Homework Assignment 3 – 601.455/655 Fall 2023

Instructions and Score Sheet (hand in with answers)

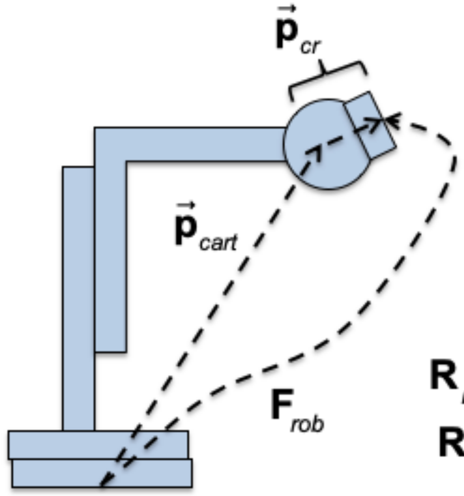
Name	Name
Email	Email
Other contact information (optional)	Other contact information (optional)
Signature (required) I/We have followed the rules in completing this assignment ____________________________	Signature (required) I/We have followed the rules in completing this assignment ____________________________

Question	Points		Score	Totals
1A	5			
1B	5			
1C	15			
1D	15			
1E	10			
	Subtotal	50		
2A	15			
2B	5			
2C	10			
2D	10			
2E	10			
3F	20			
	Subtotal	70		
Total		120		

Note: There 120 total possible points in this assignment, but at most 100 will count toward your final letter grade.

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone** or **in teams of two** and are **not to discuss the problems with anyone** other than the TAs or the instructor.
3. It is otherwise open book, notes, and web. But you should cite any references you consult.
4. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
5. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
6. Sign and hand in the score sheet as the first sheet of your assignment.

Question 1 (Calibration)



$$\mathbf{F}_{rob} = [\mathbf{R}_{rob}, \vec{\mathbf{p}}_{rob}]$$
$$= [\mathbf{R}_{rob}, \vec{\mathbf{p}}_{cart} + \mathbf{R}_{rob}\vec{\mathbf{p}}_{cr}]$$

$$\vec{\mathbf{p}}_{cart} = [x, y, z]^T \quad \vec{\theta} = [\theta_4, \theta_5, \theta_6, \theta_7]^T \quad \vec{\mathbf{q}} = \begin{bmatrix} \vec{\mathbf{p}}_{cart} \\ \vec{\theta} \end{bmatrix}$$

$$\mathbf{R}_{rob}(\vec{\theta}) = \mathbf{R}_4(\theta_4)\mathbf{R}_5(\theta_5)\mathbf{R}_6(\theta_6)\mathbf{R}_7(\theta_7)$$
$$\mathbf{R}_k(\theta_k) = Rot(\vec{\mathbf{a}}_k, \theta_k)$$

Fig. 1: Robot Kinematics

Consider the robot shown in Fig. 1This robot consists of a Cartesian base, a multi-axis wrist mechanism, and a tool holder. The transformation between the robot's base and the tool holder is given by $\mathbf{F}_{rob}(\vec{\mathbf{q}})$, as shown in the figure.

