CIS I 2023 Homework Assignment 1

Instructions and Cover Sheet (hand in with answers)

Name	Name
Email	Email
Other contact information (optional)	Other contact information (optional)
Signature (required) I/We have followed the rules in completing this assignment ______________________________	Signature (required) I/We have followed the rules in completing this assignment ______________________________

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone** or in **teams of two** and are not to discuss the problems with anyone other than the TAs or the instructor.
3. It is otherwise open book, notes, and web. But you should cite any references you consult.
4. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
5. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
6. Submit the assignment on Gradescope as a neat and legible PDF file. We will not insist on typesetting your answers, but we must be able to read them. We will not go to extraordinary lengths to decipher what you write. If the graders cannot make out an answer, the score will be 0.
7. Sign and hand in this page as the first sheet of your assignment. If you work with a partner, then you both should sign the sheet, but you should only submit one PDF file for both of you. Indicate clearly who it is from.
8. This assignment has 110 points, but the most that will be applied to your grade is 100.

Problem Scenario: Pedicle Screw Insertion

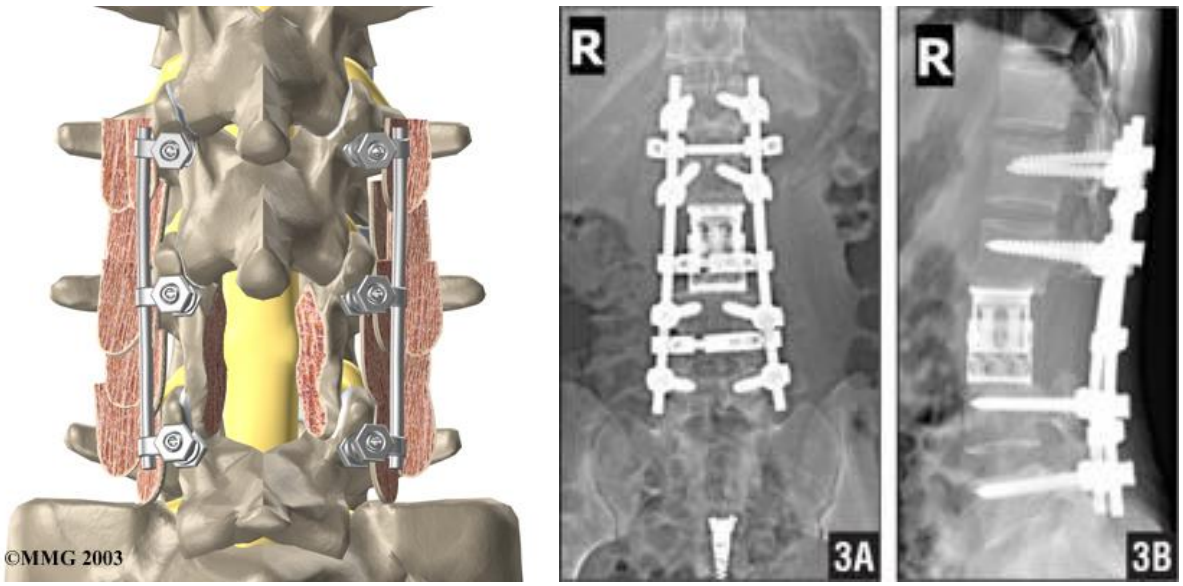

Fig. 1~~1~~: Pedicle Screws and Rods in Spine Surgery

Pedicle screws and rods are often used to stabilize the spine as part of spine surgery. Fig. 1 (above) shows an example. In these procedures, a drill or awl is used to make a guide hole down the pedicle into the vertebral body and a screw is inserted to anchor hardware used to stabilize the spine to the vertebral bodies.

