Homework Assignment 1 – 601.455/655 Fall 2025

Name	Name
Email	Email
Other contact information (optional)	Other contact information (optional)
Signature (required) I/We have followed the rules in completing this assignment ________________________________	Signature (required) I/We have followed the rules in completing this assignment ________________________________

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone** or in **teams of two** and are not to discuss the problems with anyone other than the TAs or the instructor.
3. **IMPORTANT NOTE:** If you work in teams of two, you are **not** to split up the questions and each answer a subset individually. You are to work **together**. I encourage teaming on these problems because I believe that it encourages learning, not as a way to reduce the required work for students taking the course. By signing this sheet you are asserting that each of you has contributed equally to each answer and can individually explain the answer as well as if you had answered the question alone. I view this as a question of trust and ethics.
4. It is otherwise open book, notes, and web. But you should cite any references you consult.
5. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
6. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
7. Submit the assignment on GradeScope as a neat and legible PDF file. We will not insist on typesetting your answers, but we must be able to read them. We will not go to extraordinary lengths to decipher what you write. If the graders cannot make out an answer, the score will be 0.
8. Sign and hand in this page as the first sheet of your assignment. If you work with a partner, then you both should sign the sheet, but you should only submit one PDF file for both of you, using the GradeScope teaming feature. Indicate clearly who it is from.
9. This assignment has more than 100 points, but the most that will be applied to your grade is 100.

