275, 2010.

Rigid and Similarity Transform CPD

Comparison to ICP with outliers and noise



A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 32- 12, pp. 2262-2275, 2010.

Affine CPD

Affine point set registration algorithm:

- Initialization: $\mathbf{B} = \mathbf{I}, \mathbf{t} = 0, 0 \le w \le 1$
  $\sigma^2 = \frac{1}{DNM} \sum_{n=1}^{N} \sum_{m=1}^{M} \|\mathbf{x}_n - \mathbf{y}_m\|^2$
- EM optimization, repeat until convergence:
  - E-step: Compute $\mathbf{P}$,
    $$p_{mn} = \frac{\exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (\mathbf{B}\mathbf{y}_m + \mathbf{t})\|^2}}{\sum_{k=1}^{M} \exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (\mathbf{B}\mathbf{y}_k + \mathbf{t})\|^2} + (2\pi\sigma^2)^{D/2} \frac{w}{1-w} \frac{M}{N}}$$
  - M-step: Solve for $\mathbf{B}, \mathbf{t}, \sigma^2$:
    - $\cdot\ N_{\mathbf{P}} = \mathbf{1}^T \mathbf{P} \mathbf{1}, \mu_{\mathbf{x}} = \frac{1}{N_{\mathbf{P}}} \mathbf{X}^T \mathbf{P}^T \mathbf{1}, \mu_{\mathbf{y}} = \frac{1}{N_{\mathbf{P}}} \mathbf{Y}^T \mathbf{P} \mathbf{1},$
    - $\cdot\ \hat{\mathbf{X}} = \mathbf{X} - \mathbf{1}\mu_{\mathbf{x}}^T,\ \hat{\mathbf{Y}} = \mathbf{Y} - \mathbf{1}\mu_{\mathbf{y}}^T,$
    - $\cdot\ \mathbf{B} = (\hat{\mathbf{X}}^T \mathbf{P}^T \hat{\mathbf{Y}})(\hat{\mathbf{Y}}^T \mathrm{d}(\mathbf{P}\mathbf{1})\hat{\mathbf{Y}})^{-1},$
    - $\cdot\ \mathbf{t} = \mu_{\mathbf{x}} - \mathbf{B}\mu_{\mathbf{y}},$
    - $\cdot\ \sigma^2 = \frac{1}{N_{\mathbf{P}} D}(\mathrm{tr}(\hat{\mathbf{X}}^T \mathrm{d}(\mathbf{P}^T \mathbf{1})\hat{\mathbf{X}}) - \mathrm{tr}(\hat{\mathbf{X}}^T \mathbf{P}^T \hat{\mathbf{Y}} \mathbf{B}^T)).$
- The aligned point set is $\mathcal{T}(\mathbf{Y}) = \mathbf{Y}\mathbf{B}^T + 1\mathbf{t}^T$,
- The probability of correspondence is given by $\mathbf{P}$.

A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 32- 12, pp. 2262-2275, 2010.

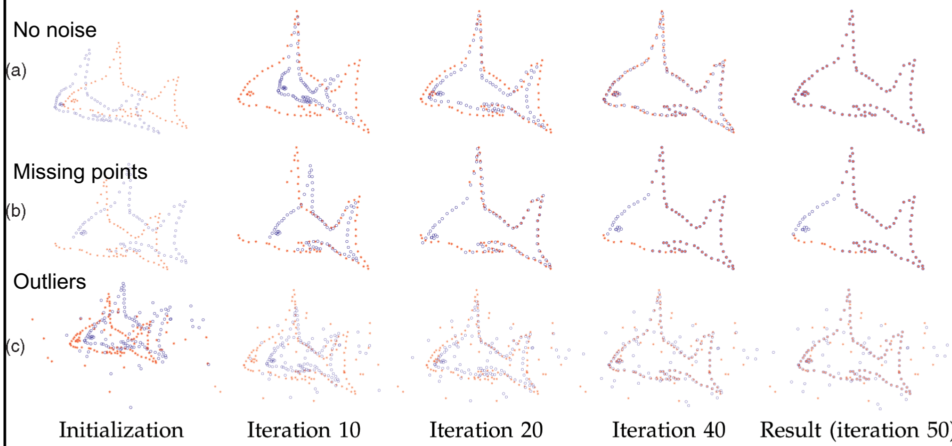
Deformable CPD

Non-rigid point set registration algorithm:

- Initialization: $\mathbf{W} = 0, \sigma^2 = \frac{1}{DNM} \sum_{m,n=1}^{M,N} \|\mathbf{x}_n - \mathbf{y}_m\|^2$
- Initialize $w (0 \le w \le 1)$, $\beta > 0$, $\lambda > 0$,
- Construct $\mathbf{G}$: $g_{ij} = \exp^{-\frac{1}{2\beta^2}\|\mathbf{y}_i - \mathbf{y}_j\|^2}$,
- EM optimization, repeat until convergence:
  - E-step: Compute $\mathbf{P}$,
    $$p_{mn} = \frac{\exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (\mathbf{y}_m + \mathbf{G}(m,\cdot)\mathbf{W})\|^2}}{\sum_{k=1}^{M} \exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (\mathbf{y}_k + \mathbf{G}(k,\cdot)\mathbf{W})\|^2} + \frac{w}{1-w} \frac{(2\pi\sigma^2)^{D/2} M}{N}}$$
  - M-step:
    - $\cdot$ Solve $(\mathbf{G} + \lambda\sigma^2 d(\mathbf{P}\mathbf{1})^{-1})\mathbf{W} = d(\mathbf{P}\mathbf{1})^{-1}\mathbf{P}\mathbf{X} - \mathbf{Y}$
    - $\cdot\ N_{\mathbf{P}} = \mathbf{1}^T \mathbf{P} \mathbf{1}$, $\mathbf{T} = \mathbf{Y} + \mathbf{G}\mathbf{W}$,
    - $\cdot\ \sigma^2 = \frac{1}{N_{\mathbf{P}} D}(\mathrm{tr}(\mathbf{X}^T \mathrm{d}(\mathbf{P}^T \mathbf{1})\mathbf{X}) - 2\,\mathrm{tr}((\mathbf{P}\mathbf{X})^T \mathbf{T}) +$
      $\mathrm{tr}(\mathbf{T}^T \mathrm{d}(\mathbf{P}\mathbf{1})\mathbf{T})),$
- The aligned point set is $\mathbf{T} = \mathcal{T}(\mathbf{Y}, \mathbf{W}) = \mathbf{Y} + \mathbf{G}\mathbf{W}$,
- The probability of correspondence is given by $\mathbf{P}$.

A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 32- 12, pp. 2262-2275, 2010.

Deformable CPD



A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 32- 12, pp. 2262-2275, 2010.

Fast Implementation

- Uses the “Fast Gauss Transform” to reduce the computational complexity to linear time (up to a multiplicative constant).

Compute $\mathbf{P}^T\mathbf{1}, \mathbf{P1}$ and $\mathbf{PX}$:
- Compute $\mathbf{K}^T\mathbf{1}$ (using FGT),
- $\mathbf{a} = 1./(\mathbf{K}^T\mathbf{1} + c\mathbf{1})$,
- $\mathbf{P}^T\mathbf{1} = \mathbf{1} - c\mathbf{a}$,
- $\mathbf{P1} = \mathbf{Ka}$ (using FGT),
- $\mathbf{PX} = \mathbf{K}(\mathbf{a}. * \mathbf{X})$ (using FGT),

- See the paper for more details

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Fast Implementation

$$f(\mathbf{y}_m) = \sum_{n=1}^{N} z_n \exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - \mathbf{y}_m\|^2}, \ \forall \mathbf{y}_m, \ m = 1, \ldots, M. \quad (26)$$

The naive approach takes $\mathcal{O}(MN)$ operations, while FGT takes only $\mathcal{O}(M+N)$. The basic idea of FGT is to expand the Gaussians in terms of truncated Hermit expansion and approximate (26) up to the predefined accuracy. Rewriting (26) in matrix form, we obtain $\mathbf{f} = \mathbf{K}\mathbf{z}$, where $\mathbf{z}$ is some vector and $\mathbf{K}_{M\times N}$ is a Gaussian affinity matrix with elements: $k_{mn} = \exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - \mathcal{T}(\mathbf{y}_m)\|^2}$, which we have already used in our notations. We simplify the matrix-vector products $\mathbf{P1}$, $\mathbf{P}^T\mathbf{1}$, and $\mathbf{PX}$, to the form of $\mathbf{Kz}$ and apply FGT. Matrix $\mathbf{P}$ (6) can be partitioned into

$$\mathbf{P} = \mathbf{K}\mathrm{d}(\mathbf{a}), \ \ \mathbf{a} = 1./(\mathbf{K}^T\mathbf{1} + c\mathbf{1}), \quad (27)$$

A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 32- 12, pp. 2262-2275, 2010.

Fast Implementation

where $\mathrm{d}(\mathbf{a})$ is diagonal matrix with a vector $\mathbf{a}$ along the diagonal. Here, we use Matlab element-wise division (./) and element-wise multiplication (.$*$) notations. We show the algorithm to compute the bottleneck matrix-vector products $\mathbf{P1}$, $\mathbf{P}^T\mathbf{1}$, and $\mathbf{PX}$ using FGT in Fig. 5. We note that for dimensions higher than three, we can use the improved fast Gauss transform (IFGT) method [44], which is a faster alternative to FGT for higher dimensions.

A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 32- 12, pp. 2262-2275, 2010.