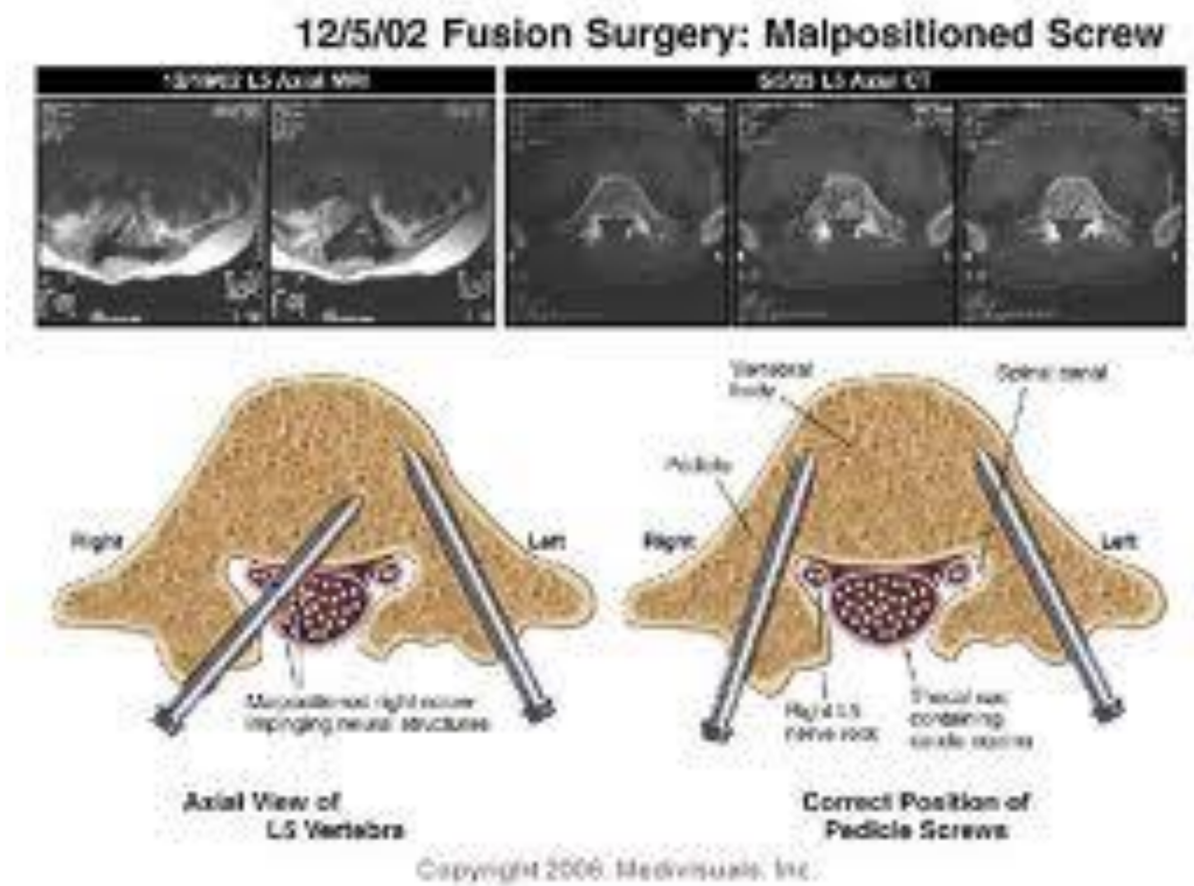

Fig. 2~~2~~: Screws pass close to critical structures like nerve roots and the spinal cord.

As shown in Fig. 2, accurate placement of these guide holes is crucial to safe and successful performance of this procedure. Malpositioned screws can cause severe injury or paralysis for the patient. Surgical navigation systems are often used to assist the surgeon in correctly positioning the guide holes (Fig. 3). Typically a reference body is attached to the vertebral body, the body is then localized by touching anatomic landmarks or features using a tracked pointing device, and displays are used to assist the surgeon in aligning the drill correctly, based on CT images of the patient.

