Homework Assignment 4 – 601.455/655 (CIRCLE ONE)

Fall 2023

Instructions and Score Sheet (hand in with answers)

Name	
Email	
Other contact information (optional)	
Signature (required)	I have followed the rules in completing this assignment ______________________________
Name	
Email	
Other contact information (optional)	
Signature (required)	I have followed the rules in completing this assignment ______________________________

Question	Points	Points	Subtotal
1A	5		
1B	5		
1C	10		
1D	10		
1E	10		
2A	10		
2B	10		
2C	20		
2D	15		
2E	10		
3A	5		
3B	5		
3C	15		
3D	5		
Total	135		
	Min (Total, 100)		

NOTE: Note: There 135 total possible points in this assignment, but at most 100 will count toward your final letter grade.

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone** or **in teams of two** and are **not to discuss the problems with anyone** other than the TAs or the instructor.
3. It is otherwise open book, notes, and web. But you should cite any references you consult.
4. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
5. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
6. Sign and hand in the score sheet as the first sheet of your assignment.

Question 1

Consider the robot manipulator shown in Fig. 1, which comprises a 6 degree-of-freedom parallel-link section carrying a multiple degree-of-freedom serial-link arm. The parallel link mechanism has six linear actuators, each connected by a spherical joint to the base of the robot at positions $\vec{\mathbf{b}}_i$ in the coordinate system of the robot. Each of these linear actuators is attached to a corresponding spherical joint on a moving plate, as shown in the figure, The length of each actuator is given by a distance q_i between the corresponding attachment points. The position and orientation of the moving plate with respect to the base of the robot is given by $\mathbf{F}_p(\vec{\mathbf{q}})$, where $\vec{\mathbf{q}} = [q_1, \cdots, q_6]^T$. The position of the attachment point to the moving plate corresponding to base attachment point $\vec{\mathbf{b}}_i$ is given by $\vec{\mathbf{a}}_i$ in the plate coordinate system.

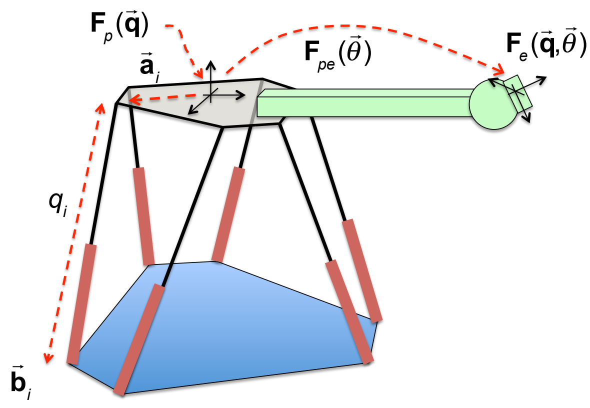

Fig. 1: Parallel-Serial Link Manipulator

The serial link second stage is attached rigidly to the plate and has N_{serial} joints with a known kinematic design. The position and orientation of the end effector relative to the plate coordinate system is given by $\mathbf{F}_{pe}(\vec{\theta}) = Kins_{pe}(\vec{\theta})$ where $\vec{\theta} = [\theta_1, \cdots, \theta_{N_{serial}}]^T$. Thus the position and orientation of the end effector with respect to the robot base is $\mathbf{F}_e(\vec{\mathbf{q}},\vec{\theta}) = \mathbf{F}_p(\vec{\mathbf{q}})\mathbf{F}_{pe}(\vec{\theta})$. The “right side: Jacobean of $Kins_{pe}(\vec{\theta})$ is given by $\mathbf{J}_{pe}^{(right)}(\vec{\theta})$, so that

$$
\begin{aligned}
\mathbf{F}_{pe}(\vec{\theta} + \Delta\vec{\theta}) &= \mathbf{F}_{pe}(\vec{\theta})\Delta\mathbf{F}_{pe}^{(right)}(\vec{\theta},\Delta\vec{\theta}) \\
&\approx \mathbf{F}_{pe}(\vec{\theta}) \bullet \left[\mathbf{I} + sk(\vec{\alpha}_{pe}), \vec{\varepsilon}_{pe}\right]
\end{aligned}
$$

$$
\begin{bmatrix} \vec{\alpha}_{pe} \\ \vec{\varepsilon}_{pe} \end{bmatrix} = \gamma_{pe} = \mathbf{J}_{pe}(\vec{\theta})\Delta\vec{\theta} = \begin{bmatrix} \mathbf{J}_{pe}^{\mathbf{R}}(\vec{\theta}) \\ \mathbf{J}_{pe}^{\mathbf{P}}(\vec{\theta}) \end{bmatrix} \Delta\vec{\theta}
$$

The robot’s end effector is equipped with a force/torque sensor that is able to resolve forces $\vec{\mathbf{f}}_e$ and torques $\vec{\tau}_e$ resolved in the end-effector coordinate system. For convenience, we will use the symbol $\vec{\phi}_e = [\vec{\tau}_e^T, \vec{\mathbf{f}}_e^T]^T$.