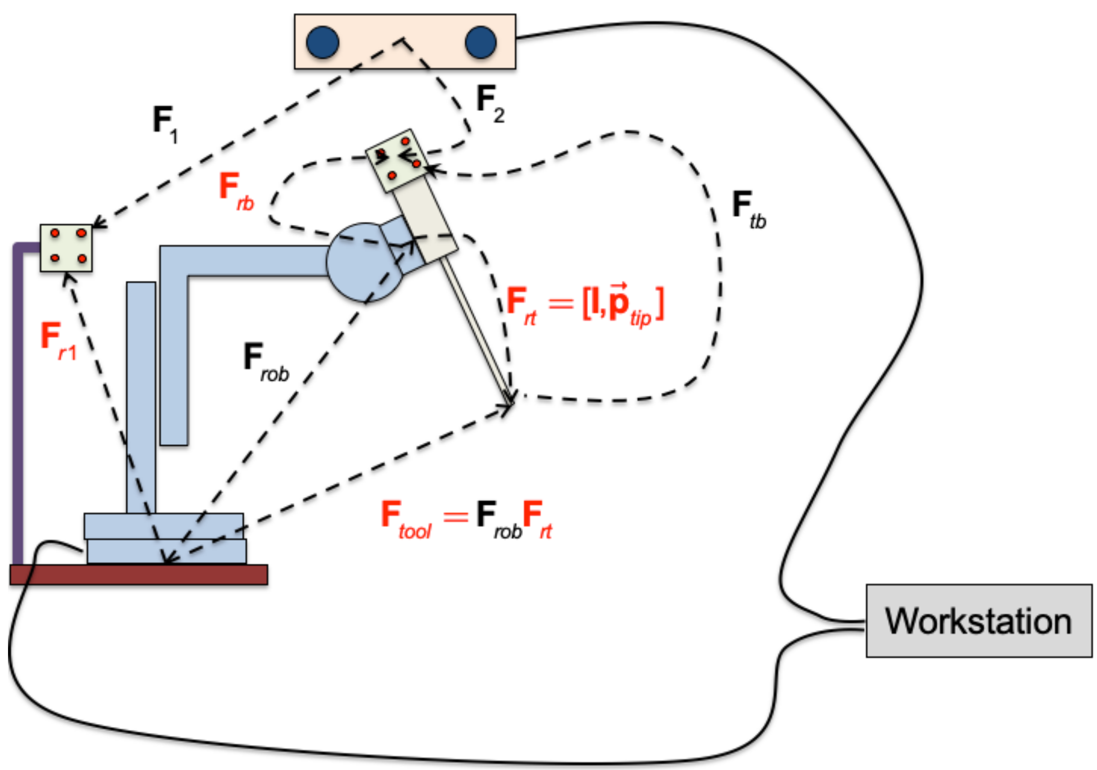


Fig. 2: Robot with tool and tracker bodies

Now consider the setup in Fig. 2, in which the robot from Question 1 has been equipped with a surgical tool which has been attached to the robot's wrist at an unknown pose $\mathbf{F}_{rb} = [\mathbf{R}_{rb}, \vec{\mathbf{p}}_{rb}]$. The origin of the tool's coordinate system is located at the tip of the tool. An optical tracking system marker body is attached to the tool at a known pose $\mathbf{F}_{tb} = [\mathbf{R}_{tb}, \vec{\mathbf{p}}_{tb}]$. The (time varying) pose of this marker body relative to the tracking system is $\mathbf{F}_2$, as shown in the figure. An additional tracker marker body has been attached to the base frame of the robot at a fixed, but unknown pose $\mathbf{F}_{r1}$. The (time varying) pose of this marker body relative to the tracking system is $\mathbf{F}_2$, as shown in the figure.

A. Assume that $\mathbf{F}_{rb}$ and $\mathbf{F}_{r1}$ are both known, but that $\mathbf{F}_1$ is not available (e.g., it could be obscured). Give a formula for computing the predicted value of $\mathbf{F}_1$, based on $\mathbf{F}_{rob}$, $\mathbf{F}_2$ and other quantities in the problem.

B. Assume that $\mathbf{F}_{rb}$ and $\mathbf{F}_{r1}$ are both known, but that that $\mathbf{F}_{rob}$ is not available (e.g., its controller may be offline). Give a formula for computing $\mathbf{F}_{rob}$ of the tool marker body relative to the tracking system, based on $\mathbf{F}_1$, $\mathbf{F}_2$ and other quantities in the problem.

