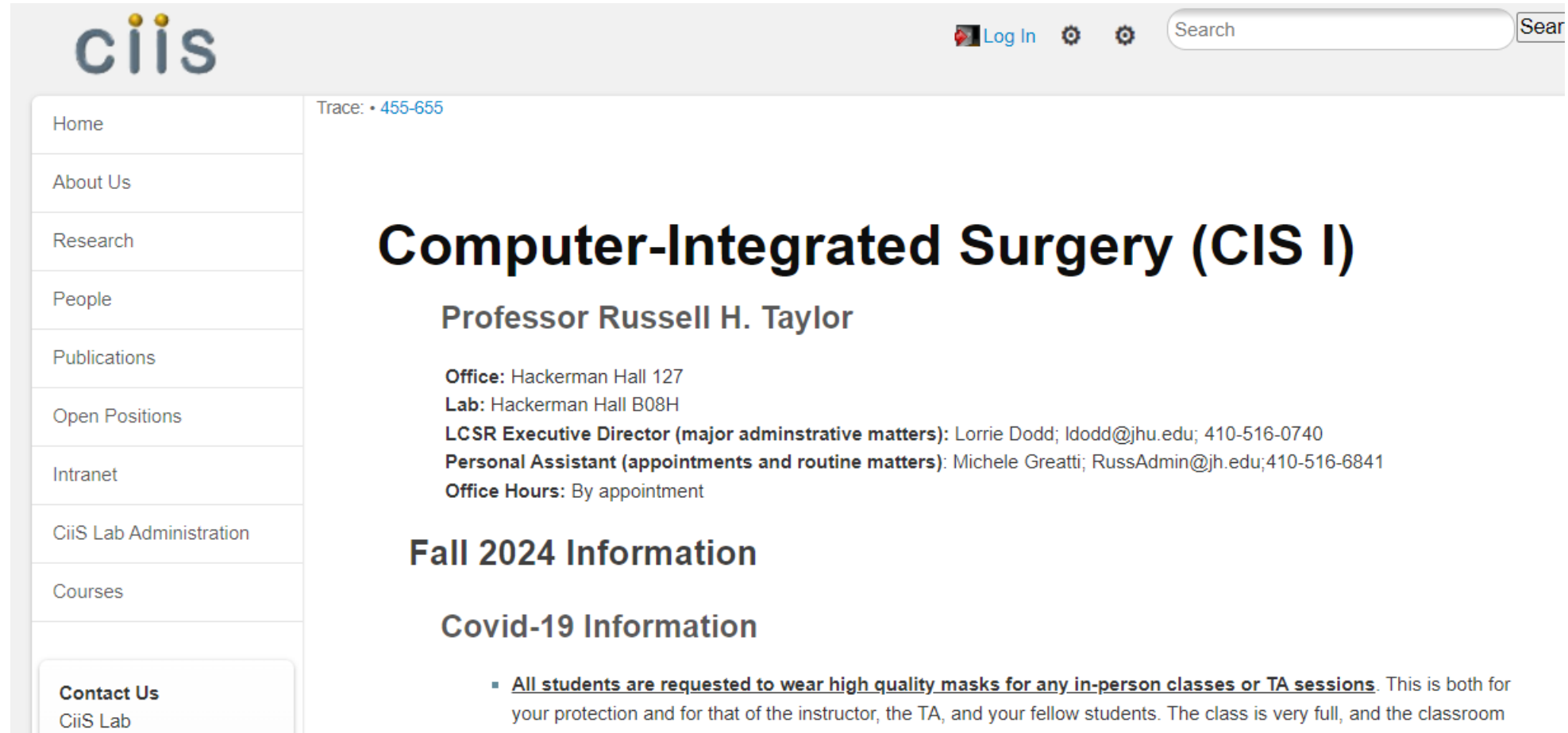


Course Wiki: <https://ciis.lcsr.jhu.edu/doku.php?id=courses:455-655:455-655>



The screenshot shows the CiIS website interface. At the top left is the CiIS logo. To the right are links for 'Log In', two gear icons, and a search bar with the text 'Search' and a 'Search' button. A breadcrumb trail reads 'Trace: • 455-655'. On the left is a vertical navigation menu with links: Home, About Us, Research, People, Publications, Open Positions, Intranet, CiiS Lab Administration, Courses, and a 'Contact Us' button above 'CiiS Lab'. The main content area features the title 'Computer-Integrated Surgery (CIS I)' in large bold font, followed by 'Professor Russell H. Taylor'. Below this is contact information: 'Office: Hackerman Hall 127', 'Lab: Hackerman Hall B08H', 'LCSR Executive Director (major administrative matters): Lorrie Dodd; ldodd@jhu.edu; 410-516-0740', 'Personal Assistant (appointments and routine matters): Michele Greatti; RussAdmin@jh.edu; 410-516-6841', and 'Office Hours: By appointment'. Further down are sections for 'Fall 2024 Information' and 'Covid-19 Information'. The Covid-19 section contains a bullet point: 'All students are requested to wear high quality masks for any in-person classes or TA sessions. This is both for your protection and for that of the instructor, the TA, and your fellow students. The class is very full, and the classroom...'

