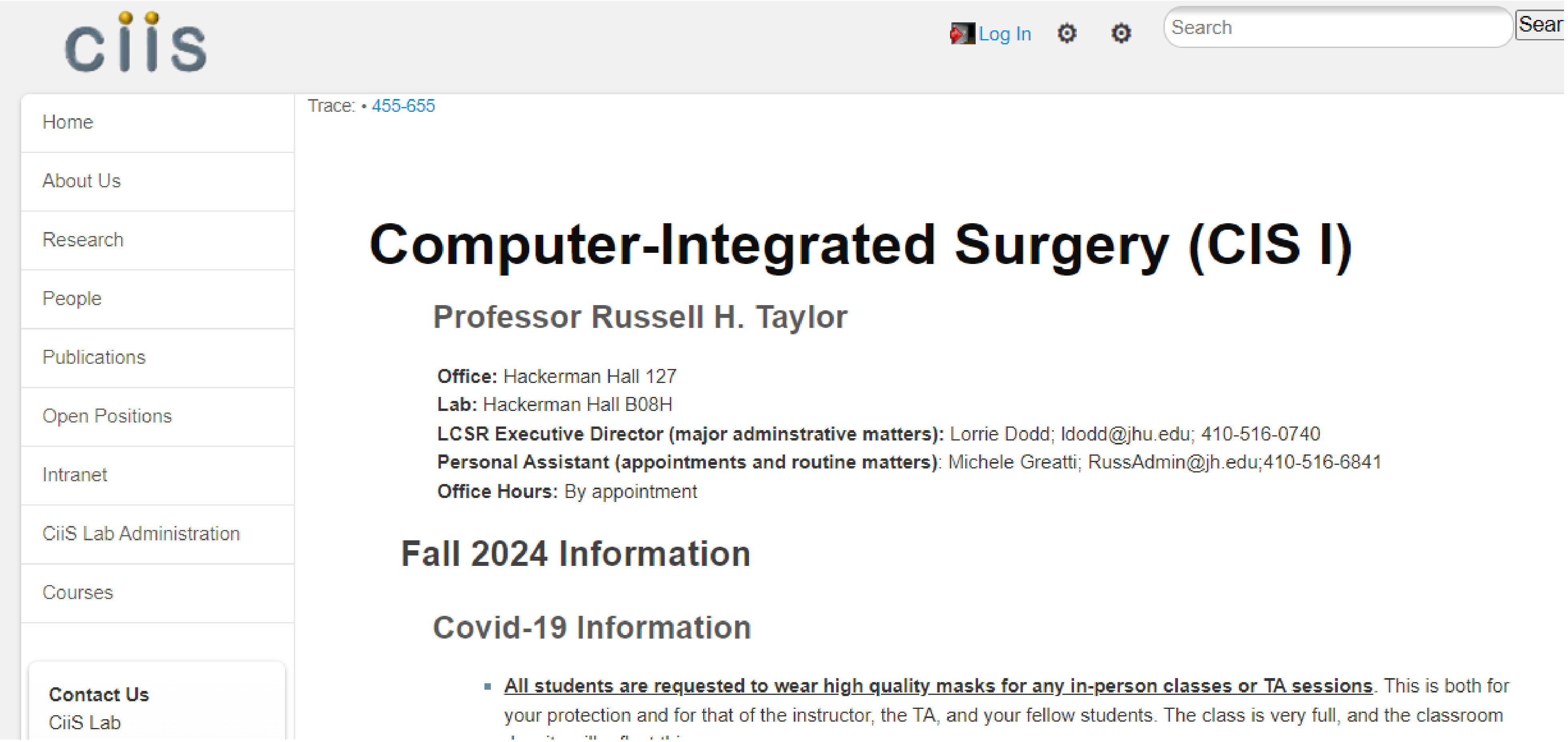
Course Wiki: https://ciis.lcsr.jhu.edu/doku.php?id=courses:455-655:455-655



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Computer-Integrated Surgery (CIS I)

Professor Russell H. Taylor

Office: Hackerman Hall 127
Lab: Hackerman Hall B08H
LCSR Executive Director (major adminstrative matters): Lorrie Dodd; ldodd@jhu.edu; 410-516-0740
Personal Assistant (appointments and routine matters): Michele Greatti; RussAdmin@jh.edu;410-516-6841
Office Hours: By appointment

Fall 2024 Information

Covid-19 Information

- **All students are requested to wear high quality masks for any in-person classes or TA sessions**. This is both for your protection and for that of the instructor, the TA, and your fellow students. The class is very full, and the classroom

Course Wiki: https://ciis.lcsr.jhu.edu/doku.php?id=courses:455-655:455-655

Section and TA office hour times and location

Tatiana Kashtanova

- Email: tkashta1@jhu.edu
- Office Hours (starting September 9)
  - Time: Monday, 10 am - 11 am
  - Location: Zoom
  - https://JHUBlueJays.zoom.us/j/99052831625?pwd=AFMJsShozAjWfMKpYv47JqLqjz3hAN.1
  - Meeting ID: 990 5283 1625
  - Passcode: 096011
- Discussion sections will be announced on Piazza