C. Suppose that the values measured by the encoders of the robot are highly repeatable and that they are equipped with a reliable index mark so that the reported encoder values $\vec{\mathbf{q}}_k = 0$ when joint k is at the index point. However, the encoders readings are subject to small systematic distortions, so that the actual value at any given time the true positions (i.e., translations of the prismatic joints and angles for the revolute joints) of the robot joints are given by $\vec{\mathbf{q}}^* = \vec{\mathbf{q}} + \Delta\vec{\mathbf{q}}(\vec{\mathbf{q}})$. The error associated with each joint is independent of all the other joints of the robot. Also, the rate of change is small, so that

$$\left| \frac{d\Delta(q_k)}{dq_k} \right| \le \rho_k$$

For now, you can assume that the optical tracking system is highly accurate. Describe an efficient calibration procedure that can be used to produce a correction function for estimating the true value for each joint value to within a specified accuracy. If q_k^{est} is the corrected value and δ_k is the desired maximum error, then you need to ensure that $\left| q_k^* - q_k^{est} \right| \le \delta_k$.

Here, you should give a step-by-step workflow describing any commanded robot motions and calculations that you need to perform. **Hint:** The amount of data required should be roughly linear in the number of joints.

D. Suppose now that you have successfully calibrated the joint encoders and incorporated the results into the robot controller software, so that you can simply use the reported values of the joint positions in your calculations. But the robot's mechanical structure is slightly deformed, so that the true direction of each cartesian axis relative to the base of the robot is $\vec{\mathbf{n}}_j^* = \Delta\mathbf{R}_j\vec{\mathbf{n}}_j$ and the true direction of each rotational axis is $\vec{\mathbf{a}}_k^* = \Delta\mathbf{R}_k\vec{\mathbf{a}}_k$. Describe an efficient calibration method for correcting for these structural deformations. Again, you should give a step-by-step workflow describing any commanded robot motions and calculations that you need to perform. **Hint:** The choice of the base coordinate system is somewhat arbitrary and can depend on how the base frame of the robot is defined. Try to align the corrected cartesian axes as close as possible to the canonical cartesian axes $[\vec{\mathbf{x}},\vec{\mathbf{y}},\vec{\mathbf{z}}]$. I.e., assume that the base frame of the robot is $\mathbf{F}_{base} = [\mathbf{I},\vec{\mathbf{0}}]$ in robot coordinates. **Hint:** This will lead to an estimate of $\mathbf{F}_{r1}$.

E. Assume now that your robot is calibrated and that all results have been incorporated into the robot controller software, so that values reported for $\mathbf{F}_{rob}$ have at most small random errors. However, suppose that the marker body associated $\mathbf{F}_1$ has been removed from the robot and reattached in an unknown pose relative to the robot. Give a procedure for estimating the unknown values of $\mathbf{F}_{r1}$ and $\mathbf{F}_{rb}$. Here you may assume that $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_{rob}$ are subject to only small random errors. Again, you should give a step-by-step workflow describing any commanded robot motions and calculations that you need to perform. You may identify known algorithms but should clearly identify the formulations of the inputs and outputs.

Question 3 (Robot-Assisted Surgery)