Course Wiki: <https://ciis.lcsr.jhu.edu/doku.php?id=courses:455-655:455-655>

Section and TA office hour times and location

Tatiana Kashtanova

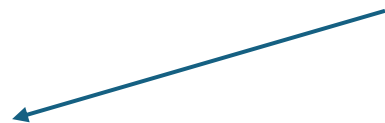
- Email: tkashta1@jhu.edu
- Office Hours (starting September 9)
 - Time: Monday, 10 am - 11 am
 - Location: Zoom
 - <https://JHUBlueJays.zoom.us/j/99052831625?pwd=AFMJsShozAjWfMKpYv47JqLqjz3hAN.1>
 - Meeting ID: 990 5283 1625
 - Passcode: 096011
- Discussion sections will be announced on Piazza

Meetings with a TA (subject to change):

- 9-Sep: Math
- 16-Sep: HW1
- 23-Sep: HW2
- 30-Sep: PA general, PA1
- TBD: PA Intro
- 7-Oct: HW1 Review
- 14-Oct: PA2
- 21-Oct: PA3
- 28-Oct: HW3
- 4-Nov: HW2 Review
- 11-Nov: HW4, PA4
- 18-Nov: HW3 Review
- 25-Nov: N/A (Thanksgiving)
- 2-Dec: PA5



I will upload my slides here



Course Wiki: <https://ciis.lcsr.jhu.edu/doku.php?id=courses:455-655:455-655>

Organizational Information

- [Fall 2024 Schedule](#)

Schedule: <https://ciis.lcsr.jhu.edu/doku.php?id=courses:455-655:2024:fall-2024-schedule>

CIS I (601.455/655) Fall 2024 Schedule

Note: This page is subject to change

- Lecture slides
- Supplementary material
- Assignments (hand-out & due dates)

Piazza: <https://piazza.com/jhu/fall2024/601455655>

- Course announcements
- Find a partner
- Q & A
- Typos / errors
- Private message to the professor & TA



Piazza App

Partner:

- Can be changed between assignments
- Do not abandon anybody before the due date!

Special circumstances / Accommodations:

- Sport / science competitions
- Conferences
- Marriage
- Scheduled health-related procedures
- Etc. expected



Tell us in advance!

- Sickness
- Fire
- Etc. unexpected



Do not come to the class. Tell us!

- Disability



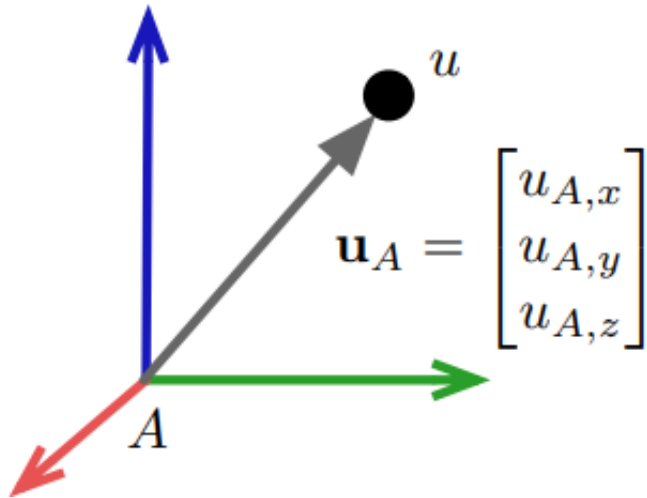
Contact “Student Disability Services”!
They will contact us.

CIS Math Tutorial

References:

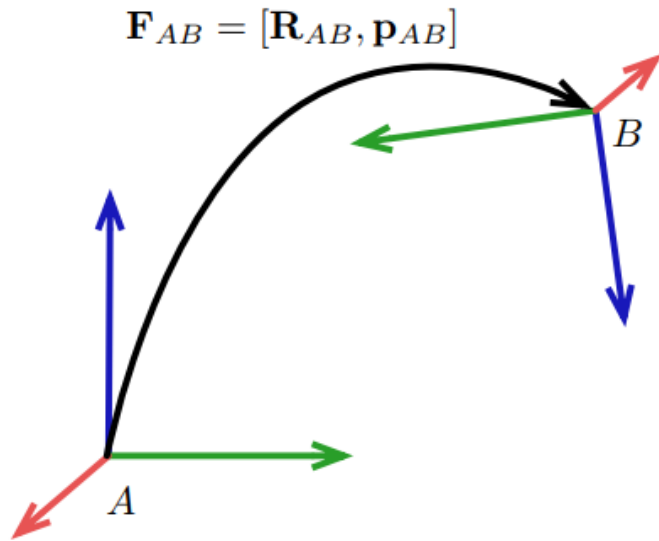
Benjamin D. Killeen (2022). Frame Transformations in Computer Integrated Surgery: A Graphical Introduction

Russell H. Taylor (2024). 600.455/655 Lecture Notes: Basic Mathematical Methods for CIS



A point u as measured in frame A

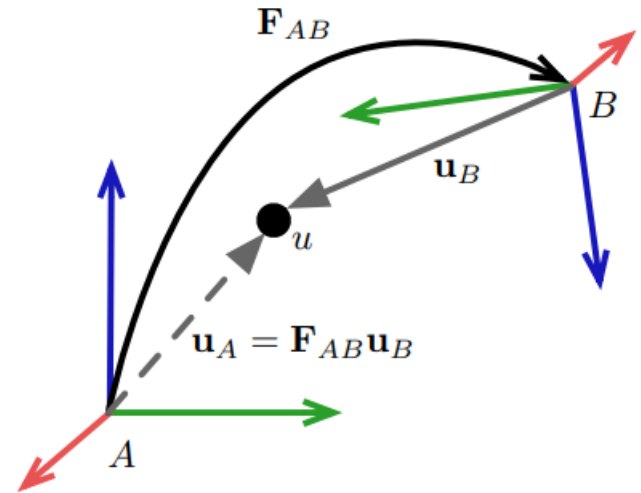
- A **frame** is a basis for numerical measurements of object locations, orientations, or poses
- A **point** is a singular location in space
- Vector \mathbf{u}_A defines the position of u relative to frame A



