Homework Assignment 1 – 600.455/655 Fall 2024

Name	Name
Email	Email
Other contact information (optional)	Other contact information (optional)
Signature (required) I/We have followed the rules in completing this assignment ______________________________	Signature (required) I/We have followed the rules in completing this assignment ______________________________

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone** or in **teams of two** and are not to discuss the problems with anyone other than the TAs or the instructor.
3. It is otherwise open book, notes, and web. But you should cite any references you consult.
4. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
5. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
6. Submit the assignment on Gradescope as a neat and legible PDF file. We will not insist on typesetting your answers, but we must be able to read them. We will not go to extraordinary lengths to decipher what you write. If the graders cannot make out an answer, the score will be 0.
7. Sign and hand in this page as the first sheet of your assignment. If you work with a partner, then you both should sign the sheet, but you should only submit one PDF file for both of you. Indicate clearly who it is from.
8. This assignment has more than 100 points, but the most that will be applied to your grade is 100.

Problem Scenario: Stereotactic system

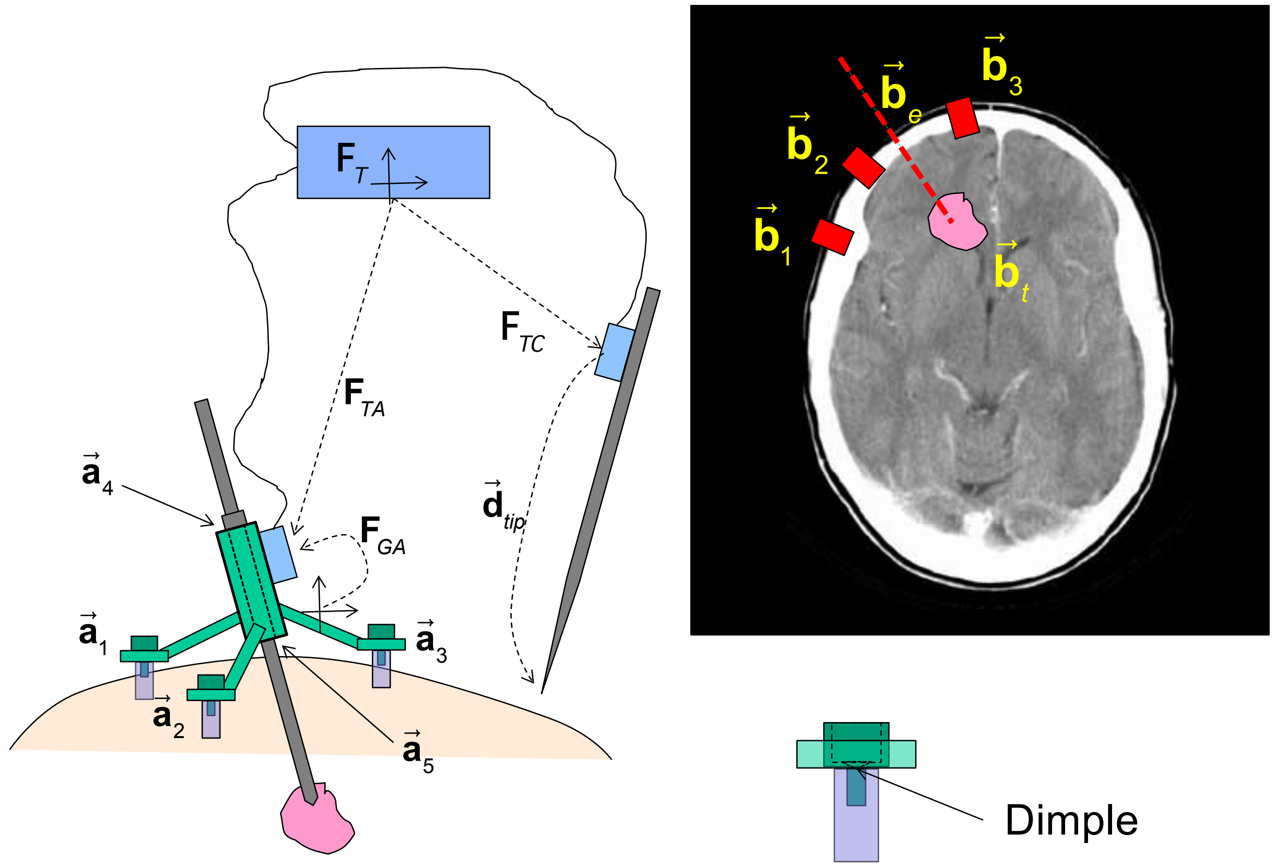

Fig. 1

Consider the stereotactic system shown in Fig. 1, which is based loosely on work from Vanderbilt University (e.g., Henderson JM, Holloway KL, Gaede SE, Rosenow JM: "The application accuracy of a skull-mounted trajectory guide for image-guided functional neurosurgery". *Computer Aided Surgery* 2004;9:155–160). The work flow for this system is as follows:

1. Three small fiducial devices are screwed into the patient's skull. Each fiducial device has a threaded hole in its top.
2. A 3D volumetric scan (CT or MRI) is taken of the patient's head. For simplicity, we will assume CT. For the purpose of this problem, you can assume that all CT coordinates are expressed in mm.
3. The CT coordinates $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2$, and $\vec{\mathbf{b}}_3$ of the tops (outer ends) of the threaded holes in the fiducials are determined by image processing.
4. A CT coordinates of desired entry point $\vec{\mathbf{b}}_e$ and target point $\vec{\mathbf{b}}_t$ for a stereotactic needle insertion are determined by the surgeon, using planning software.
5. A custom needle guide is fabricated. This guide has three holes located at points $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2$, and $\vec{\mathbf{a}}_3$ in a local coordinate system associated with the guide. The guide contains a 25 mm long tube with

an entry point at position $\vec{\mathbf{a}}_4$ and exit point at position $\vec{\mathbf{a}}_5$ relative to the guide coordinate system $\mathbf{F}_G$. The guide is constructed so that when it is attached to the fiducials, the guide tube axis will pass through the planned entry and target points on the patient's head. Furthermore, the design of the guide ensures that point $\vec{\mathbf{a}}_4$ will be at a fixed distance d_t from the target. The coordinates of all of the $\vec{\mathbf{a}}_k$ are expressed in mm.

6. In surgery, the guide is placed so that the holes in the guide are lined up with the corresponding fiducials and secured with three small screws. The tube is used to guide a drill for creating a burr hole at the entry point. Then, a needle with a stop at distance d_t from the tip is passed through the tube to hit the planned target.

[The next part is a bit artificial, but is here for the assignment]: The guide is also constructed so that a navigational marker is placed at a pose $\mathbf{F}_{GA} = [\mathbf{R}_{GA}, \vec{\mathbf{p}}_{GA}]$ relative to the guide coordinate system. A navigational tracking system is able to track the position and orientation $\mathbf{F}_{TA} = [\mathbf{R}_{TA}, \vec{\mathbf{p}}_{TA}]$ of this marker relative to the base unit of the tracking system. It is also able to track the position and orientation $\mathbf{F}_{TC} = [\mathbf{R}_{TC}, \vec{\mathbf{p}}_{TC}]$ of a second marker attached to a pointer device. The tip of this pointer is located at a point $\vec{\mathbf{d}}_{tip}$ relative to the tracker marker on the pointer, so that the tip of the pointer is at position $\mathbf{F}_{TC} \bullet \vec{\mathbf{d}}_{tip}$ relative to the tracker base unit.

Questions

1. Suppose that a needle has a stopper placed on its shaft at distance d_t from the needle tip. Then when the needle is inserted into the tube up to the stopper, the tip will be at a target point along the tube axis at a distance d_t from the point $\vec{\mathbf{a}}_4$. Give an expression for the 3D location $\vec{\mathbf{a}}_{tip}$ of the needle tip after insertion in the coordinate system of the needle guide (*i.e.,* in terms of $\vec{\mathbf{a}}_4$ and $\vec{\mathbf{a}}_5$).

2. Suppose that there are some manufacturing errors in the needle guide, so that the actual positions of the ends of the guide tube are

$$\vec{\mathbf{a}}_4^* = \vec{\mathbf{a}}_4 + \vec{\varepsilon}_4$$
$$\vec{\mathbf{a}}_5^* = \vec{\mathbf{a}}_5 + \vec{\varepsilon}_5$$

where $\vec{\varepsilon}_4$ and $\vec{\varepsilon}_5$ are small. The actual position of the inserted needle will then differ from the computed position by $\vec{\varepsilon}_{tip} = \vec{\mathbf{a}}_{tip}^* - \vec{\mathbf{a}}_{tip}$. Give an expression for $\vec{\varepsilon}_{tip}$. **Note:** Here, I am looking for a conservative linearized approximation for $\vec{\varepsilon}_{tip}$, i.e. one for which $\left\|\vec{\mathbf{a}}_{tip}^* - \vec{\mathbf{a}}_{tip}\right\| \le \left\|\vec{\varepsilon}_{tip}\right\|$.