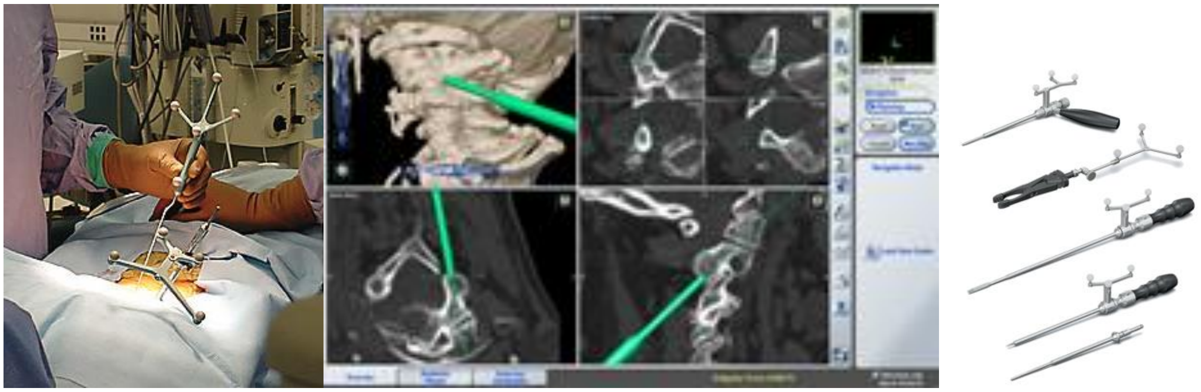

Fig. 3~~3~~: Surgical Navigation for Spine Surgery. (Left) OR scene; (Middle) typical display; (Right) typical instruments. (Pictures are captured from web and show Brainlab system, but there are many similar systems).