Frame transformation “A from B”

$$F_{AB} = [R_{AB}, p_{AB}]$$

A measurement of frame **B** pose (rotation + translation) with respect to frame **A**



$$u_A = F_{AB} u_B = [R_{AB}, p_{AB}]u_B = R_{AB} u_B + p_{AB}$$

- u_B is the measurement of u in frame **B**
- F_{AB} known

The right-hand subscript of the transform should match the subscript of the point $F_{AB}F_{BC} = F_{AC}$

Inverse Transformations

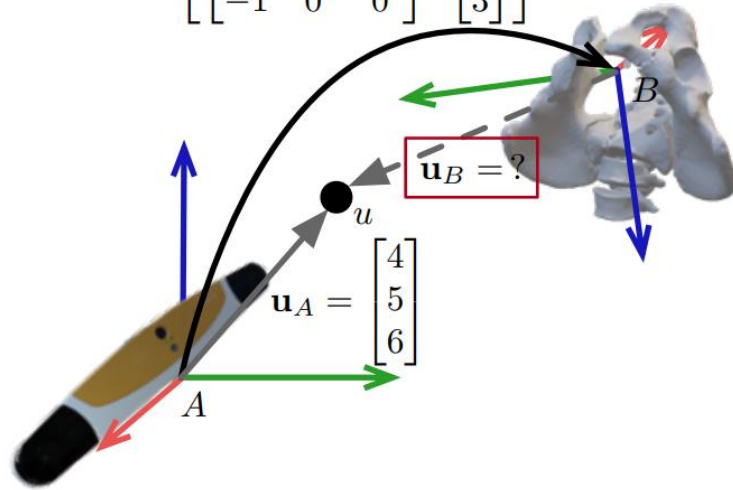
$$\mathbf{F}_{AB}^{-1} = [\mathbf{R}_{AB}^{-1}, -\mathbf{R}_{AB}^{-1} \mathbf{p}_{AB}]$$

$$\mathbf{F}_{AB}^{-1} = \mathbf{F}_{BA}$$

Rotation matrices are orthonormal:

$$\mathbf{R}^{-1} = \mathbf{R}^T$$

$$\mathbf{F}_{AB} = [\mathbf{R}_{AB}, \mathbf{p}_{AB}] = \left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right]$$



$$\begin{aligned} \mathbf{u}_B &= \mathbf{F}_{BA} \mathbf{u}_A = \mathbf{F}_{AB}^{-1} \mathbf{u}_A \\ &= [\mathbf{R}_{AB}^{-1}, -\mathbf{R}_{AB}^{-1} \mathbf{p}_{AB}] \mathbf{u}_A \\ &= \mathbf{R}_{AB}^{-1} \mathbf{u}_A - \mathbf{R}_{AB}^{-1} \mathbf{p}_{AB} \end{aligned}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 4 \\ -5 \end{bmatrix} - \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix}$$