3. Suppose that the positions of $\vec{\mathbf{a}}_4$ and $\vec{\mathbf{a}}_5$ are accurate, but the inner diameter d_{tube} of the tube is larger than the diameter d_{needle} of the needle, so that $d_{tube} - d_{needle} = \delta$. Give an expression giving linearized bounds on $\left\| \vec{\varepsilon}_{tip} \right\|_2$ in terms of δ.

4. Suppose that you can adjust the length of the guide tube. How long does the tube need to be in order to guarantee that $\left\| \vec{\varepsilon}_{tip} \right\|_2 \le \varepsilon_{\max}$, where we know that $\delta \le 2\varepsilon_{\max}$.

5. Give expression for computing the positions $\vec{\mathbf{p}}_{ptr}$ of the pointer tip relative to the coordinate systems of the tracking marker attached to the guide. Also, give an expression for the position $\vec{\mathbf{p}}_{Gp}$ of the pointer tip relative to the coordinate system of the guide. Give these first in terms of the **F** values and then expand it out into an expression involving the corresponding **R** and $\vec{\mathbf{p}}$ values.

6. Suppose that we have computed a transformation $\mathbf{F}_{reg}$ such that any point $\vec{\mathbf{x}}_G$ in guide coordinates corresponds to a point $\vec{\mathbf{x}}_{CT} = \mathbf{F}_{reg} \bullet \vec{\mathbf{x}}_G$ in CT coordinates for computing the CT coordinates $\vec{\mathbf{b}}_{ptr}$ corresponding to $\vec{\mathbf{p}}_{ptr}$, assuming that the navigational tracker is perfectly accurate. Give an expression for $\vec{\mathbf{b}}_{ptr}$ in terms of the $\mathbf{R}$'s and $\vec{\mathbf{p}}$'s known to the system.

7. Suppose now that the navigational tracker is not perfectly accurate, so that the actual values for $\mathbf{F}_{TA}$ and $\mathbf{F}_{TC}$ are $\mathbf{F}_{TA}^{*} = \mathbf{F}_{TA} \bullet \Delta\mathbf{F}_{TA}$ and $\mathbf{F}_{TC}^{*} = \mathbf{F}_{TC} \bullet \Delta\mathbf{F}_{TC}$. What is the error $\Delta\vec{\mathbf{p}}_{ptr}$ between the computed and actual values of $\vec{\mathbf{p}}_{ptr}$. Express this in terms first of the $\Delta\mathbf{F}$'s and then in terms of the $\Delta\mathbf{R}$'s and $\Delta\vec{\mathbf{p}}$'s. **Hint:** first compute expressions for $\Delta\mathbf{F}_{AC}$, where $\mathbf{F}_{AC}{*} = \mathbf{F}_{AC}\Delta\mathbf{F}_{AC}$. You may use $\mathbf{F}_{AC} = \left[\mathbf{R}_{AC}, \vec{\mathbf{p}}_{AC}\right]$ in your answer provided that you first show the formula for computing it in terms of the $\mathbf{F}$'s.

8. Assume now that we know that the $\Delta\mathbf{R}$'s and $\Delta\vec{\mathbf{p}}$'s are "small", so that $\Delta\mathbf{F}_i \approx [\mathbf{I} + sk(\vec{\alpha}_i), \vec{\varepsilon}_i]$ for each of the measured frames $\mathbf{F}_i$. Compute a simplified expression for $\Delta\mathbf{p}_{ptr}$. Please express your answer as a sum of linearized products $\mathbf{M}_{TC\alpha}\vec{\alpha}_{TC} + \mathbf{M}_{TC\varepsilon}\vec{\varepsilon}_{TC} + \mathbf{M}_{TA\alpha}\vec{\alpha}_{TA} + \mathbf{M}_{TA\varepsilon}\vec{\varepsilon}_{TA}$ where each of the **M**'s can be expressed as a 3x3 matrix of quantities known to the computer. **Hint:** Some of these M's may involve skew matrices.

9. Suppose now that, in addition to the errors above, there is a small error in the value of $\vec{\mathbf{d}}_{tip}$, so that the actual value is $\vec{\mathbf{d}}_{tip}^{\,*} = \vec{\mathbf{d}}_{tip} + \vec{\varepsilon}_{tip}$. Give an approximate expression for $\Delta\mathbf{p}_{ptr}$. Express your answer as a sum of linearized products, as above. (You may use appropriate approximations, assuming that the errors are small).