Meetings with a TA (subject to change):

- 9-Sep: Math
- 16-Sep: HW1
- 23-Sep: HW2
- 30-Sep: PA general, PA1
- TBD: PA Intro
- 7-Oct: HW1 Review
- 14-Oct: PA2
- 21-Oct: PA3
- 28-Oct: HW3
- 4-Nov: HW2 Review
- 11-Nov: HW4, PA4
- 18-Nov: HW3 Review
- 25-Nov: N/A (Thanksgiving)
- 2-Dec: PA5

I will upload my slides here

Course Wiki: https://ciis.lcsr.jhu.edu/doku.php?id=courses:455-655:455-655

Organizational Information

- Fall 2024 Schedule

Schedule: https://ciis.lcsr.jhu.edu/doku.php?id=courses:455-655:2024:fall-2024-schedule

CIS I (601.455/655) Fall 2024 Schedule

Note: This page is subject to change

- Lecture slides
- Supplementary material
- Assignments (hand-out & due dates)

Piazza: https://piazza.com/jhu/fall2024/601455655

- Course announcements
- Find a partner
- Q & A
- Typos / errors
- Private message to the professor & TA

Piazza App

Partner:

- Can be changed between assignments
- Do not abandon anybody before the due date!

Special circumstances / Accommodations:

- Sport / science competitions
- Conferences
- Marriage
- Scheduled health-related procedures
- Etc. expected

→ Tell us in advance!

- Sickness
- Fire
- Etc. unexpected

→ Do not come to the class. Tell us!

- Disability

→ Contact "Student Disability Services"! They will contact us.

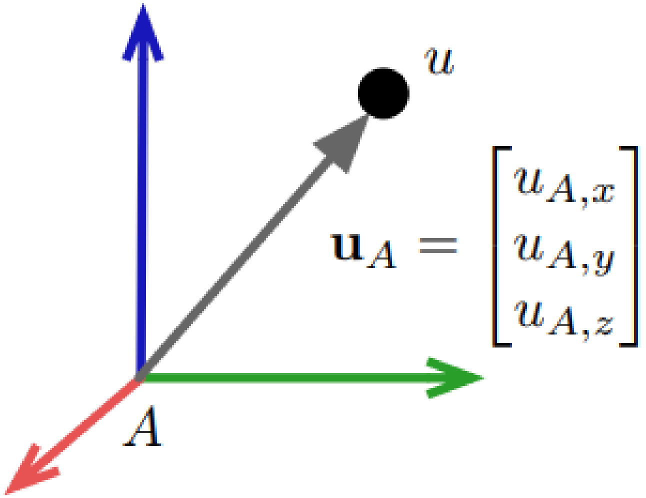
Review – Russell H. Taylor (2024). Fall 2024 Organization Lecture – for details

CIS Math Tutorial

References:

Benjamin D. Killeen (2022). Frame Transformations in Computer Integrated Surgery: A Graphical Introduction

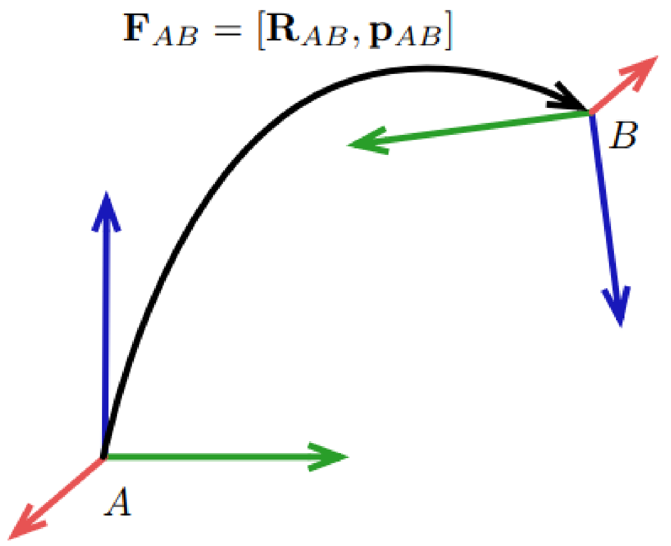
Russell H. Taylor (2024). 600.455/655 Lecture Notes: Basic Mathematical Methods for CIS

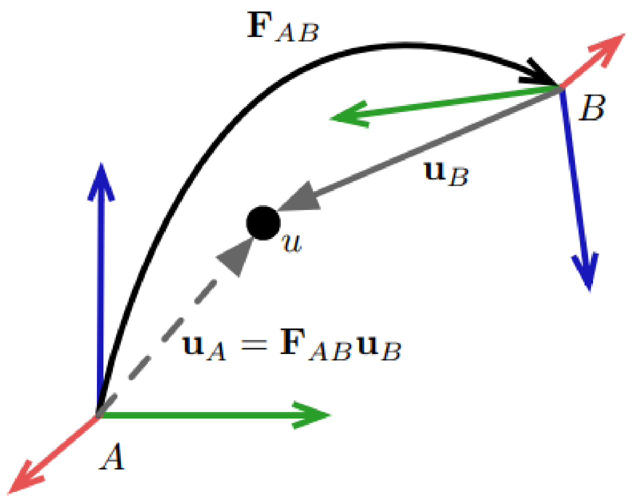


A point $\boldsymbol{u}$ as measured in frame $\boldsymbol{A}$

- A **frame** is a basis for numerical measurements of object locations, orientations, or poses
- A **point** is a singular location in space
- Vector $\boldsymbol{u}_{\boldsymbol{A}}$ defines the position of $\boldsymbol{u}$ relative to frame $\boldsymbol{A}$