$$0*4 + 0*5 - 1*6 = -6$$

$$1*4 + 0*5 + 0*6 = 4$$

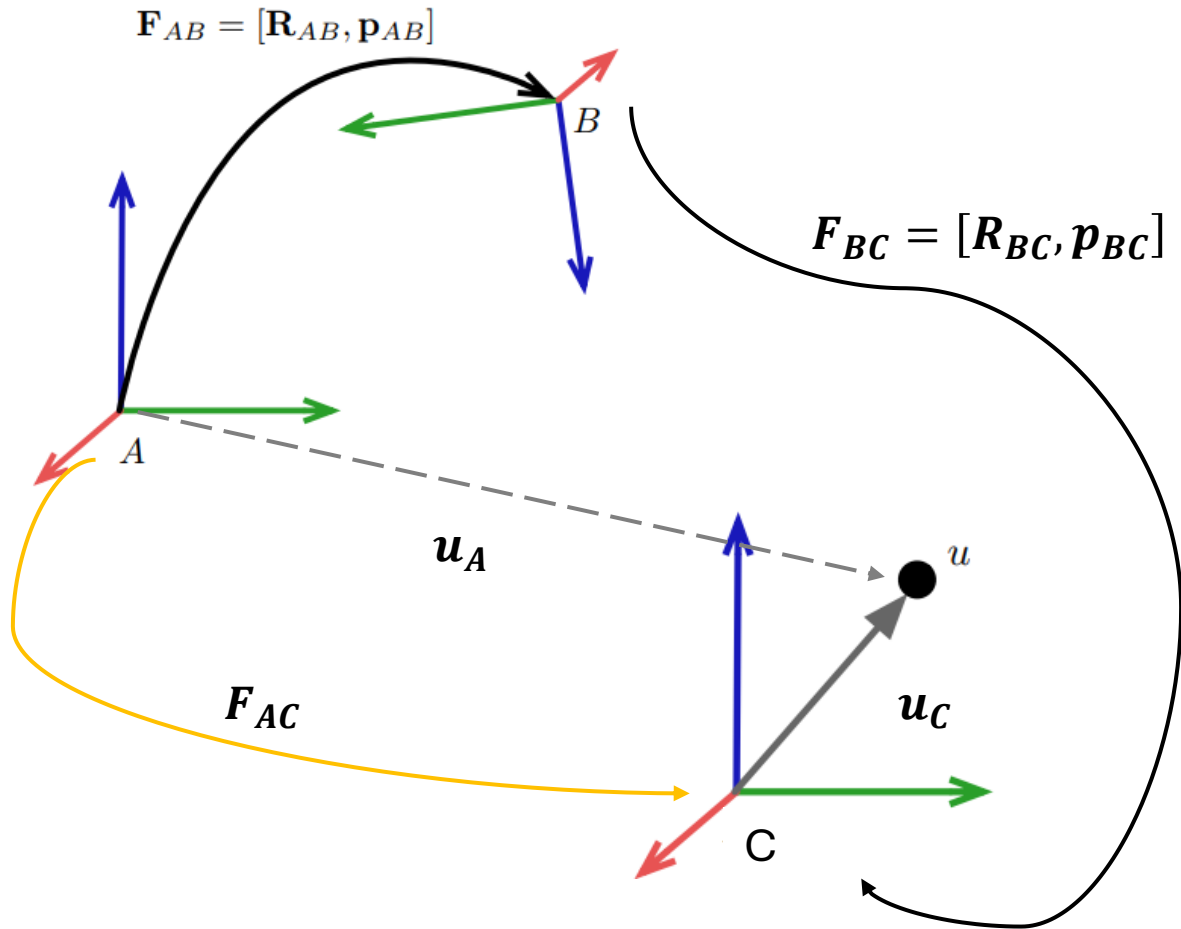
$$0*4 - 1*5 + 0*6 = -5$$

$$-6 + 3 = -3$$

$$4 - 1 = 3$$

$$-5 + 2 = -3$$

Frame Composition



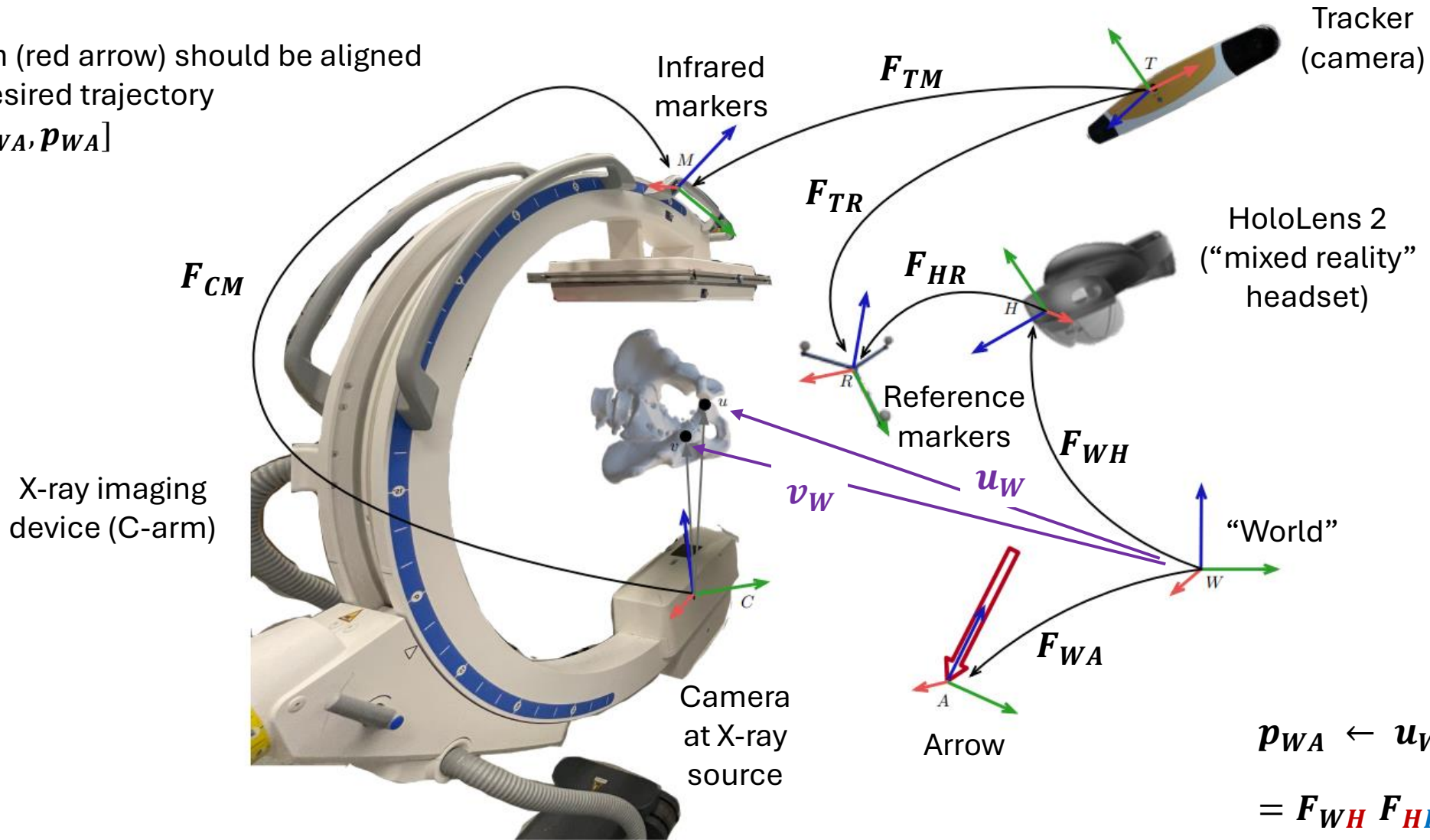
$$\mathbf{F}_{AC} = \mathbf{F}_{AB}\mathbf{F}_{BC} = [\mathbf{R}_{AB}\mathbf{R}_{BC}, \mathbf{R}_{AB}\mathbf{p}_{BC} + \mathbf{p}_{AB}]$$