10. Suppose now that, in addition to the errors above, there is some small error in the attachment of the marker onto the guide, so that $\mathbf{F}_{GA}^{\,*} = \mathbf{F}_{GA} \bullet \Delta\mathbf{F}_{GA}$, where $\Delta\mathbf{F}_{GA} \approx [\mathbf{I} + sk(\vec{\alpha}_{GA}), \vec{\varepsilon}_{GA}]$. Give an expression for $\Delta\mathbf{p}_{Gp}$. (You may use appropriate approximations, assuming that the errors are small). Express your answer in terms of $\vec{\varepsilon}_{ptr} = \Delta\vec{\mathbf{p}}_{ptr}$, $\vec{\alpha}_{GA}$ and $\vec{\varepsilon}_{GA}$. **Note:** Since we want answer in terms of $\Delta\vec{\mathbf{p}}_{ptr}$, this answer doesn't depend on earlier answers for computing $\Delta\vec{\mathbf{p}}_{ptr}$.

11. Suppose that the screws used to secure the guide to the fiducials are constructed so that they have a small dimple in their tops so that the pointer can be placed so that it corresponds exactly to the points $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2$, and $\vec{\mathbf{a}}_3$ relative to the guide. Explain how to perform the registration process to find $\mathbf{F}_{reg} = [\mathbf{R}_{reg}, \vec{\mathbf{p}}_{reg}]$. Give sufficient mathematical and work-flow detail so that it is clear how this computation would be implemented. You do not need to recite individual steps of known algorithms given in class, but you should identify which algorithms are used.

12. Suppose, now, that we have bought a new tracking system, which is very accurate. Unfortunately, the marker has not been attached carefully to the guide and pointer, and (in any case) the pointer has been slightly bent. Describe a procedure for determining $\mathbf{F}_{reg}$ and $\vec{\mathbf{d}}_{tip}$. Give sufficient detail so that it is clear what measurements need to be taken, how they will be taken, and what methods will be used to compute $\mathbf{F}_{GA}$ and $\vec{\mathbf{d}}_{tip}$.

13. This question concerns how you would design the guide, given the processed CT image. Given the values of $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3, \vec{\mathbf{b}}_t$, and $\vec{\mathbf{b}}_e$ from the planning system explain how to compute values

$$\vec{\mathbf{a}}_1 = [x_{1,} y_1, 0]^T, \quad \vec{\mathbf{a}}_2 = [x_{2,} y_2, 0]^T, \quad \vec{\mathbf{a}}_3 = [x_{3,} y_3, 0]^T$$

$$\vec{\mathbf{a}}_4 = [x_{4,} y_4, 35]^T, \text{ and } \vec{\mathbf{a}}_5 = [x_{5,} y_5, z_5]^T$$

so that the inserted needle will hit the target and will pass through the point $[0,0,0]^T$ in the implant coordinate system, with $\left\|\vec{\mathbf{a}}_5 - \vec{\mathbf{a}}_4\right\| = 25$. Also, specify the distance d_t that the stop needs to be put from the end of the needle. Give formulas in sufficient detail so that it is clear how this process will be implemented.

14. Suppose that the navigational tracker system has both a systematic and a “random” component, so that $\mathbf{F}_k^* = \mathbf{F}_k \Delta \mathbf{F}_k = \mathbf{F}_k \Delta \mathbf{F}_{sys} \Delta \mathbf{F}_{rk}$ for any measurement taken. Suppose that the guide tube apparatus and pointer have been manufactured or measured to very high accuracy so

that all the $\vec{\mathbf{a}}_i$, $\mathbf{F}_{GA}$, and $\vec{\mathbf{d}}_{tip}$ are known with negligible error. Explain how you can calibrate this system to determine $\Delta\mathbf{F}_{sys}$. Here, you can

$$\Delta\mathbf{F}_{sys} \approx [\mathbf{I} + sk(\vec{\alpha}_{sys}), \vec{\varepsilon}_{sys}]$$

use the notation:

$$\Delta\mathbf{F}_{rk} \approx [\mathbf{I} + sk(\vec{\alpha}_{rk}), \vec{\varepsilon}_{rk}]$$

$$\Delta\mathbf{F}_{sys}\Delta\mathbf{F}_{rk} \approx [\mathbf{I} + sk(\vec{\alpha}_{sys} + \vec{\alpha}_{rk}), \vec{\varepsilon}_{sys} + \vec{\varepsilon}_{rk}]$$

Explain your method in sufficient workflow and mathematical detail so that it will be clear how one of ordinary skill can implement it.