Benjamin D. Killeen (2022). Frame Transformations in Computer Integrated Surgery: A Graphical Introduction





Frame transformation "A from B"

$$F_{AB} = [R_{AB}, p_{AB}]$$

A measurement of frame $\boldsymbol{B}$ pose (rotation + translation) with respect to frame $\boldsymbol{A}$

$$u_A = F_{AB}\, u_B = [R_{AB}, p_{AB}]u_B = R_{AB}\, u_B + p_{AB}$$

- u_B is the measurement of $\boldsymbol{u}$ in frame $\boldsymbol{B}$
- F_{AB} known

The right-hand subscript of the transform should match the subscript of the point $\mathbf{F}_{AB}\mathbf{F}_{BC} = \mathbf{F}_{AC}$

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Inverse Transformations

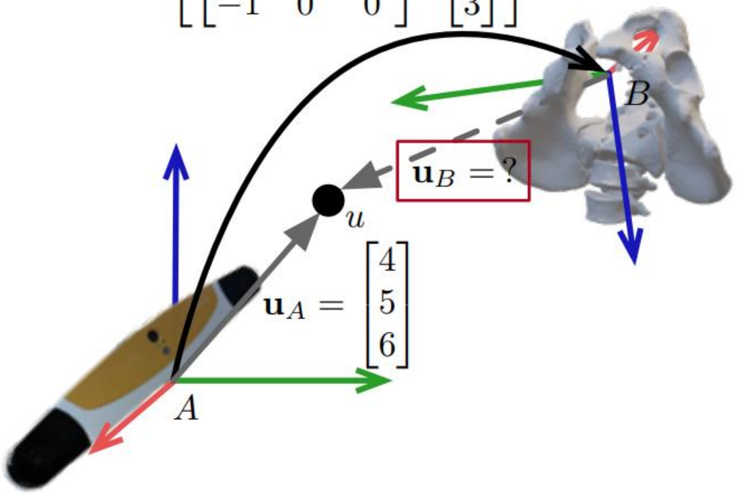
$$F_{AB}^{-1} = [R_{AB}^{-1}, -R_{AB}^{-1}\, p_{AB}]$$

$$F_{AB}^{-1} = F_{BA}$$

Rotation matrices are orthonormal:

$$R^{-1} = R^{T}$$

$$\mathbf{F}_{AB} = [\mathbf{R}_{AB}, \mathbf{p}_{AB}] = \left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right]$$



$$\begin{aligned}
\mathbf{u}_B = \mathbf{F}_{BA}\mathbf{u}_A &= \mathbf{F}_{AB}^{-1}\mathbf{u}_A \\
&= [\mathbf{R}_{AB}^{-1}, -\mathbf{R}_{AB}^{-1}\mathbf{p}_{AB}]\mathbf{u}_A \\
&= \mathbf{R}_{AB}^{-1}\mathbf{u}_A - \mathbf{R}_{AB}^{-1}\mathbf{p}_{AB} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 4 \\ -5 \end{bmatrix} - \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$$