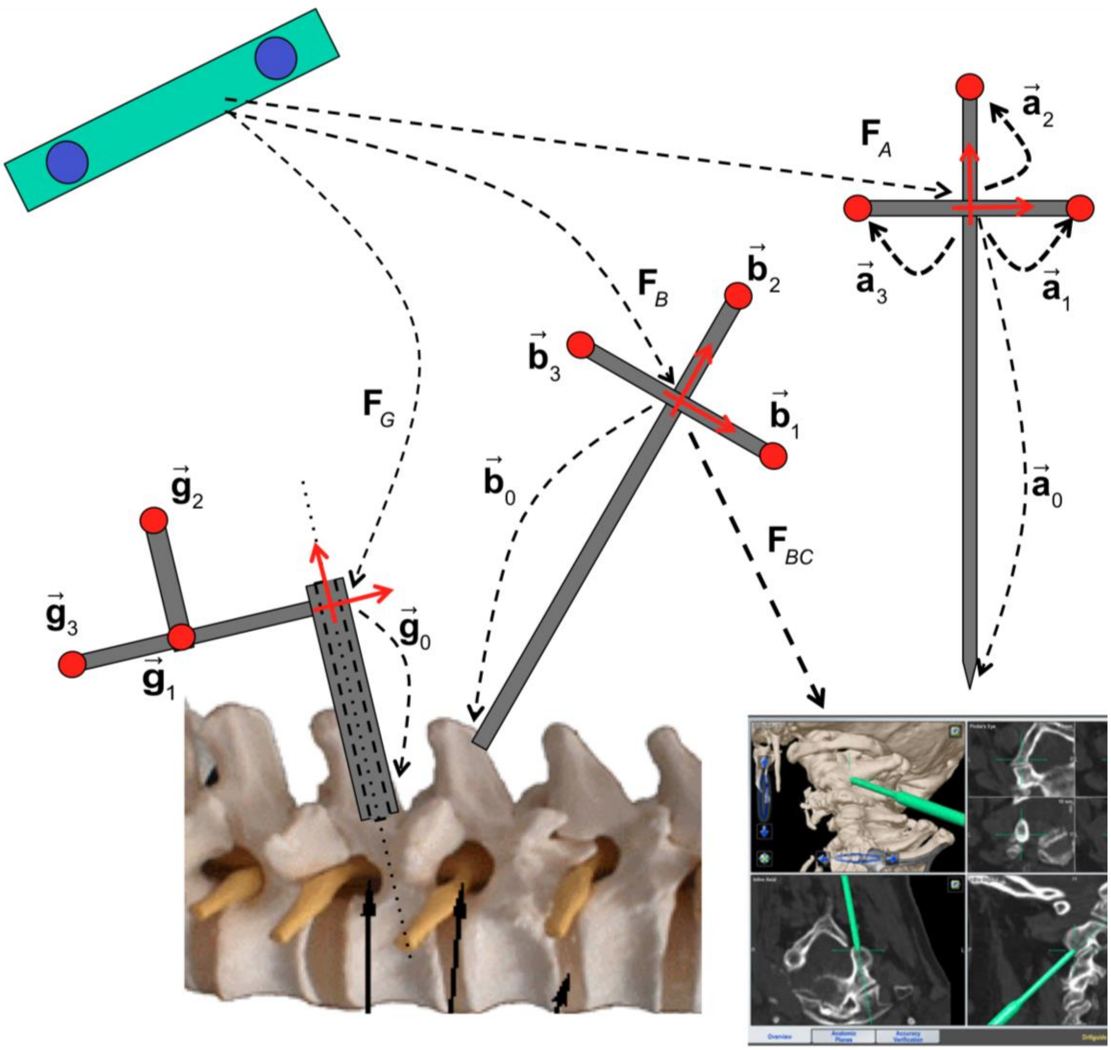

Fig. 44: Coordinate systems and positions associated with assignment

Consider the spinal navigation system illustrated in Figs. 3 and 4. This system has a pointing device, a “reference body” which may be clamped to a vertebra, and a drill guide. These tools are equipped with optical markers placed at known positions relative to the tools, and an optical tracking system similar to the Northern Digital Polaris® is used to track the tools. We have:

$\mathbf{F}_A = \left[\mathbf{R}_A, \vec{\mathbf{p}}_A\right]$	Pose of the pointer device relative to the optical tracker
$\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{a}}_3$	Positions of pointer optical markers relative to $\mathbf{F}_A$
$\vec{\mathbf{a}}_0 = [0,0,-L_A]^T$	Position of pointer tip relative to $\mathbf{F}_A$
$\mathbf{F}_B = \left[\mathbf{R}_B, \vec{\mathbf{p}}_B\right]$	Pose of the reference body attached to the vertebra relative to the optical tracker
$\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3$	Positions of reference body optical markers relative to $\mathbf{F}_B$
$\vec{\mathbf{b}}_0 = [0,0,-L_B]^T$	Position of marker clamp relative to $\mathbf{F}_B$
$\mathbf{F}_G = \left[\mathbf{R}_G, \vec{\mathbf{p}}_G\right]$	Pose of the reference body attached to the drill guide relative to the optical tracker
$\vec{\mathbf{g}}_1, \vec{\mathbf{g}}_2, \vec{\mathbf{g}}_3$	Positions of pointer optical markers relative to $\mathbf{F}_G$
$\vec{\mathbf{g}}_0 = [0,0,-L_G]^T$	Position of the end of the guide tube relative to $\mathbf{F}_G$
$\mathbf{F}_{BC}$	Registration transformation between $\mathbf{F}_B$ and CT image. Any point $\mathbf{F}_B\vec{\mathbf{v}}$ relative to

	the tracker corresponds to $\mathbf{F}_{BC}\vec{\mathbf{v}}$ in CT coordinates

Question 1

A. (5 points) Suppose that the guide tool is positioned against the spine. Give formulas for computing the position $\vec{\mathbf{g}}_{C0}$ of the guide tip and the direction $\vec{\mathbf{n}}_{CG}$ of the guide channel relative to CT coordinates. The guide channel axis runs from the origin of $\mathbf{F}_G$ through $\vec{\mathbf{g}}_0$. So $\vec{\mathbf{n}}_g = \vec{\mathbf{g}}_0 / \left\| \vec{\mathbf{g}}_0 \right\|$ in the coordinate system of the guide.

Express your answer in terms of the $\mathbf{R}$ and $\vec{\mathbf{p}}$ variables.