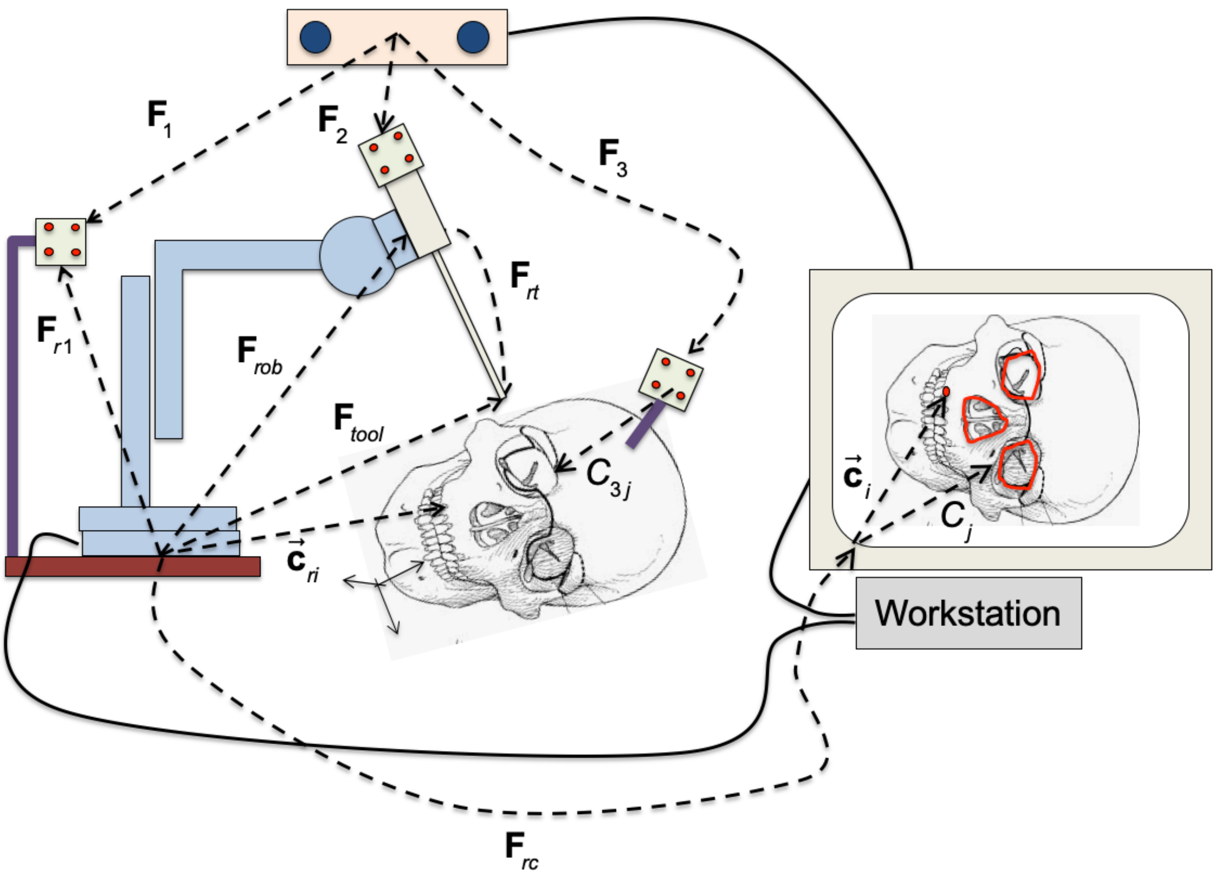


Fig. 3: Robot-Assisted Surgery

Consider now the situation shown in Fig. , in which the robot described in Question 2 has been integrated into a surgical application. All calibration steps have been performed correctly, so that $\mathbf{F}_{r1}$ and $\mathbf{F}_{rt}$ are known. A third marker body has been attached to the patient. The pose of this marker body relative to the tracking system is $\mathbf{F}_3$.

A CT scan of the patient is available and image processing has been performed to locate a number of anatomical features at positions $\vec{\mathbf{c}}_i$ relative to CT coordinates and several 3D contour features C_j represented by linked short line segments in CT coordinates. Through the use of a tracked pointer (not shown), the corresponding contour features C_{3j} have been located on the patient, represented as dense sets of 3D points relative to marker $\mathbf{F}_3$. I.e., if a point $\vec{\mathbf{a}} \in C_{3j}$ it lies somewhere on the anatomic contour C_j identified in the CT image.

A. Describe a method for computing the transformation $\mathbf{F}_{rc}$ between robot coordinates and CT coordinates, so that the position of an anatomical feature at $\vec{\mathbf{c}}_i$ in CT coordinates is given by $\vec{\mathbf{c}}_{ri} = \mathbf{F}_{rc}\vec{\mathbf{c}}_i$ in robot coordinates. Here, you should give a step-by-step procedure explaining what data you will use together with the inputs and outputs of any known algorithms that you will use. **Note:** This question requires you to perform a registration between $\mathbf{F}_3$ and CT coordinates and also to provide formulas that will account for the fact that $\mathbf{F}_{rc}$ will vary as the head is moved.