$$\begin{aligned} \mathbf{u}_A &= \mathbf{F}_{AB}\mathbf{F}_{BC}\mathbf{u}_C = \mathbf{F}_{AB}[\mathbf{R}_{BC}, \mathbf{p}_{BC}]\mathbf{u}_C \\ &= \mathbf{F}_{AB}(\mathbf{R}_{BC}\mathbf{u}_C + \mathbf{p}_{BC}) = [\mathbf{R}_{AB}, \mathbf{p}_{AB}](\mathbf{R}_{BC}\mathbf{u}_C + \mathbf{p}_{BC}) \\ &= \mathbf{R}_{AB}(\mathbf{R}_{BC}\mathbf{u}_C + \mathbf{p}_{BC}) + \mathbf{p}_{AB} \\ &= (\mathbf{R}_{AB}\mathbf{R}_{BC})\mathbf{u}_C + (\mathbf{R}_{AB}\mathbf{p}_{BC} + \mathbf{p}_{AB}) \\ &= [\mathbf{R}_{AB}\mathbf{R}_{BC} \quad \mathbf{R}_{AB}\mathbf{p}_{BC} + \mathbf{p}_{AB}]\mathbf{u}_C \\ &= [\mathbf{R}_{AB}\mathbf{R}_{BC}, \mathbf{R}_{AB}\mathbf{p}_{BC} + \mathbf{p}_{AB}]\mathbf{u}_C \end{aligned}$$

Insert a rigid metal rod (K-wire) into pelvis from u to v

A hologram (red arrow) should be aligned with the desired trajectory

$$F_{WA} = [R_{WA}, p_{WA}]$$



X-ray imaging device (C-arm)

Infrared markers

Tracker (camera)

HoloLens 2 ("mixed reality" headset)

Reference markers

"World"

Camera at X-ray source

Arrow

$$p_{WA} \leftarrow u_W$$

$$= F_{WH} F_{HR} F_{TR}^{-1} F_{TM}^{-1} F_{CM}^{-1} u_C$$

$$= F_{WC} u_C$$

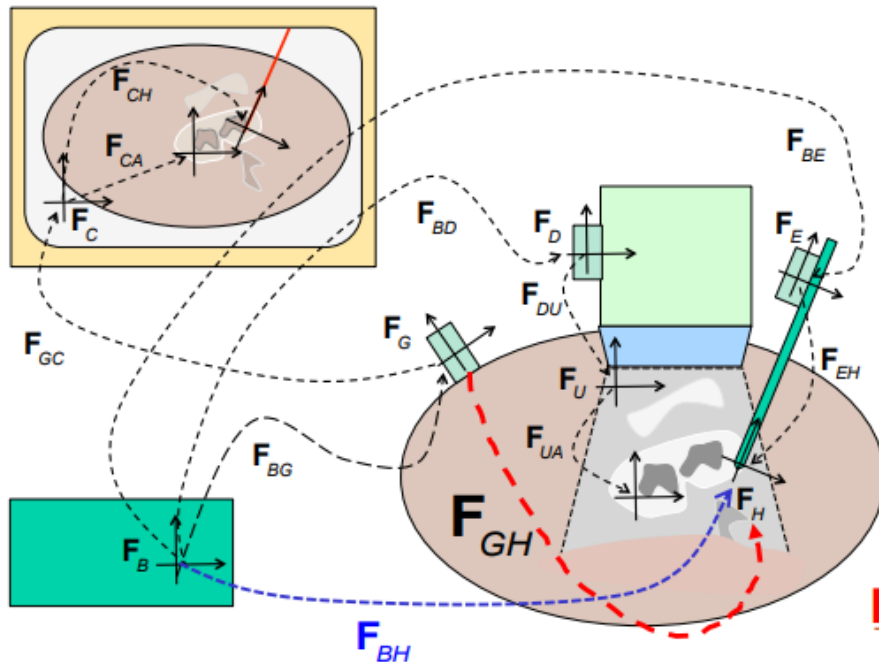
$$\mathbf{p}_{WA} \leftarrow \mathbf{u}_W$$