0*4 + 0*5 -1*6 = -6

1*4+0*5+0*6 = 4

0*4-1*5+0*6 = -5

$$= \begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix}$$

-6+3 = -3

4-1 = 3

-5+2 = -3

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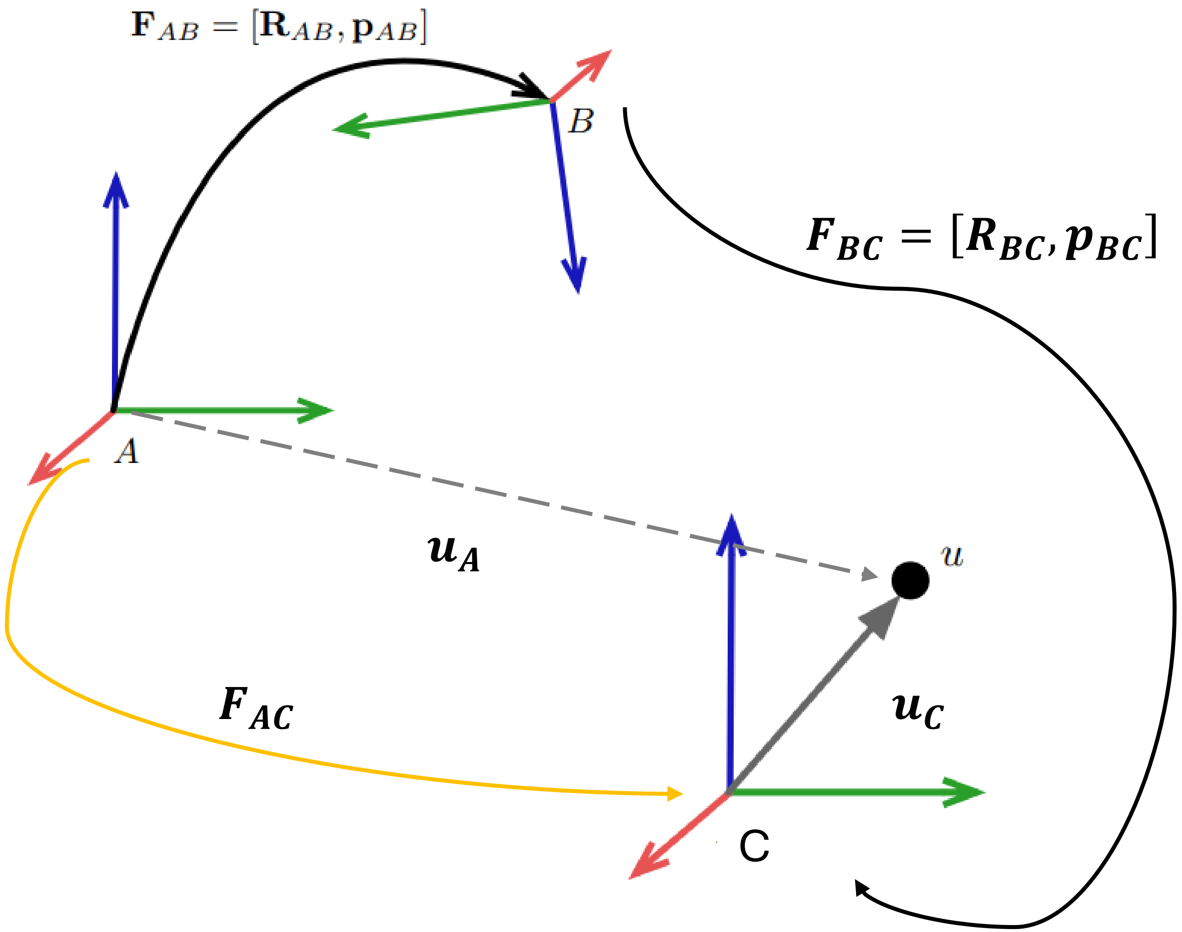
Frame Composition



$$\mathbf{F}_{AC} = \mathbf{F}_{AB}\mathbf{F}_{BC} = [\mathbf{R}_{AB}\mathbf{R}_{BC}, \mathbf{R}_{AB}\mathbf{p}_{BC} + \mathbf{p}_{AB}]$$

$$\begin{aligned}
\mathbf{u}_A &= \mathbf{F}_{AB}\mathbf{F}_{BC}\mathbf{u}_C \quad = \boldsymbol{F}_{AB}\,[\boldsymbol{R}_{BC}, \boldsymbol{p}_{BC}]\boldsymbol{u}_C \\
&= \mathbf{F}_{AB}(\mathbf{R}_{BC}\mathbf{u}_C + \mathbf{p}_{BC}) \quad = [\boldsymbol{R}_{AB}, \boldsymbol{p}_{AB}](\boldsymbol{R}_{BC}\boldsymbol{u}_C + \boldsymbol{p}_{BC}) \\
&= \mathbf{R}_{AB}(\mathbf{R}_{BC}\mathbf{u}_C + \mathbf{p}_{BC}) + \mathbf{p}_{AB} \\
&= (\mathbf{R}_{AB}\mathbf{R}_{BC})\mathbf{u}_C + (\mathbf{R}_{AB}\mathbf{p}_{BC} + \mathbf{p}_{AB}) \\
&= [\mathbf{R}_{AB}\mathbf{R}_{BC}, \mathbf{R}_{AB}\mathbf{p}_{BC} + \mathbf{p}_{AB}]\mathbf{u}_C \\
&= [\mathbf{R}_{AB}\mathbf{R}_{BC}, \mathbf{R}_{AB}\mathbf{p}_{BC} + \mathbf{p}_{AB}]\mathbf{u}_C
\end{aligned}$$