B. What values of $\mathbf{F}_{rob}$ will place the tool tip at a position $\vec{\mathbf{c}}_{ri} = \mathbf{F}_{rc}\vec{\mathbf{c}}_i$? **Note:** There are actually many possible values. Please provide the condition that all such values must obey.

C. Suppose now that $\mathbf{F}_{rob}$ and $\mathbf{F}_{r1}$ are known accurately, but that the tracker's accuracy is limited, so that $\mathbf{F}_m^* = \mathbf{F}_m \Delta\mathbf{F}_m \approx \mathbf{F}_m \bullet [\mathbf{I} + sk(\vec{\alpha}_m), \vec{\varepsilon}_m]$ for $m \in \{1,2,3\}$ and that we know that $\|\vec{\alpha}_m\| \le \gamma$ and $\|\vec{\varepsilon}_m\| \le \sigma$. Provide an estimated bound $\|\vec{\varepsilon}_{ri}\| \le \rho$ on the error $\vec{\varepsilon}_{ri} = \vec{\mathbf{p}}_{tool}^* - \vec{\mathbf{c}}_{ri}$ when the robot is placed so that $\vec{\mathbf{c}}_{ri} = \mathbf{F}_{rc}\vec{\mathbf{c}}_i$. For now, you can assume that the robot is perfectly accurate. **Note:** In answering this question, you should assume that the value of $\mathbf{F}_2$ is not available.

D. Suppose now that the robot is subject to some error, so that $\mathbf{F}_{rob}^* = \mathbf{F}_{rob}\Delta\mathbf{F}_{rob}$ where $\Delta\mathbf{F}_{rob} \approx [\mathbf{I} + sk(\vec{\alpha}_{rob}), \vec{\varepsilon}_{rob}]$ and that the tracker's accuracy is as in Question 3C. Suppose that you also have bounds $\|\vec{\alpha}_{rob}\| \le \gamma_{rob}$ and $\|\vec{\varepsilon}_{rob}\| \le \sigma_{rob}$. Provide an estimated bound $\|\vec{\varepsilon}_{ri}\| \le \rho$ on the error $\vec{\varepsilon}_{ri} = \vec{\mathbf{p}}_{tool}^* - \vec{\mathbf{c}}_{ri}$ when the robot is placed so that $\vec{\mathbf{c}}_{ri} = \mathbf{F}_{rc}\vec{\mathbf{c}}_i$. Here, you may express your answer in terms of your bound ρ_c from Question 2C, together with γ_{rob} and σ_{rob}. **Note:** In answering this question, you should assume that the value of $\mathbf{F}_2$ is not available.

E. The pose $\mathbf{F}_{tool}$ can actually be measured in two separate ways, one using $\mathbf{F}_{rob}$ and another using $\mathbf{F}_1$ and $\mathbf{F}_2$. Suppose that $\mathbf{F}_{rob}$ is not available. Produce a formula for $\vec{\varepsilon}_{tool} = \Delta\vec{\mathbf{p}}_{tool}$ based on $\mathbf{F}_{r1}$, $\mathbf{F}_m$, $\vec{\alpha}_m$, and $\vec{\varepsilon}_m$ for $m \in \{1,2\}$ and the other quantities in the problem.

F. Suppose now that we have statistical information about our errors, rather than strict bounds, so that

$$Ex(\vec{\eta}) = \vec{\mathbf{0}} \text{ and } Cov(\vec{\eta}) = \mathbf{C}_\eta \text{, where } \vec{\eta} = \begin{bmatrix} \vec{\alpha} \\ \vec{\varepsilon} \end{bmatrix}$$

Produce formulas for $\mathbf{C}_{tool} = Cov(\vec{\varepsilon}_{tool})$ based on $\mathbf{C}_{rob} = Cov(\vec{\eta}_{rob})$ and $\mathbf{C}_m = Cov(\eta_m)$ for $m \in \{1,2\}$ and other variables in the problem.