$$\begin{aligned}
 &= \mathbf{F}_{WH} \mathbf{F}_{HR} \mathbf{F}_{TR}^{-1} \mathbf{F}_{TM} \mathbf{F}_{CM}^{-1} \mathbf{u}_C \\
 &= [\mathbf{R}_{WH}, \mathbf{p}_{WH}] [\mathbf{R}_{HR}, \mathbf{p}_{HR}] [\mathbf{R}_{TR}^{-1}, -\mathbf{R}_{TR}^{-1} \mathbf{p}_{TR}] [\mathbf{R}_{TM}, \mathbf{p}_{TM}] [\mathbf{R}_{CM}^{-1}, -\mathbf{R}_{CM}^{-1} \mathbf{p}_{CM}] \mathbf{u}_C \\
 &= \left[\begin{array}{l} \mathbf{R}_{WH} \mathbf{R}_{HR} \mathbf{R}_{TR}^{-1} \mathbf{R}_{TM} \mathbf{R}_{CM}^{-1} \\ -\mathbf{R}_{WH} \mathbf{R}_{HR} \mathbf{R}_{TR}^{-1} \mathbf{R}_{TM} \mathbf{R}_{CM}^{-1} \mathbf{p}_{CM} \\ +\mathbf{R}_{WH} \mathbf{R}_{HR} \mathbf{R}_{TR}^{-1} \mathbf{p}_{TM} \\ -\mathbf{R}_{WH} \mathbf{R}_{HR} \mathbf{R}_{TR}^{-1} \mathbf{p}_{TR} \\ +\mathbf{R}_{WH} \mathbf{p}_{HR} \\ +\mathbf{p}_{WH} \end{array} \right] \mathbf{u}_C \\
 &\equiv \mathbf{F}_{WC} \mathbf{u}_C \\
 &= \mathbf{R}_{WC} \mathbf{u}_C + \mathbf{p}_{WC}
 \end{aligned}$$

$$\mathbf{u}_W = \mathbf{F}_{WH} \mathbf{F}_{HR} \mathbf{F}_{TR}^{-1} \mathbf{F}_{TM} \mathbf{F}_{CM}^{-1} \mathbf{u}_C = \mathbf{F}_{WC} \mathbf{u}_C$$

$$\mathbf{v}_W = \mathbf{F}_{WC} \mathbf{v}_C$$

Similarly, we note $\mathbf{v}_W = \mathbf{F}_{WC} \mathbf{v}_C$.



$$\begin{aligned} \mathbf{F}_{GH} &= \mathbf{F}_{BG}^{-1} \mathbf{F}_{BE} \mathbf{F}_{EH} \\ &= \mathbf{F}_{BG}^{-1} \mathbf{F}_{BH} \end{aligned}$$

$$\mathbf{R}_{GH} = \mathbf{R}_{BG}^{-1} \mathbf{R}_{BE} \mathbf{R}_{EH}$$

$$\vec{\mathbf{p}}_{GH} = \mathbf{F}_{BG}^{-1} \vec{\mathbf{p}}_{BH} = \mathbf{F}_{BG}^{-1} \mathbf{F}_{BE} \vec{\mathbf{p}}_{EH}$$

$$\vec{\mathbf{p}}_{GH} = \mathbf{F}_{BG}^{-1} (\mathbf{R}_{BE} \vec{\mathbf{p}}_{EH} + \vec{\mathbf{p}}_{BE})$$

$$\mathbf{p}_{GH} = [\mathbf{R}_{BG}^{-1}, -\mathbf{R}_{BG}^{-1} \mathbf{p}_{BG}] (\mathbf{R}_{BE} \mathbf{p}_{EH} + \mathbf{p}_{BE})$$

$$\mathbf{p}_{GH} = \mathbf{R}_{BG}^{-1} (\mathbf{R}_{BE} \mathbf{p}_{EH} + \mathbf{p}_{BE}) - \mathbf{R}_{BG}^{-1} \mathbf{p}_{BG}$$

$$\vec{\mathbf{p}}_{GH} = \mathbf{R}_{BG}^{-1} (\mathbf{R}_{BE} \vec{\mathbf{p}}_{EH} + \vec{\mathbf{p}}_{BE} - \vec{\mathbf{p}}_{BG})$$

Give a formula for computing the pose \mathbf{F}_{GH} of the surgical tool coordinate system relative to the patient rigid body coordinate system \mathbf{F}_G ?

What are the components $\mathbf{F}_{GH} = [\mathbf{R}_{GH}, \vec{\mathbf{p}}_{GH}]$?



Forward and Inverse Frame Transformations

Forward

$$\mathbf{F} = [\mathbf{R}, \bar{\mathbf{p}}]$$

$$\begin{aligned}\bar{\mathbf{v}} &= \mathbf{F} \bullet \bar{\mathbf{b}} \\ &= [\mathbf{R}, \bar{\mathbf{p}}] \bullet \bar{\mathbf{b}} \\ &= \mathbf{R} \bullet \bar{\mathbf{b}} + \bar{\mathbf{p}}\end{aligned}$$

Inverse

$$\begin{aligned}\mathbf{F}^{-1} \bar{\mathbf{v}} &= \bar{\mathbf{b}} \\ \bar{\mathbf{b}} &= \mathbf{R}^{-1} \bullet (\bar{\mathbf{v}} - \bar{\mathbf{p}}) \\ &= \mathbf{R}^{-1} \bullet \bar{\mathbf{v}} - \mathbf{R}^{-1} \bullet \bar{\mathbf{p}}\end{aligned}$$

$$\mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1} \bullet \bar{\mathbf{p}}]$$

Composition

Assume $\mathbf{F}_1 = [\mathbf{R}_1, \bar{\mathbf{p}}_1]$, $\mathbf{F}_2 = [\mathbf{R}_2, \bar{\mathbf{p}}_2]$