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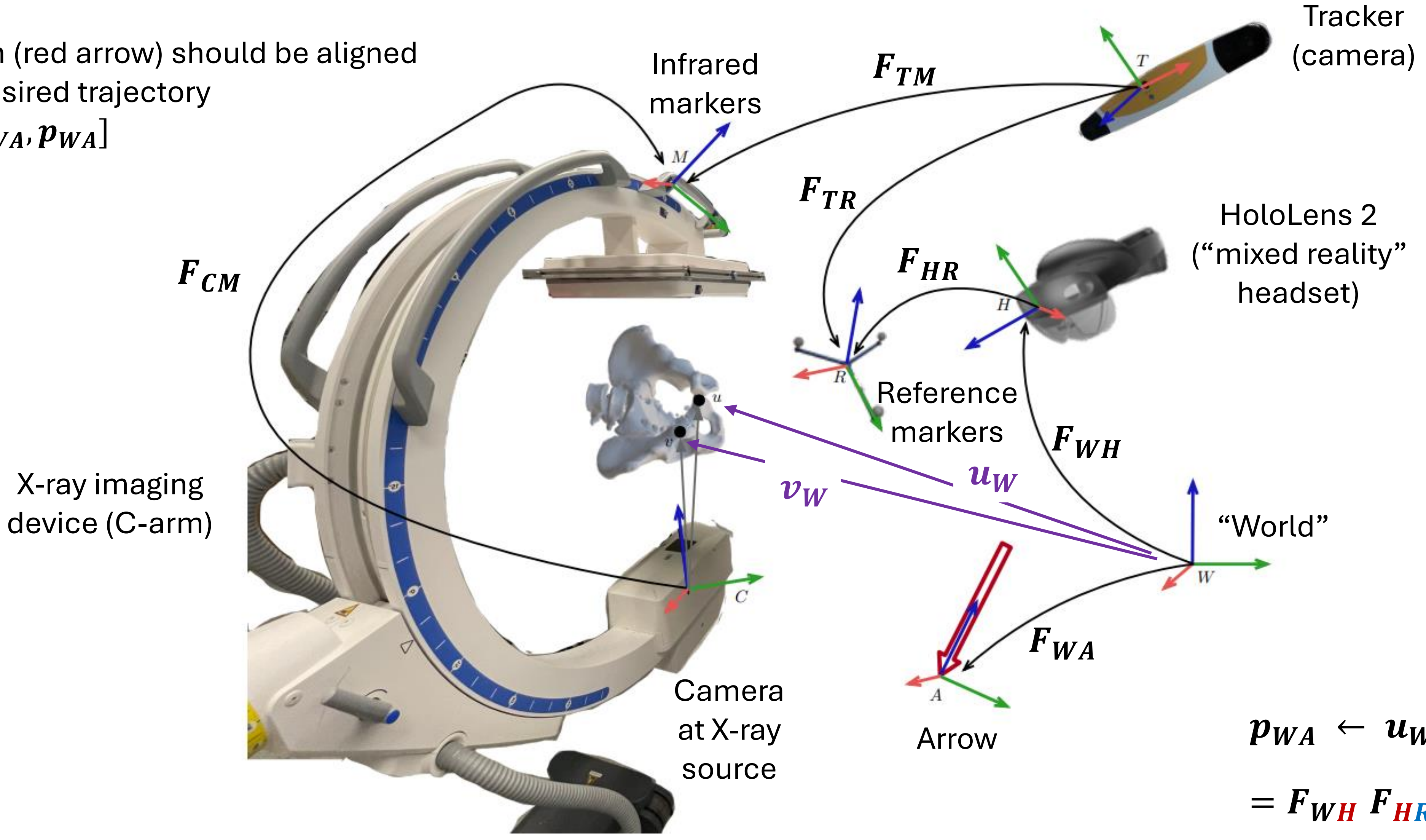
Insert a rigid metal rod (K-wire) into pelvis from *u* to *v*

A hologram (red arrow) should be aligned with the desired trajectory $\boldsymbol{F}_{WA} = [\boldsymbol{R}_{WA}, \boldsymbol{p}_{WA}]$



$$\boldsymbol{p}_{WA} \leftarrow \boldsymbol{u}_W$$

$$= \boldsymbol{F}_{WH}\,\boldsymbol{F}_{HR}\boldsymbol{F}_{TR}^{-1}\boldsymbol{F}_{TM}\boldsymbol{F}_{CM}^{-1}\boldsymbol{u}_c$$

$$= \boldsymbol{F}_{WC}\,\boldsymbol{u}_c$$

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$$\mathbf{p}_{WA} \leftarrow \mathbf{u}_W$$

$$
\begin{aligned}
&= \mathbf{F}_{WH}\mathbf{F}_{HR}\mathbf{F}_{TR}^{-1}\mathbf{F}_{TM}\mathbf{F}_{CM}^{-1}\mathbf{u}_C \\
&= [\mathbf{R}_{WH}, \mathbf{p}_{WH}][\mathbf{R}_{HR}, \mathbf{p}_{HR}][\mathbf{R}_{TR}^{-1}, -\mathbf{R}_{TR}^{-1}\mathbf{p}_{TR}][\mathbf{R}_{TM}, \mathbf{p}_{TM}][\mathbf{R}_{CM}^{-1}, -\mathbf{R}_{CM}^{-1}\mathbf{p}_{CM}]\mathbf{u}_C \\
&= \Big[\mathbf{R}_{WH}\mathbf{R}_{HR}\mathbf{R}_{TR}^{-1}\mathbf{R}_{TM}\mathbf{R}_{CM}^{-1}, \\
&\qquad - \mathbf{R}_{WH}\mathbf{R}_{HR}\mathbf{R}_{TR}^{-1}\mathbf{R}_{TM}\mathbf{R}_{CM}^{-1}\mathbf{p}_{CM} \\
&\qquad + \mathbf{R}_{WH}\mathbf{R}_{HR}\mathbf{R}_{TR}^{-1}\mathbf{p}_{TM} \\
&\qquad - \mathbf{R}_{WH}\mathbf{R}_{HR}\mathbf{R}_{TR}^{-1}\mathbf{p}_{TR} \\
&\qquad + \mathbf{R}_{WH}\mathbf{p}_{HR} \\
&\qquad + \mathbf{p}_{WH}\Big]\mathbf{u}_C \\
&\equiv \mathbf{F}_{WC}\mathbf{u}_C \\
&= \mathbf{R}_{WC}\mathbf{u}_C + \mathbf{p}_{WC}
\end{aligned}
$$