Hint: Let $\vec{\mathbf{x}} \sim N(\vec{\mathbf{0}}, \mathbf{C}_{XX})$ and $\vec{\mathbf{y}} \sim N(\vec{\mathbf{0}}, \mathbf{C}_{YY})$ be zero-mean multivariate gaussian random variables corresponding to your two answers to Questions 2D and 2E, respectively.

Produce formulas for computing $\mathbf{C}_D = \mathbf{C}_{XX}$ and $\mathbf{C}_E = \mathbf{C}_{YY}$. Note that the two methods for computing $\vec{\varepsilon}_{tool}$ are not independent of each other, since both paths use $\mathbf{F}_1$.

However, if you have three independent random variables, $\vec{\mathbf{a}} \sim N(\vec{\mathbf{0}},\mathbf{C}_a)$, $\vec{\mathbf{b}} \sim N(\vec{\mathbf{0}},\mathbf{C}_b)$, and $\vec{\mathbf{c}} \sim N(\vec{\mathbf{0}},\mathbf{C}_c)$ then we have

$$
\begin{aligned}
Cov(\vec{\mathbf{a}}+\vec{\mathbf{c}},\vec{\mathbf{b}}+\vec{\mathbf{c}}) &= Ex\left(\left(\vec{\mathbf{a}}+\vec{\mathbf{c}}\right)\left(\vec{\mathbf{b}}+\vec{\mathbf{c}}\right)^T \right) \\
&= Ex\left(\vec{\mathbf{a}}\ \vec{\mathbf{b}}^T + \vec{\mathbf{a}}\vec{\mathbf{c}}^T + \vec{\mathbf{c}}\vec{\mathbf{b}}^T + \vec{\mathbf{c}}\vec{\mathbf{c}}^T\right) \\
&= Cov(\vec{\mathbf{a}},\vec{\mathbf{b}}) + Cov(\vec{\mathbf{a}},\vec{\mathbf{c}}) + Cov(\vec{\mathbf{b}},\vec{\mathbf{c}}) + Cov(\vec{\mathbf{c}},\mathbf{c}) \\
&= Cov(\vec{\mathbf{c}},\mathbf{c}) \ \text{ if } \vec{\mathbf{a}},\mathbf{b},\vec{\mathbf{c}} \text{ are independent}
\end{aligned}
$$

Similarly,

$$Cov(\mathbf{M}_a\vec{\mathbf{a}}+\mathbf{M}_b\vec{\mathbf{b}}) = \mathbf{M}_a\mathbf{C}_a\mathbf{M}_a^T + \mathbf{M}_b\mathbf{C}_b\mathbf{M}_b^T + \mathbf{M}_a\mathbf{C}_{ab}\mathbf{M}_b^T + \mathbf{M}_b\mathbf{C}_{ba}\mathbf{M}_b^T$$

where $\mathbf{C}_{ab} = \mathbf{C}_{ba}^T = Cov(\vec{\mathbf{a}},\vec{\mathbf{b}})$. If $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are independent, then $\mathbf{C}_{ab} = \mathbf{C}_{ba}^T = \mathbf{0}$.

Now form a new random variable $\vec{\mathbf{z}} = \vec{\mathbf{x}} - \vec{\mathbf{y}}$ and compute

$$\mathbf{C}_{ZZ} = Cov(\vec{\mathbf{z}}) = Ex\left((\vec{\mathbf{x}}-\vec{\mathbf{y}})\bullet(\vec{\mathbf{x}}-\vec{\mathbf{y}})^T\right) = \mathbf{C}_{XX} + \mathbf{C}_{YY} + \mathbf{C}_{XY} + \mathbf{C}_{XY}^T$$