Problem Scenario: Computer-Assisted Osteotomy

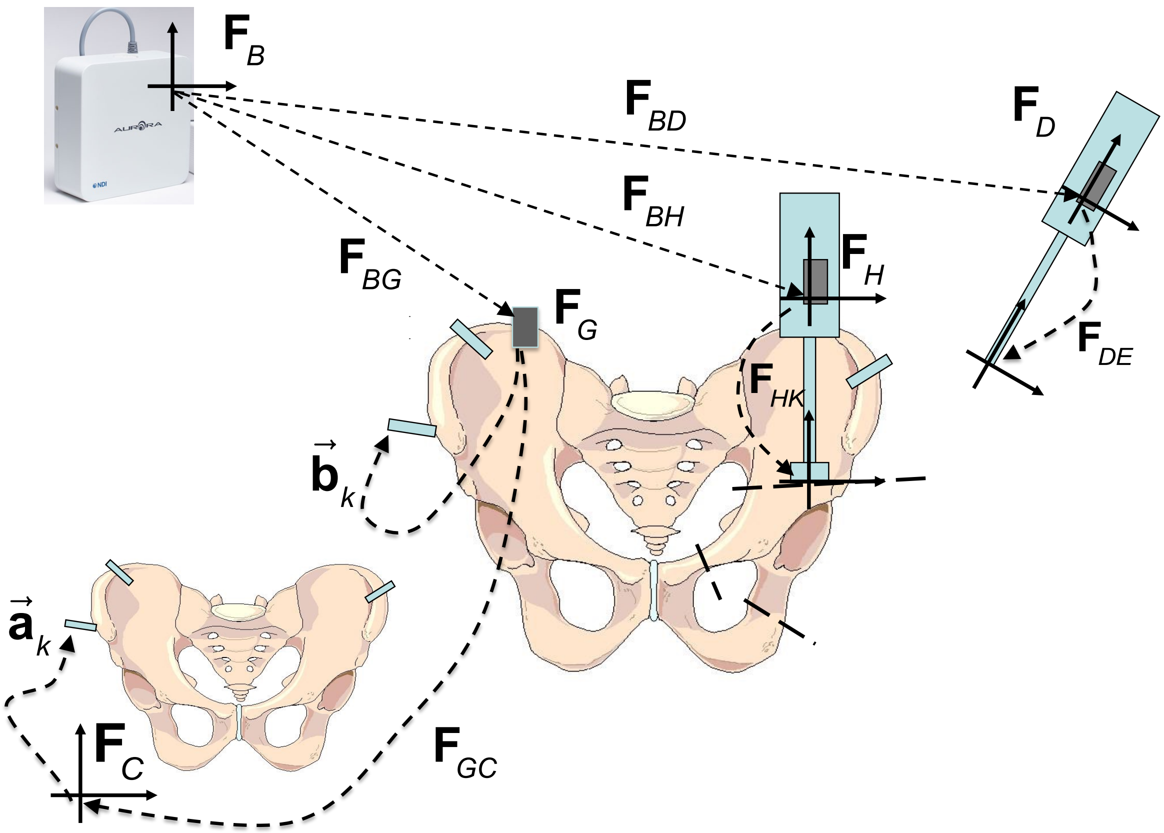

Fig. 1: Computer-Assisted Osteotomy

Consider the pelvic osteotomy situation illustrated in Fig. 1. Here we assume that a three locating pins (similar to those used in Robodoc) have been inserted into the patient's pelvis, and that a CT scan of the pelvis with the pins inserted has been produced. The patient has been placed onto the operating table. Also, an electromagnetic magnetic navigation system (here, the Northern Digital Aurora) is present in the room. Two surgical tools are available:

- A probe/pointer device
- An osteotome (essentially a fancy chisel) or saw that will be used to cut the pelvis.

6 DOF Aurora tracking sensors have been attached to the handle of each tool and an additional 6 DOF sensor has been affixed rigidly to the pelvis. The Aurora is capable of determining the position and orientation of each sensor relative to the Aurora base unit. We will define the following coordinate systems:

$\mathbf{F}_B =$ Coordinate system of tracking system base unit

$\mathbf{F}_D =$ Coordinate system of tracking device on pointer handle

$\mathbf{F}_H =$ Coordinate system of tracking device on osteotome handle

$\mathbf{F}_G =$ Coordinate system of tracking device attached to pelvis

$\mathbf{F}_C =$ Coordinate system of CT image

We also have the following relationships

$\mathbf{F}_{Bx} =$ Measured 6 DOF pose of tracking device x relative to base unit

$\mathbf{F}_{HK} =$ 6 DOF pose of osteotome blade relative to osteotome handle tracking device

$\mathbf{F}_{DE} =$ 6 DOF pose of pointer tip relative to pointer handle tracking device

$\vec{\mathbf{a}}_k =$ Position of the top of pin k in CT coordinates

$\vec{\mathbf{b}}_k =$ Position of the top of pin k relative to tracking device G

We will follow our usual conventions where frame position and orientation components are represented by $\mathbf{F} = [\mathbf{R}, \vec{\mathbf{p}}]$, and errors are represented by $\Delta\mathbf{F} = [\Delta\mathbf{R}, \Delta\mathbf{p}]$. We will also use the approximation convention $\Delta\mathbf{R} \approx \mathbf{I} + sk(\vec{\alpha})$.

Question 1

A. (5 points) Let $\vec{\mathbf{p}}_{tip} = \vec{\mathbf{p}}_{GE}$ be the position of the tip of the pointer tool relative to the reference marker coordinate system $\mathbf{F}_G$. Give a formula for computing $\vec{\mathbf{p}}_{tip}$, based on the available tracking system measurements $\mathbf{F}_{Bx}$. Express your answers in terms of the $\mathbf{F}_{Bx}$'s and $\mathbf{F}_{DE}$ in terms of the individual $[\mathbf{R}_{xx},\vec{\mathbf{p}}_{xx}]$ components.

B. (5 points) Suppose that we have touched the tops of the three fiducial pins and used the results to compute a registration transformation $\mathbf{F}_{GC}$ such that $\mathbf{F}_{GC}\vec{\mathbf{a}}_k = \vec{\mathbf{b}}_k$. Give an expression for computing the position and orientation $\mathbf{F}_{CK}$ of the osteotome blade in CT coordinates, based on the available tracking system measurements $\mathbf{F}_{Bx}$. Express your answers in terms of the $\mathbf{F}_{xx}$'s and in terms of the individual $[\mathbf{R}_{xx},\vec{\mathbf{p}}_{xx}]$ components.