$$\boldsymbol{u}_W = \boldsymbol{F}_{WH}\boldsymbol{F}_{HR}\boldsymbol{F}_{TR}^{-1}\boldsymbol{F}_{TM}\boldsymbol{F}_{CM}^{-1}\boldsymbol{u}_c = \boldsymbol{F}_{WC}\,\boldsymbol{u}_c$$

$$\boldsymbol{v}_W = \boldsymbol{F}_{WC}\,\boldsymbol{v}_c$$

Similarly, we note $\mathbf{v}_W = \mathbf{F}_{WC}\mathbf{v}_C$.

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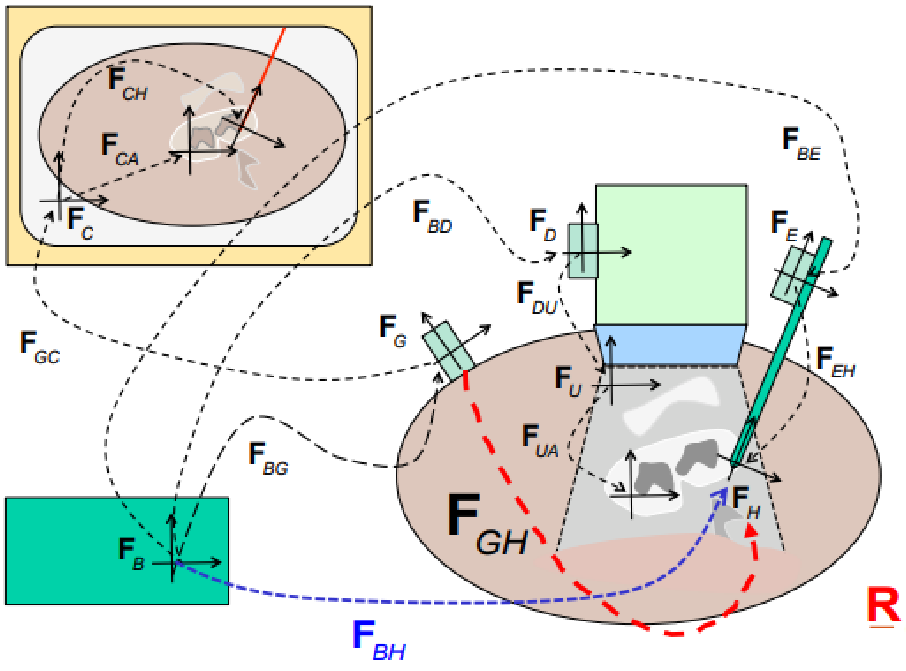

$$\mathbf{F}_{GH} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\mathbf{F}_{EH}$$
$$= \mathbf{F}_{BG}^{-1}\mathbf{F}_{BH}$$

$$\mathbf{R}_{GH} = \mathbf{R}_{BG}^{-1}\mathbf{R}_{BE}\mathbf{R}_{EH}$$

$$\vec{\mathbf{p}}_{GH} = \mathbf{F}_{BG}^{-1}\vec{\mathbf{p}}_{BH} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\vec{\mathbf{p}}_{EH}$$

$$\vec{\mathbf{p}}_{GH} = \mathbf{F}_{BG}^{-1}(\mathbf{R}_{BE}\vec{\mathbf{p}}_{EH} + \vec{\mathbf{p}}_{BE})$$

$$p_{GH} = [R_{BG}^{-1}, -R_{BG}^{-1}\, p_{BG}](R_{BE}p_{EH} + p_{BE})$$

$$p_{GH} = R_{BG}^{-1}(R_{BE}p_{EH} + p_{BE}) - R_{BG}^{-1}\, p_{BG}$$

$$\vec{\mathbf{p}}_{GH} = \mathbf{R}_{BG}^{-1}(\mathbf{R}_{BE}\vec{\mathbf{p}}_{EH} + \vec{\mathbf{p}}_{BE} - \vec{\mathbf{p}}_{BG})$$

Give a formula for computing the pose $\mathbf{F}_{GH}$ of the surgical tool coordinate system relative to the patient rigid body coordinate system $\mathbf{F}_G$?

What are the components $\mathbf{F}_{GH} = [\mathbf{R}_{GH}, \vec{\mathbf{p}}_{GH}]$?