A. Suppose that the value of $\vec{\theta}$ is fixed to a known value. Give formulas for computing the values $\vec{\mathbf{q}}_{des} = [q_1, \cdots, q_6]_{des}^T$ such that $\mathbf{F}_e(\vec{\mathbf{q}},\vec{\theta}) = \mathbf{F}_{des}$, where $\mathbf{F}_{des}$ is some desired target pose for the end effector. **Hint:** Note that the values can be computed directly from $\mathbf{F}_p$.

B. Suppose, now that there is a new desired pose for the end effector $\mathbf{F}_e^{(new)} = \mathbf{F}_e^{(prev)}\Delta\mathbf{F}_e$. Assuming that $\Delta\mathbf{F}_e \approx [\mathbf{I} + sk(\vec{\alpha}_e), \vec{\varepsilon}_e]$, provide a linearized approximation for the required change in plate pose such that $\Delta\mathbf{F}_p\mathbf{F}_p^{(prev)}\mathbf{F}_{pe} = \mathbf{F}_e^{(new)}$, where $\Delta\mathbf{F}_p \approx [\mathbf{I} + sk(\vec{\alpha}_p), \vec{\varepsilon}_p]$. I.e., express $\vec{\alpha}_p$ and $\vec{\varepsilon}_p$ in terms of $\vec{\alpha}_e$, $\vec{\varepsilon}_e$, and the other quantities in the system. Your answer should be expanded sufficiently so that terms like $sk(\vec{\alpha}_p)$ have been simplified out. Terms of the general form $\mathbf{M} \bullet \vec{\alpha}_p$ where $\mathbf{M}$ represents some sort of matrix are fine. **Hint:** Remember that $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}$.

C. Similarly, give expressions for $\vec{\alpha}_e$ and $\vec{\varepsilon}_e$ corresponding to a small change in plate position $\mathbf{F}_p^{new} = \Delta\mathbf{F}_p(\vec{\alpha}_p, \vec{\varepsilon}_p)\mathbf{F}_p(\vec{\mathbf{q}})$. Again, your answer should be expanded sufficiently so that terms like $sk(\vec{\alpha}_p)$ or $sk(\vec{\alpha}_e)$ have been simplified out.

D. For the moment, keeping $\vec{\theta}$ fixed, compute $\Delta\vec{\mathbf{q}}$ required to move the end effector to a new pose $\mathbf{F}_{new} = \mathbf{F}_{des}\Delta\mathbf{F}_e$. Express in terms of $\vec{\alpha}_e$, $\vec{\varepsilon}_e$, and the other quantities in the system. Again, your answer should be expanded sufficiently so that terms like $sk(\vec{\alpha}_p)$ or $sk(\vec{\alpha}_e)$ have been simplified out. **Hint:** You may want to consider producing a formula relating small motions $\vec{\gamma}_p = [\vec{\alpha}_p^{\,T}, \vec{\varepsilon}_p^{\,T}]^T$ of the plate to small changes $\Delta\vec{\mathbf{q}}$ in $\vec{\mathbf{q}}$ and incorporating this formula into your answer.

E. For convenience, we can define $\vec{\gamma}_p = [\vec{\alpha}_p^T, \vec{\varepsilon}_p^T]^T$ and $\vec{\gamma}_e = [\vec{\alpha}_e^T, \vec{\varepsilon}_e^T]^T$. Suppose now that the values of $\vec{\theta}$ can change as well as the values of $\vec{\mathbf{q}}$, give linearized formulas for computing $\vec{\gamma}_e(\vec{\mathbf{q}}, \vec{\theta}; \vec{\alpha}_p, \vec{\varepsilon}_p, \Delta\vec{\theta})$, i.e., for computing $\vec{\alpha}_e(\vec{\mathbf{q}}, \vec{\theta}; \vec{\alpha}_p, \vec{\varepsilon}_p, \Delta\vec{\theta})$ and $\vec{\varepsilon}_e(\vec{\mathbf{q}}, \vec{\theta}; \vec{\alpha}_p, \vec{\varepsilon}_p, \Delta\vec{\theta})$. Again, your answer should be expanded sufficiently so that terms like $sk(\vec{\alpha}_p)$ or $sk(\vec{\alpha}_e)$ have been simplified out.