B. (5 points) Suppose now that the tracking system produces some error in the returned values of $\mathbf{F}_A$, $\mathbf{F}_B$, and $\mathbf{F}_G$, so that the true values are given by $\mathbf{F}_X^* = \mathbf{F}_X \Delta \mathbf{F}_X$ for X = A, B, G. Produce formulas for the actual values for $\vec{\mathbf{g}}_{C0}$ and $\vec{\mathbf{n}}_G$. **Hint:** You may find it useful to define $\mathbf{F}_{BG} = \mathbf{F}_B^{-1}\mathbf{F}_G$

C. (5 points) Suppose now that you know that the errors in Question 1B are "small", so that the linear approximation $\Delta \mathbf{F}_X \approx \left[\mathbf{I} + sk(\vec{\alpha}_X), \vec{\varepsilon}_X \right]$ holds. Give formulas for estimating the error in your computation of $\vec{\mathbf{g}}_{C0}$ and $\vec{\mathbf{n}}_G$. I.e., provide formulas for estimating $\vec{\varepsilon}_{C0} = \Delta \vec{\mathbf{g}}_{C0} = \vec{\mathbf{g}}_{C0}^* - \vec{\mathbf{g}}_{C0}$ and $\vec{\mathbf{n}}_{CG}^* = \vec{\mathbf{n}}_{CG} + \Delta \vec{\mathbf{n}}_{CG}$.

Express your answer in "normalized" form, where

$$\begin{bmatrix} \Delta \vec{\mathbf{g}}_{C0} \\ \Delta \vec{\mathbf{n}}_{CG} \end{bmatrix} \approx \mathbf{M} \bullet \vec{\eta}_{BC} = \mathbf{M} \bullet \begin{bmatrix} \vec{\alpha}_{BC} \\ \vec{\varepsilon}_{BC} \end{bmatrix}$$

and $\mathbf{M}$ is a suitable matrix. Alternatively, you can express your formulas as sums of terms in which the $\vec{\alpha}_X$ and $\vec{\varepsilon}_X$ terms appear on the right.

D. (5 points) Suppose, now, that the registration transformation $\mathbf{F}_{BC}$ is subject to a small error $\mathbf{F}_{BC}^{*} = \mathbf{F}_{BC}\Delta\mathbf{F}_{BC}$, with $\Delta\mathbf{F}_{BC} \approx \left[\mathbf{I} + sk(\vec{\alpha}_{BC}), \vec{\varepsilon}_{BC}\right]$, in addition to the other errors . Give formulas for estimating the error in your computation of $\vec{\mathbf{g}}_{C0}$ and $\vec{\mathbf{n}}_{CG}$ Express your answer in "normalized" linear form. **Hint:** Start with your answer to Question 1C and substitute the elements of $\mathbf{F}_{BC}^{*} = \mathbf{F}_{BC}\Delta\mathbf{F}_{BC}$ for the elements of $\mathbf{F}_{BC} = [\mathbf{R}_{BC}, \vec{\mathbf{p}}_{BC}]$.

Question 2

Suppose that the accuracy of the optical tracking system is limited, such that if the tracking system reports a value $\vec{\mathbf{v}}_t$ for an optical marker the true value is $\vec{\mathbf{v}}_k^{*} = \vec{\mathbf{v}}_k + \vec{\varepsilon}_k$, where $\left\|\vec{\varepsilon}_k\right\| \le \delta$. Suppose also that we have the following dimensions for the pointer tool: $\vec{\mathbf{a}}_0 = [0,-200,0]^T, \vec{\mathbf{a}}_1 = [25,0,0]^T, \vec{\mathbf{a}}_2 = [0,50,0]^T$, and $\vec{\mathbf{a}}_3 = [-25,0,0]^T$.