Forward and Inverse Frame Transformations

Forward

$$\mathbf{F} = [\mathbf{R}, \vec{\mathbf{p}}]$$

$$\begin{aligned}\vec{\mathbf{v}} &= \mathbf{F} \bullet \vec{\mathbf{b}} \\ &= [\mathbf{R}, \vec{\mathbf{p}}] \bullet \vec{\mathbf{b}} \\ &= \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}\end{aligned}$$

Inverse

$$\begin{aligned}\mathbf{F}^{-1}\vec{\mathbf{v}} &= \vec{\mathbf{b}} \\ \vec{\mathbf{b}} &= \mathbf{R}^{-1} \bullet (\vec{\mathbf{v}} - \vec{\mathbf{p}}) \\ &= \mathbf{R}^{-1} \bullet \vec{\mathbf{v}} - \mathbf{R}^{-1} \bullet \vec{\mathbf{p}}\end{aligned}$$

$$\mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1} \bullet \vec{\mathbf{p}}]$$

Composition

Assume $\mathbf{F}_1 = [\mathbf{R}_1, \vec{\mathbf{p}}_1]$, $\mathbf{F}_2 = [\mathbf{R}_2, \vec{\mathbf{p}}_2]$

Then

$$\begin{aligned}\mathbf{F}_1 \bullet \mathbf{F}_2 \bullet \vec{\mathbf{b}} &= \mathbf{F}_1 \bullet (\mathbf{F}_2 \bullet \vec{\mathbf{b}}) \\ &= \mathbf{F}_1 \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\ &= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\ &= \mathbf{R}_1 \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) + \vec{\mathbf{p}}_1 \\ &= \mathbf{R}_1 \bullet \mathbf{R}_2 \bullet \vec{\mathbf{b}} + \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1 \\ &= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1] \bullet \vec{\mathbf{b}}\end{aligned}$$

So

$$\begin{aligned}\mathbf{F}_1 \bullet \mathbf{F}_2 &= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet [\mathbf{R}_2, \vec{\mathbf{p}}_2] \\ &= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1]\end{aligned}$$

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Vectors

dot product: $a = \vec{\mathbf{v}} \cdot \vec{\mathbf{w}} \;\; = \left(v_x w_x + v_y w_y + v_z w_z \right) \;\; = \|\vec{\mathbf{v}}\| \|\vec{\mathbf{w}}\| \cos\theta$

cross product: $\vec{\mathbf{u}} = \vec{\mathbf{v}} \times \vec{\mathbf{w}} = \begin{bmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{bmatrix}, \|\mathbf{u}\| = \|\mathbf{v}\| \|\mathbf{w}\| \sin\theta$

Matrix representation of cross product operator

Define

$$\hat{\vec{\mathbf{a}}} \stackrel{\Delta}{=} skew(\vec{\mathbf{a}}) \stackrel{\Delta}{=} \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Then

$$\vec{\mathbf{a}} \times \vec{\mathbf{v}} = skew(\vec{\mathbf{a}}) \bullet \vec{\mathbf{v}}$$

Note that rotation doesn't affect inner products

$$\left(\mathbf{R} \bullet \vec{\mathbf{b}} \right) \bullet \left(\mathbf{R} \bullet \vec{\mathbf{c}} \right) = \vec{\mathbf{b}} \bullet \vec{\mathbf{c}}$$

or lengths of vectors

$$\left\| \mathbf{R} \cdot \vec{\mathbf{v}} \right\| = \left\| \vec{\mathbf{v}} \right\|$$

“Small” Frame Transformations

Represent a "small" pose shift consisting of a small rotation $\Delta\mathbf{R}$ followed by a small displacement $\Delta\vec{\mathbf{p}}$ as

$$\Delta\mathbf{F} = [\Delta\mathbf{R}, \Delta\vec{\mathbf{p}}]$$

Then

$$\Delta\mathbf{F} \bullet \vec{\mathbf{v}} = \Delta\mathbf{R} \bullet \vec{\mathbf{v}} + \Delta\vec{\mathbf{p}}$$

$$\Delta F \approx [\boldsymbol{I} + \boldsymbol{sk}(\boldsymbol{\alpha}), \boldsymbol{\varepsilon}]$$
$$\Delta F^{-1} \approx [\boldsymbol{I} - \boldsymbol{sk}(\boldsymbol{\alpha}), -\boldsymbol{\varepsilon}]$$

Approximations to “Small” Frames

$$\Delta\mathbf{R}(\vec{\mathbf{a}}) \approx \mathbf{I} + skew(\vec{\mathbf{a}})$$
$$\Delta\mathbf{R}(\vec{\mathbf{a}})^{-1} \approx \mathbf{I} - skew(\vec{\mathbf{a}}) = \mathbf{I} + skew(-\vec{\mathbf{a}})$$

Notational NOTE:

We often use $\vec{\alpha}$ to represent a vector of small angles and $\vec{\varepsilon}$ to represent a vector of small displacements

In using these approximations, we typically ignore second order terms. I.e.,

$$\vec{\alpha}_A\vec{\alpha}_B \approx \vec{0},\ \vec{\alpha}_A\vec{\varepsilon}_B \approx \vec{0},\ \vec{\varepsilon}_A\vec{\varepsilon}_B \approx \vec{0}, \text{ etc.}$$