Now define one more random variable

$$\vec{\mathbf{w}} = \begin{bmatrix} \vec{\mathbf{x}} \\ \vec{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{x}} \\ \vec{\mathbf{x}} - \vec{\mathbf{y}} \end{bmatrix}$$

and compute the matrix corresponding to $Cov(\vec{\mathbf{w}})$ in terms of $\mathbf{C}_{XX}$, $\mathbf{C}_{XY}$ and $\mathbf{C}_{YY}$. Now recall the following rule for conditional distributions. If

$$\begin{bmatrix} \vec{\mathbf{a}} \\ \vec{\mathbf{b}} \end{bmatrix} \sim N\left(\begin{bmatrix} \vec{\mu}_a \\ \mu_b \end{bmatrix}, \begin{bmatrix} \mathbf{C}_{aa} & \mathbf{C}_{ab} \\ \mathbf{C}_{ba} & \mathbf{C}_{bb} \end{bmatrix} \right)$$

then

$$
\begin{aligned}
Ex(\vec{\mathbf{a}} \,|\, \vec{\mathbf{b}} = \vec{\mathbf{b}}_0) &= \vec{\mu}_a + \mathbf{C}_{ab}\mathbf{C}_{bb}^{-1}(\vec{\mathbf{b}}_0 - \vec{\mu}_b) \\
Cov(\vec{\mathbf{a}} \,|\, \vec{\mathbf{b}} = \vec{\mathbf{b}}_0) &= \mathbf{C}_{aa} - \mathbf{C}_{ab}\mathbf{C}_{bb}^{-1}\mathbf{C}_{ba}
\end{aligned}
$$

Use this relationship to compute $Cov(\vec{\varepsilon}_{tool} \,|\, \vec{\mathbf{z}} = \vec{\mathbf{0}})$. Express your answer in terms of $\mathbf{C}_D$, $\mathbf{C}_E$, and $\mathbf{C}_1,\mathbf{C}_2,\mathbf{C}_3,\mathbf{C}_{rob}$. You are encouraged to define and use intermediate variables for things that are constant. For example, if $\vec{\varepsilon}_A = \mathbf{R}_1 sk(\vec{\mathbf{a}})\vec{\alpha} + \mathbf{R}_2\vec{\varepsilon} = [\mathbf{R}_1 sk(\vec{\mathbf{a}}),\mathbf{R}_2]\vec{\eta}$, you can write this as $\vec{\varepsilon}_A = \mathbf{M}_A\vec{\eta}$ where $\mathbf{M}_A$=$[\mathbf{R}_1 sk(\vec{\mathbf{a}}),\mathbf{R}_2]$. So $Cov(\vec{\varepsilon}_A) = \mathbf{M}_A Cov(\vec{\varepsilon})\mathbf{M}_A^T$.

Note: Although we don't ask you to work out the details here, the conditional expectation formulas above can provide a principled way to combine the two estimates for $\vec{\mathbf{p}}_{tool}$, based on the relative uncertainties of the two paths.

Simplified and slightly easier version (15 points)

Develop two formulas for the actual position of the tool tip relative to the tracker:

$$\vec{\mathbf{p}}^*_{1t} = \vec{\mathbf{p}}_{1t} + \vec{\varepsilon}_{1t} = \text{ a formula based on } \mathbf{F}_1$$

$$\vec{\mathbf{p}}^*_{2t} = \vec{\mathbf{p}}_{2t} + \vec{\varepsilon}_{2t} = \text{ a formula based on } \mathbf{F}_2$$

In this case, $\vec{\varepsilon}_{1t}$ and $\vec{\varepsilon}_{2t}$ are independent zero-mean variables: $\vec{\varepsilon}_{1t} \sim N(\vec{\mathbf{0}},\mathbf{C}_{1t})$, $\vec{\varepsilon}_{2t} \sim N(\vec{\mathbf{0}},\mathbf{C}_{2t})$, and $Cov(\vec{\varepsilon}_{1t},\vec{\varepsilon}_{2t}) = \vec{\mathbf{0}}$.

Develop formulas for $\mathbf{C}_{1t}$ and $\mathbf{C}_{2t}$. Then develop a formula to estimate the covariance associated with the uncertainty after combining the two estimates for the position of the tool tip relative to the tracker.