A. (5 points) Give simple formulas for computing $\mathbf{F}_A = [\mathbf{R}_A, \vec{\mathbf{p}}_A]$ from the measured positions $[\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3]$ of $[\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{a}}_3]$. **Hint:** You may find it useful to compute $\vec{\mathbf{p}}_A$ and then compute a couple intermediate vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ from the $\vec{\mathbf{v}}_k$ values and $\vec{\mathbf{p}}_A$.

B. (10 points) Assuming that the values of the $\vec{\mathbf{a}}_k$ are as given above, give formulas to *estimate* bounds on the coordinates of

$$\vec{\eta}_A = \begin{bmatrix} \vec{\alpha}_A \\ \vec{\varepsilon}_A \end{bmatrix}$$

in terms of d. Express the answer as a set of linear inequalities. **Hint:** Make a sketch and think about the geometry of the situation before making a deep dive into formulas.

C. (5 points) Estimate how accurately the pointer tool tip can be located relative to the tracking system. I.e., if the computed value for the pointer tip location is $\vec{\mathbf{v}}_{tip}$, give a formula for $\vec{\varepsilon}_{tip} = \Delta\vec{\mathbf{v}}_{tip}$ in terms of $\mathbf{F}_A = [\mathbf{R}_A, \vec{\mathbf{p}}_A]$, the values of the $\vec{\mathbf{a}}_k$, $\vec{\alpha}_A$, and $\vec{\varepsilon}_A$, expressing your answer in normalized linear form.

D. (5 points) Suppose (for the present purposes) that $\mathbf{R}_A = \mathbf{I}$. Based on your answer to Questions 2B and 2C, produce bounds on the components of the error $\vec{\varepsilon}_{tip} = \Delta\vec{\mathbf{v}}_{tip}$ in terms of d.

E. (5 points) Give formulas for computing the position $\vec{\mathbf{v}}_{Bt}$ of the pointer tip relative to $\mathbf{F}_B$ and the error $\vec{\varepsilon}_{Bt} = \Delta\vec{\mathbf{v}}_{Bt}$ assuming that

$$\vec{\eta}_A = \begin{bmatrix} \vec{\alpha}_A \\ \vec{\varepsilon}_A \end{bmatrix} \text{ and } \vec{\eta}_B = \begin{bmatrix} \vec{\alpha}_B \\ \vec{\varepsilon}_B \end{bmatrix}$$

Here, you may find it convenient to define $\mathbf{F}_{BA} = \mathbf{F}_B^{-1}\mathbf{F}_A$.

F. (5 points) Suppose that we know something about the magnitude of $\vec{\eta}_A$ and $\vec{\eta}_B$. In particular,

$$\left\|\vec{\varepsilon}_A\right\| \leq \mu,\ \left\|\vec{\varepsilon}_B\right\| \leq \mu,\ \left\|\vec{\alpha}_A\right\| \leq 2\nu,\ \left\|\vec{\alpha}_B\right\| \leq \nu$$

Give a reasonable estimate limiting the magnitude of $\vec{\varepsilon}_{Bt}$. I.e., what is a reasonable value of b to ensure $\left\|\vec{\varepsilon}_{Bt}\right\| \leq \beta$?

G. (5 points) **[CLARIFIED]** Suppose that the pointer has been placed within a distance s of three anatomic landmarks identified in the CT images. I.e., if $\vec{\mathbf{v}}_k^*$ is the actual position of the pointer tip and $\vec{\mathbf{u}}_k$ is the position of the landmark in CT coordinates, then the point $\vec{\mathbf{v}}_k^*$ is within a distance s of the actual point whose corresponding CT coordinate is $\vec{\mathbf{u}}_k$. The inequality in the original version of this problem was mis-stated as $\left\|\vec{\mathbf{v}}_k^* - \vec{\mathbf{u}}_k\right\| \leq \sigma$. More correctly, this should be stated as $\left\|\vec{\mathbf{v}}_k^* - \mathbf{F}_B^*\mathbf{F}_{BC}^*\vec{\mathbf{u}}_k\right\| \leq \sigma$. Suppose that we have the following values for $\vec{\mathbf{p}}_A$ and $\vec{\mathbf{p}}_B$, but (for simplicity) $\mathbf{R}_A = \mathbf{R}_B = \mathbf{I}$ for all measurements.