Question 2

For now, we will continue to refer to the robot in Question 1. For this question, you can also assume that functions to compute $\vec{\gamma}_p(\vec{\mathbf{q}}, \vec{\theta}; \vec{\alpha}_e, \vec{\varepsilon}_e, \Delta\vec{\theta})$ and $\vec{\gamma}_e(\vec{\mathbf{q}}, \vec{\theta}; \vec{\alpha}_p, \vec{\varepsilon}_p, \Delta\vec{\theta})$ are available. Suppose that each of the linear actuators has a (very stiff, but non-negligible) degree of compliance, so that a linear force f_i along the direction of the actuator will produce a small length change $\Delta q_i = \delta_i = f_i / \kappa_i$, where κ_i is a spring constant associated with actuator i. We are interested in assessing the stiffness of the system at the end effector. Let $\vec{\gamma}_{e,k}$ represent the k'th component of $\vec{\gamma}_e$ (e.g., $\vec{\gamma}_{e,1} = \vec{\alpha}_{e,1}$, etc.) and let $\vec{\phi}_{e,k}$ represent a torque or force in the corresponding direction. If we ignore friction and other complicating factors, then $\vec{\phi}_{e,k}$ will produce a deflection $\vec{\gamma}_{e,k} = \phi_{e,k} / \kappa_{e,k}$, where $\kappa_{e,k}$ is an effective spring constant. In general this value will depend on the configuration variables $[\vec{\mathbf{q}}, \vec{\theta}]$. In the questions below, you should assume that the deflection is not measurable by the encoders used in the linear actuator control. I.e., if the encoders say that the parallel actuators are at $\vec{\mathbf{q}}$, the actual positions will be $\vec{\mathbf{q}} + \Delta\vec{\mathbf{q}}_{deflect}$. Further, you can assume that $\Delta\vec{\mathbf{q}}_{deflect}$ is small.

A. Produce a formula for computing $\kappa_{e,k}(\vec{\mathbf{q}},\vec{\theta})$ for $1 \le k \le 6$. **Hint:** Remember that the energy expended in deflecting a linear spring with spring constant κ by an amount σ is $\kappa\sigma^2/2$ and recognize that you are expending energy by deflecting the end effector. You know the κ_i.

B. Suppose that you have successfully derived a formula for $\kappa_{e,k}(\vec{\mathbf{q}},\vec{\theta})$, and that the force/torque sensor has reported values $\vec{\phi}_e$. This will produce some deflection of the end effector. Describe a method for using the linear actuators to compensate for this deflection. For the purposes of problem 2.B, you can ignore any complications that arise from the fact that the robot is actually controlled with some sort of real time periodic process. We are just looking for a one-time adjustment.

C. Suppose now that the robot design has been improved, so that compliance of the linear actuators may be ignored. I.e., suppose that the κ_i values are extremely high. We wish to implement a simple admittance controller for the robot. I.e., given measured force/torque sensor values $\vec{\phi}_e$, we wish the robot's velocity (relative to the end effector coordinate system) to be given by $\dot{\vec{\gamma}}_e^{(des)} = \mathbf{C}\vec{\phi}_e$. The basic run loop should look something like this:

   Step 0: Wait for the next time step Δt.

   Step 1: Measure the state $[\vec{\mathbf{q}},\vec{\theta},\vec{\phi}_e]$. Also, either measure or estimate velocities $\dot{\vec{\mathbf{q}}}$ and $\dot{\vec{\theta}}$

   Step 2: Compute incremental actuator motions $\Delta\vec{\mathbf{q}}$ and $\Delta\vec{\theta}$ for the current Δt time step

   Step 3: Output actuator velocities $\Delta\vec{\mathbf{q}}/\Delta t$ and $\Delta\vec{\theta}/\Delta t$.

   Step 4: Go back to Step 0.

   Your job is to describe how to implement Step 2. Note that the robot is redundant, so you cannot simply invert a matrix. You need to observe the following position and velocity constraints:

$$\vec{\mathbf{q}}_{\min} \le \vec{\mathbf{q}}(t) \le \vec{\mathbf{q}}_{\max} \quad \vec{\theta}_{\min} \le \vec{\theta}(t) \le \vec{\theta}_{\max}$$

$$\dot{\vec{\mathbf{q}}}_{\min} \le \dot{\vec{\mathbf{q}}}(t) \le \dot{\vec{\mathbf{q}}}_{\max} \quad \dot{\vec{\theta}}_{\min} \le \dot{\vec{\theta}}(t) \le \dot{\vec{\theta}}_{\max}$$