Then

$$\begin{aligned}\mathbf{F}_1 \bullet \mathbf{F}_2 \bullet \bar{\mathbf{b}} &= \mathbf{F}_1 \bullet (\mathbf{F}_2 \bullet \bar{\mathbf{b}}) \\ &= \mathbf{F}_1 \bullet (\mathbf{R}_2 \bullet \bar{\mathbf{b}} + \bar{\mathbf{p}}_2) \\ &= [\mathbf{R}_1, \bar{\mathbf{p}}_1] \bullet (\mathbf{R}_2 \bullet \bar{\mathbf{b}} + \bar{\mathbf{p}}_2) \\ &= \mathbf{R}_1 \bullet (\mathbf{R}_2 \bullet \bar{\mathbf{b}} + \bar{\mathbf{p}}_2) + \bar{\mathbf{p}}_1 \\ &= \mathbf{R}_1 \bullet \mathbf{R}_2 \bullet \bar{\mathbf{b}} + \mathbf{R}_1 \bullet \bar{\mathbf{p}}_2 + \bar{\mathbf{p}}_1 \\ &= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \bullet \bar{\mathbf{p}}_2 + \bar{\mathbf{p}}_1] \bullet \bar{\mathbf{b}}\end{aligned}$$

So

$$\begin{aligned}\mathbf{F}_1 \bullet \mathbf{F}_2 &= [\mathbf{R}_1, \bar{\mathbf{p}}_1] \bullet [\mathbf{R}_2, \bar{\mathbf{p}}_2] \\ &= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \bullet \bar{\mathbf{p}}_2 + \bar{\mathbf{p}}_1]\end{aligned}$$

Vectors

$$\text{dot product: } \mathbf{a} = \vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = (v_x w_x + v_y w_y + v_z w_z) = \|\vec{\mathbf{v}}\| \|\vec{\mathbf{w}}\| \cos \theta$$

$$\text{cross product: } \vec{\mathbf{u}} = \vec{\mathbf{v}} \times \vec{\mathbf{w}} = \begin{bmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{bmatrix}, \|\mathbf{u}\| = \|\mathbf{v}\| \|\mathbf{w}\| \sin \theta$$

Matrix representation of cross product operator

Define

$$\hat{\mathbf{a}} \triangleq \text{skew}(\vec{\mathbf{a}}) \triangleq \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Then

$$\vec{\mathbf{a}} \times \vec{\mathbf{v}} = \text{skew}(\vec{\mathbf{a}}) \bullet \vec{\mathbf{v}}$$

Note that rotation doesn't affect inner products

$$(\mathbf{R} \bullet \vec{\mathbf{b}}) \bullet (\mathbf{R} \bullet \vec{\mathbf{c}}) = \vec{\mathbf{b}} \bullet \vec{\mathbf{c}}$$

or lengths of vectors

$$\|\mathbf{R} \bullet \vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\|$$

“Small” Frame Transformations

Represent a "small" pose shift consisting of a small rotation $\Delta\mathbf{R}$ followed by a small displacement $\Delta\vec{\mathbf{p}}$ as

$$\Delta\mathbf{F} = [\Delta\mathbf{R}, \Delta\vec{\mathbf{p}}]$$

Then

$$\Delta\mathbf{F} \bullet \vec{\mathbf{v}} = \Delta\mathbf{R} \bullet \vec{\mathbf{v}} + \Delta\vec{\mathbf{p}}$$



$$\begin{aligned} \Delta\mathbf{F} &= [\mathbf{I} + \mathbf{sk}(\boldsymbol{\alpha}), \boldsymbol{\varepsilon}] \\ \Delta\mathbf{F}^{-1} &= [\mathbf{I} - \mathbf{sk}(\boldsymbol{\alpha}), -\boldsymbol{\varepsilon}] \end{aligned}$$

Approximations to “Small” Frames

$$\Delta\mathbf{R}(\vec{\mathbf{a}}) \approx \mathbf{I} + \mathbf{skew}(\vec{\mathbf{a}})$$

$$\Delta\mathbf{R}(\vec{\mathbf{a}})^{-1} \approx \mathbf{I} - \mathbf{skew}(\vec{\mathbf{a}}) = \mathbf{I} + \mathbf{skew}(-\vec{\mathbf{a}})$$

Notational NOTE:

We often use $\vec{\alpha}$ to represent a vector of small angles and $\vec{\varepsilon}$ to represent a vector of small displacements

In using these approximations, we typically ignore second order terms. I.e.,

$$\vec{\alpha}_A \vec{\alpha}_B \approx \vec{0}, \vec{\alpha}_A \vec{\varepsilon}_B \approx \vec{0}, \vec{\varepsilon}_A \vec{\varepsilon}_B \approx \vec{0}, \text{ etc.}$$

Errors & sensitivity

Often, we do not have an accurate value for a transformation, so we need to model the error. We model this as a composition of a "nominal" frame and a small displacement

$$\mathbf{F}_{\text{actual}} = \mathbf{F}_{\text{nominal}} \bullet \Delta \mathbf{F}$$

Often, we will use the notation \mathbf{F}^* for $\mathbf{F}_{\text{actual}}$ and will just use \mathbf{F} for $\mathbf{F}_{\text{nominal}}$. Thus we may write something like

$$\mathbf{F}^* = \mathbf{F} \bullet \Delta \mathbf{F}$$

or (less often) $\mathbf{F}^* = \Delta \mathbf{F} \bullet \mathbf{F}$. We also use $\vec{\mathbf{v}}^* = \vec{\mathbf{v}} + \Delta \vec{\mathbf{v}}$, etc.