$\vec{\mathbf{u}}_k$	$\vec{\mathbf{p}}_A^T$	$\vec{\mathbf{p}}_B^T$
[150,200,150]	[550,650,650]	[550,450,850]
[100,250,150]	[510,710,670]	[510,510,870]
[100,200,200]	[490,700,710]	[490,450,910]

What is the nominal value of $\mathbf{F}_{BC}$, assuming that the pointer and reference body are tracked perfectly.

In view of the confusion from copy errors, here is the answer: $\vec{\mathbf{p}}_{BC} = \left[-100,-150,0\right]^T$ and $\mathbf{R}_{BC} = \mathbf{I}$.

H. (10 points) Assume for the moment, that $s = 0$, i.e., that you can place the pointer tip exactly on desired physical landmarks, but that there is some tracking error, so your values for $\vec{\mathbf{v}}_k$ are not accurate. However, you have been told a bounding value b such that the tracking error $\vec{\varepsilon}_{Bt}$ of the pointer tip relative to $\mathbf{F}_B$ is bounded by b. I.e.

$$\left\|\vec{\varepsilon}_{Bt,k}\right\| = \left\|\Delta\mathbf{F}_B^{-1}\mathbf{F}_B^{-1}\vec{\mathbf{v}}_k^* - \mathbf{F}_B^{-1}\vec{\mathbf{v}}_k\right\| \le \beta$$

<u>*Estimate*</u> bounds on the error in your answer for Question 2G

$$\Delta\mathbf{F}_{BC} = \left(\mathbf{F}_{BC}\right)^{-1}\mathbf{F}_{BC}^* \approx \left[\mathbf{I} + sk(\vec{\alpha}_{BC}), \vec{\varepsilon}_{BC}\right].$$

Express these as constraints on the components of $\vec{\alpha}_{BC}$ and $\vec{\varepsilon}_{BC}$ in terms of b. **Hint:** Again, make a sketch and do some simple geometric reasoning. This problem is similar to one you have seen before.

I. (10 points) Now, suppose that your tip tracking and your ability to place the pointer exactly on landmarks are subject to small errors. I.e. $s > 0$ and $b > 0$. Give constraints on the components of $\vec{\varepsilon}_{BC}$ and $\vec{\alpha}_{BC}$ in terms of s and b.

J. (10 points) Assume that $\sigma \leq 1$ mm and that you have placed the pointer on a physical anatomic target at a position $\vec{\mathbf{p}}_t^* = \left(\mathbf{F}_B^*\right)^{-1}\mathbf{F}_A^*\vec{\mathbf{a}}_0$ with $\left\|\vec{\mathbf{p}}_t^*\right\| \leq 100$. The corresponding point in CT coordinates would be $\vec{\mathbf{u}}_t^* = \left(\mathbf{F}_{BC}^*\right)^{-1}\vec{\mathbf{p}}_t^*$ and the computed position would be $\vec{\mathbf{u}}_t = \mathbf{F}_{BC}^{-1}\vec{\mathbf{p}}_t$. What value (if any) of b would be needed to ensure that $\left\|\vec{\mathbf{u}}_t^* - \vec{\mathbf{u}}_t\right\| \leq 3\text{mm}$?

K. (10 points) Suppose that instead of knowing bounds on the error associated with $\vec{\varepsilon}_k = \Delta\vec{\mathbf{v}}_k$, we have a gaussian error model $\vec{\eta}_A \sim \mathrm{N}(\vec{\mathbf{0}}, \mathbf{C}_A)$ where $\mathbf{C}_A = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}$. Give a formula for $\mathbf{C}_{tip} = \text{cov}(\vec{\varepsilon}_{tip}) = E(\vec{\varepsilon}_{tip} \bullet \vec{\varepsilon}_{tip}^T)$ in terms of the components of $\mathbf{C}_A$ and other variables in this problem.