   You can assume that the robot accelerates instantly and that the computation requires negligible time, so that the robot actuator velocities $\dot{\vec{\mathbf{q}}}$ and $\dot{\vec{\theta}}$ will be the commanded values throughout the time step (and thus that the actuator motion increments for the time step will be the computed $\Delta\vec{\mathbf{q}}$ and $\Delta\vec{\theta}$. Although the accelerations are assumed to be instantaneous, you are also asked to minimize the change in joint velocities from time step to time step, and you should try to keep all the actuators as close as possible to the midpoint of their range. If you cannot achieve the desired admittance, you should try to achieve it as closely as possible, subject to the other constraints. The relative importance of the three desired behaviors are as follows:

   - Importance of providing the desired admittance behavior: η_{admit}
   - Importance of keeping actuators near midpoint: $\eta_{mid,\vec{\mathbf{q}}}$ and $\eta_{mid,\vec{\theta}}$ for the two sets of actuators

- Importance of minimizing speed changes: $\eta_{vel,\vec{\mathbf{q}}}$ and $\eta_{vel,\vec{\theta}}$ for the two sets of actuators

Hint: Here, I am asking you to set up an appropriate constrained least squares optimization problem. Remember that you can have both inequality and equality constraints. You should use equality constraints to make your answers clearer. I don't want you to have to carry out extensive algebraic substitutions to get rid of equalities. Also, you should feel free to use appropriate linear approximations to simplify your constraints and objective functions. For example, you may want to find an appropriate linear approximation expressing $\Delta\vec{\mathbf{q}}$ in terms of $\vec{\alpha}_p$ and $\vec{\varepsilon}_p$ as in Question 1.D. Also, if you rely on an answer from a previous problem, you should indicate the fact, and say which answer you are relying on.

D. Suppose that the forces $\vec{\phi}_e$ is Question 2.C are produced by a combination of forces exerted by the surgeon on a tool attached at the end effector and gravitational forces due to the weight of the tool with weight μ and center of gravity $\vec{\mathbf{c}}$. How would you modify your answer to Question 2.C so that the robot only responds to forces and torques exerted by the surgeon? To simplify matters, you can assume that a function. $\phi_{grav}(\vec{q},\vec{\theta};\mu,\vec{\mathbf{c}})$ is available to compute the forces and torques due to gravity exerted by the tool at the end-effector interface. Also, you can assume that the speed of the robot is slow enough so that you can ignore inertial forces, and only consider forces due to gravity.

E. Suppose, now, that you cannot ignore the actuator stiffness that you considered in Questions 2.A and 2.B. How would you modify your answer to Question 2.C to minimize the effect of deflection due to $\vec{\phi}_e$?

Question 3

Assume that you have a correct answer to Question 2.C. Also, you can also assume that functions to compute $\vec{\gamma}_p(\vec{\mathbf{q}},\vec{\theta};\vec{\alpha}_e,\vec{\varepsilon}_e,\Delta\vec{\theta})$ and $\vec{\gamma}_e(\vec{\mathbf{q}},\vec{\theta};\vec{\alpha}_p,\vec{\varepsilon}_p,\Delta\vec{\theta})$ are available. This problem asks you what you would add to your solution to provide some virtual fixtures to assist a surgeon. In the scenario shown in Fig. 2, a surgical tool (which you can think of as a surgical cutter or drill, if you like) has been attached to the end-effector of the robot. The coordinate system associated with the tool tip, relative to the end effector, is $\mathbf{F}_{et} = [\mathbf{I},\vec{\mathbf{p}}_{tip}]$.

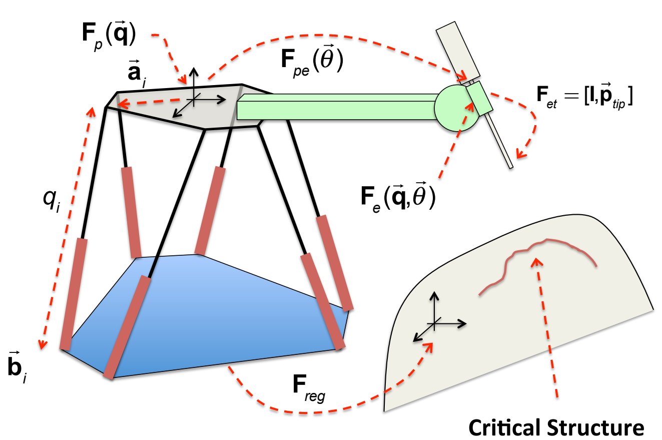

Fig. 2

A portion of a patient's anatomy has been placed in a fixed position within the work volume of the robot. A CT scan has been made; a segmentation step has been performed to identify the surface of a target structure; and an additional computation has been performed to compute a signed Euclidean distance to the structure of every location of interest within the vicinity of the structure. The distance is positive for points "outside" of the structure, zero for points on the surface, and negative for points "inside" the structure. This information is stored in a CT-like volume with the same coordinates as the CT scan. Thus, for every voxel in the CT image, the software has stored a real number corresponding to the signed Euclidean distance to the structure, together with the CT coordinates of the closest point on the structure. Software is available that interpolates this data. The following subroutines are available for any structure Ω:

$S(\Omega,\vec{\mathbf{c}})$	Signed distance from $\vec{\mathbf{c}}$ to Ω
$D(\Omega,\vec{\mathbf{c}})$	$\lvert S(\Omega,\vec{\mathbf{c}})\rvert$ I.e., the unsigned distance
$\nabla S(\Omega,\vec{\mathbf{c}})$	Gradient of signed distance at $\vec{\mathbf{c}}$
$\nabla D(\Omega,\vec{\mathbf{c}})$	Gradient of unsigned distance at $\vec{\mathbf{c}}$
$\vec{\mathbf{s}}(\Omega,\vec{\mathbf{c}})$	Closest point on Ω to $\vec{\mathbf{c}}$

For the purposes of this exercise, you can assume that the structure is fairly smooth, so that the surface has no sharp corners and that there is no ambiguity about what is the closest point and that the closest points to voxels that are close to each other are also close to each other, and vice versa. The system has also performed segmentation to identify additional anatomic structures (such as the surface of the patient) and has computed additional signed Euclidean distance maps for these structures. These structure are also assumed to be reasonably smooth but with sufficient asymmetry that they may be used for a registration algorithm.

A. Give a linearized equation or system of equations giving the position $\vec{\mathbf{p}}_{tool}$ of the tip of the tooltool relative to the base of the robot after a small motion to a new position $\mathbf{F}_e^{new} \approx \Delta\mathbf{F}_p(\vec{\alpha}_p,\vec{\varepsilon}_p)\mathbf{F}_e(\vec{\mathbf{q}},\vec{\theta}+\Delta\vec{\theta})$. Express your answer in terms of $\vec{\alpha}_p$, $\vec{\varepsilon}_p$, and $\Delta\vec{\theta}$. Again, avoid expressions with things like $sk(\vec{\alpha}_p)$. **Hint:** It is fine if you give expressions for $[\vec{\alpha}_e,\vec{\varepsilon}_e]=\vec{\gamma}_e(\vec{\mathbf{q}},\vec{\theta};\vec{\alpha}_p,\vec{\varepsilon}_p,\Delta\vec{\theta})$ and then express the new tool tip position in terms of $[\vec{\alpha}_e,\vec{\varepsilon}_e]$. Or you can substitute things out. **Hint:** You may want to consider producing a formula relating small motions $\vec{\gamma}_p=[\vec{\alpha}_p{}^T,\vec{\varepsilon}_p{}^T]^T$ of the plate to small changes $\Delta\vec{\mathbf{q}}$ in $\vec{\mathbf{q}}$ and incorporating this formula into your answer.

B. The surgeon has hand guided the robot to a number of points $\vec{\mathbf{g}}_j$ (in robot coordinates) on the surface of the anatomic structures to be used for registration. For some reason, the engineer designing the system has developed a virulent dislike of ICP and does not want to implement a "find closest point" algorithm. Please describe a suitable alternative algorithm that the engineer might use to compute a registration transformation $\mathbf{F}_{reg}$ so that any point $\vec{\mathbf{c}}$ in CT coordinates has the position $\mathbf{F}_{reg}\vec{\mathbf{c}}$ relative to the base of the robot. You may assume that a reasonably accurate initial guess $\mathbf{F}_{reg}^{(0)}$ is available. Here, you should describe the approach in sufficient detail so that it is clear how it relates to the problem data, but you do not need to provide all the algorithmic details. You should discuss some of the efficiency trade-offs involved.

C. Now, assume that a suitable algorithm for Question 3.B has been implemented, so that $\mathbf{F}_{reg}$ is known. How would you modify your answer to Question 2.C to ensure that the tool tip never penetrates to a depth greater than $\delta_{\lim}$ below the surface of the critical structure.

Hint: This will require adding some constraints to an optimization problem. Remember that you can have both equality and inequality constraints.

D. How would your further modify you answer to Question 3.C to assist the surgeon to keep the tool tip as close as possible to the surface of the critical structure while being able to move the tool freely along the surface. The importance of staying as close as possible is given by η_{surf}.