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Errors & sensitivity

Often, we do not have an accurate value for a transformation, so we need to model the error. We model this as a composition of a "nominal" frame and a small displacement

$$\mathbf{F}_{actual} = \mathbf{F}_{nominal} \bullet \Delta\mathbf{F}$$

Often, we will use the notation $\mathbf{F}^*$ for $\mathbf{F}_{actual}$ and will just use $\mathbf{F}$ for $\mathbf{F}_{nominal}$. Thus we may write something like

$$\mathbf{F}^* = \mathbf{F} \bullet \Delta\mathbf{F}$$

or (less often) $\mathbf{F}^* = \Delta\mathbf{F} \bullet \mathbf{F}$. We also use $\vec{\mathbf{v}}^* = \vec{\mathbf{v}} + \Delta\vec{\mathbf{v}}$, *etc*. Thus, if we use the former form (error on the right), and have nominal relationship $\vec{\mathbf{v}} = \mathbf{F} \bullet \vec{\mathbf{b}}$, we get

$$\vec{\mathbf{v}}^* = \mathbf{F}^* \bullet \vec{\mathbf{b}}^*$$

$$= \mathbf{F} \bullet \Delta\mathbf{F} \bullet (\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}}) = \mathbf{F} \bullet (\Delta\mathbf{R} \bullet \vec{\mathbf{b}} + \Delta\mathbf{R} \bullet \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}})$$

$$\approx \mathbf{R} \bullet ((\mathbf{I} + sk(\vec{\alpha})) \bullet (\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}}) + \Delta\vec{\mathbf{p}}) + \vec{\mathbf{p}} = \mathbf{R} \bullet (\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}) + \vec{\mathbf{p}}$$

$$\approx \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}) + \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}} = \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}) + \vec{\mathbf{v}}$$

$$\Delta\vec{\mathbf{v}} \approx \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}})$$

$$= \boldsymbol{F} \cdot [\Delta\boldsymbol{R}, \Delta\boldsymbol{p}] \cdot (\boldsymbol{b} + \Delta\boldsymbol{b})$$
$$= \boldsymbol{F} \cdot (\Delta\boldsymbol{R} \cdot (\boldsymbol{b} + \Delta\boldsymbol{b}) + \Delta\boldsymbol{p})$$
$$= \boldsymbol{F} \cdot (\Delta\boldsymbol{R} \cdot \boldsymbol{b} + \Delta\boldsymbol{R} \cdot \Delta\boldsymbol{b} + \Delta\boldsymbol{p})$$
$$= \boldsymbol{R} \cdot (\Delta\boldsymbol{R} \cdot \boldsymbol{b} + \Delta\boldsymbol{R} \cdot \Delta\boldsymbol{b} + \Delta\boldsymbol{p}) + \mathbf{p}$$
$$= \boldsymbol{R} \cdot (\Delta\boldsymbol{R} \cdot (\boldsymbol{b} + \Delta\boldsymbol{b}) + \Delta\boldsymbol{p}) + \mathbf{p}$$

$$\Delta\boldsymbol{R} \approx \mathbf{I} + \mathbf{sk}(\boldsymbol{\alpha})$$
$$\Delta\boldsymbol{p} \approx \boldsymbol{\varepsilon}$$

$$\mathbf{sk}(\boldsymbol{\alpha}) \cdot \Delta\mathbf{b} \approx \mathbf{0}$$

$$\boldsymbol{v}^* = \boldsymbol{v} + \Delta\boldsymbol{v}$$

Digression: "rotation triple product"

Expressions like $\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}}$ are linear in $\vec{\mathbf{a}}$, but are not always convenient to work with. Often we would prefer something like $\mathbf{M}(\mathbf{R}, \vec{\mathbf{b}}) \bullet \vec{\mathbf{a}}$.

$$
\begin{aligned}
\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}} &= -\mathbf{R} \bullet \vec{\mathbf{b}} \times \vec{\mathbf{a}} \\
&= \mathbf{R} \bullet skew(-\vec{\mathbf{b}}) \bullet \vec{\mathbf{a}} \\
&= \left[\mathbf{R} \bullet skew(\vec{\mathbf{b}})^T \right] \bullet \vec{\mathbf{a}}
\end{aligned}
$$

$$
\begin{aligned}
skew(\vec{\mathbf{a}}) \bullet \mathbf{R} &= \mathbf{R} \bullet skew(\mathbf{R}^{-1} \bullet \vec{\mathbf{a}}) \\
\mathbf{R}^{-1} skew(\vec{\mathbf{a}}) \bullet \mathbf{R} &= skew(\mathbf{R}^{-1} \bullet \vec{\mathbf{a}})
\end{aligned}
$$

!

A "standard form" for linearized error expressions

It is often convenient to use identities to rearrange expressions involving small error variables into sums of terms with the general form $\mathbf{M}_k\vec{\eta}_k$, where $\mathbf{M}_k$ involve things known to the computer, and the $\vec{\eta}_k$ are error variables.

For example,

$$\vec{\gamma}=\mathbf{R}sk(\vec{\alpha})\vec{\mathbf{a}}+sk(\vec{\beta})\vec{\mathbf{b}}$$

would be rewritten as

$$\vec{\gamma}=-\mathbf{R}sk(\vec{\mathbf{a}})\vec{\alpha}-sk(\vec{\mathbf{b}})\vec{\beta}$$

or

$$\vec{\gamma}=\mathbf{R}sk(-\vec{\mathbf{a}})\vec{\alpha}+sk(-\vec{\mathbf{b}})\vec{\beta}$$

!