Thus, if we use the former form (error on the right), and have nominal relationship $\vec{\mathbf{v}} = \mathbf{F} \bullet \vec{\mathbf{b}}$, we get

$$\begin{aligned} \vec{\mathbf{v}}^* &= \mathbf{F}^* \bullet \vec{\mathbf{b}}^* \\ &= \mathbf{F} \bullet \Delta \mathbf{F} \bullet (\vec{\mathbf{b}} + \Delta \vec{\mathbf{b}}) = \mathbf{F} \bullet (\Delta \mathbf{R} \bullet \vec{\mathbf{b}} + \Delta \mathbf{R} \bullet \Delta \vec{\mathbf{b}} + \Delta \vec{\mathbf{p}}) \\ &\approx \mathbf{R} \bullet ((\mathbf{I} + \mathbf{sk}(\vec{\alpha})) \bullet (\vec{\mathbf{b}} + \Delta \vec{\mathbf{b}}) + \Delta \vec{\mathbf{p}}) + \vec{\mathbf{p}} = \mathbf{R} \bullet (\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \Delta \vec{\mathbf{b}} + \Delta \vec{\mathbf{p}}) + \vec{\mathbf{p}} \\ &\approx \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta \vec{\mathbf{b}} + \Delta \vec{\mathbf{p}}) + \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}} = \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta \vec{\mathbf{b}} + \Delta \vec{\mathbf{p}}) + \vec{\mathbf{v}} \\ \Delta \vec{\mathbf{v}} &\approx \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta \vec{\mathbf{b}} + \Delta \vec{\mathbf{p}}) \end{aligned}$$

$$\begin{aligned} &= \mathbf{F} \bullet [\Delta \mathbf{R}, \Delta \mathbf{p}] \bullet (\mathbf{b} + \Delta \mathbf{b}) \\ &= \mathbf{F} \bullet (\Delta \mathbf{R} \bullet (\mathbf{b} + \Delta \mathbf{b}) + \Delta \mathbf{p}) \\ &= \mathbf{F} \bullet (\Delta \mathbf{R} \bullet \mathbf{b} + \Delta \mathbf{R} \bullet \Delta \mathbf{b} + \Delta \mathbf{p}) \\ &= \mathbf{R} \bullet (\Delta \mathbf{R} \bullet \mathbf{b} + \Delta \mathbf{R} \bullet \Delta \mathbf{b} + \Delta \mathbf{p}) + \mathbf{p} \\ &= \mathbf{R} \bullet (\Delta \mathbf{R} \bullet (\mathbf{b} + \Delta \mathbf{b}) + \Delta \mathbf{p}) + \mathbf{p} \end{aligned}$$

$$\Delta \mathbf{R} \approx \mathbf{I} + \mathbf{sk}(\alpha)$$

$$\Delta \mathbf{p} \approx \epsilon$$

$$\mathbf{sk}(\alpha) \bullet \Delta \mathbf{b} \approx \mathbf{0}$$

$$\mathbf{v}^* = \mathbf{v} + \Delta \mathbf{v}$$



Digression: “rotation triple product”

Expressions like $\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}}$ are linear in $\vec{\mathbf{a}}$, but are not always convenient to work with. Often we would prefer something like $\mathbf{M}(\mathbf{R}, \vec{\mathbf{b}}) \bullet \vec{\mathbf{a}}$.

$$\begin{aligned}\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}} &= -\mathbf{R} \bullet \vec{\mathbf{b}} \times \vec{\mathbf{a}} \\ &= \mathbf{R} \bullet \text{skew}(-\vec{\mathbf{b}}) \bullet \vec{\mathbf{a}} \\ &= \left[\mathbf{R} \bullet \text{skew}(\vec{\mathbf{b}})^T \right] \bullet \vec{\mathbf{a}}\end{aligned}$$

$$\begin{aligned}\text{skew}(\vec{\mathbf{a}}) \bullet \mathbf{R} &= \mathbf{R} \bullet \text{skew}(\mathbf{R}^{-1} \bullet \vec{\mathbf{a}}) \\ \mathbf{R}^{-1} \text{skew}(\vec{\mathbf{a}}) \bullet \mathbf{R} &= \text{skew}(\mathbf{R}^{-1} \bullet \vec{\mathbf{a}})\end{aligned}$$



A “standard form” for linearized error expressions

It is often convenient to use identities to rearrange expressions involving small error variables into sums of terms with the general form $\mathbf{M}_k \vec{\eta}_k$, where \mathbf{M}_k involve things known to the computer, and the $\vec{\eta}_k$ are error variables.

For example,

$$\vec{\gamma} = \mathbf{R} sk(\vec{\alpha}) \vec{\mathbf{a}} + sk(\vec{\beta}) \vec{\mathbf{b}}$$

would be rewritten as

$$\vec{\gamma} = -\mathbf{R} sk(\vec{\mathbf{a}}) \vec{\alpha} - sk(\vec{\mathbf{b}}) \vec{\beta}$$

or

$$\vec{\gamma} = \mathbf{R} sk(-\vec{\mathbf{a}}) \vec{\alpha} + sk(-\vec{\mathbf{b}}) \vec{\beta}$$