C. (5 points) Suppose now that the tracking system is not perfectly accurate, so that at any given time the actual value $\mathbf{F}_{Bx}^{*}$ of a measurement $\mathbf{F}_{Bx}$ is given by $\mathbf{F}_{Bx}^{*} = \Delta\mathbf{F}_B\mathbf{F}_{Bx}\Delta\mathbf{F}_{Bx}$. Thus, the measurements will include some "common mode" errors $\Delta\mathbf{F}_B$ as well as "place specific" errors associated with each marker. In both cases, these errors may come from many different sources. The specific causes are irrelevant for your analysis. In any case, they will introduce some error in your computation of $\vec{\mathbf{p}}_{tip}$, so that $\vec{\mathbf{p}}_{tip}^{*} = \vec{\mathbf{p}}_{tip} + \Delta\vec{\mathbf{p}}_{tip}$. Give an expression for $\vec{\mathbf{p}}_{tip}^{*}$ in terms of the $\mathbf{R}$'s, $\vec{\mathbf{p}}$'s, $\Delta\mathbf{R}$'s, and $\Delta\vec{\mathbf{p}}$'s. **Hint:** You can use the simplification $\mathbf{F}_{GD} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BD}$. This is commonly done with tracking subsystems.

D. (10 points) Now, make use of linear approximations to re-express your answer for 1C in terms of the $\vec{\alpha}$'s and $\vec{\varepsilon}$'s and simplify the result. Express your answer in "normal" linearized form as a sum of linear terms in which any terms of the form $sk(\vec{\alpha})\vec{\mathbf{p}}$ are written $-sk(\vec{\mathbf{p}})\vec{\alpha}$. I.e., in which all the $\vec{\alpha}_{xy}$ and $\vec{\varepsilon}_{xy}$ are on the right.

E. (5 points) Now assume again that the tracking system is not perfectly accurate, so that at any given time the actual value $\mathbf{F}_{Bx}^{*}$ of a measurement $\mathbf{F}_{Bx}$ is given by $\mathbf{F}_{Bx}^{*} = \Delta\mathbf{F}_B\mathbf{F}_{Bx}\Delta\mathbf{F}_{Bx}$. Assume that a registration has been done somehow, but that there is some error in the registration transformation, such that $\mathbf{F}_{GC}^{*} = \mathbf{F}_{GC}\Delta\mathbf{F}_{GC}$. Produce an expression for the position $\vec{\mathbf{p}}_{Ct}^{*} = \vec{\mathbf{p}}_{Ct} + \Delta\vec{\mathbf{p}}_{Ct}$ and CT image coordinates corresponding to $\vec{\mathbf{p}}_{tip}^{*} = \vec{\mathbf{p}}_{tip} + \Delta\vec{\mathbf{p}}_{tip}$. Here, you do not need to expand $\Delta\vec{\mathbf{p}}_{tip}$ based on your answer to Question 1.D. But your answer should be in terms of $\mathbf{R}_{GC},\vec{\mathbf{p}}_{GC},\Delta\mathbf{R}_{GC},\Delta\vec{\mathbf{p}}_{GC},\vec{\mathbf{p}}_{tip},\Delta\vec{\mathbf{p}}_{tip}$.

F. (10 points) Apply linear approximations to your answer to Question 1.E to produce an expression for $\Delta\vec{\mathbf{p}}_{Ct} = \vec{\varepsilon}_{Ct}$ in normalized form.

G. (10 points) Suppose that our pointer system has been used to touch the tops of the fiducial pins and the system has produced values for the $\vec{\mathbf{b}}_k$. But there is some error, so that $\vec{\mathbf{b}}_k^{*} = \vec{\mathbf{b}}_k + \Delta\vec{\mathbf{b}}_k$. These have been used to compute the registration transformation $\mathbf{F}_{GC}$, where $\mathbf{F}_{GC}^{*} = \mathbf{F}_{GC}\Delta\mathbf{F}_{GC} \approx \mathbf{F}_{GC} \bullet \left[\mathbf{I} + sk(\vec{\alpha}_{GC}),\vec{\varepsilon}_{GC}\right]$. Suppose that we have some constraint function $f(\cdot)$ and that we know $f(\Delta\vec{\mathbf{b}}_k) \le \sigma_k$ but that the CT segmentation is good enough so that we can assume that the $\vec{\mathbf{a}}_k$ are known exactly. Develop a system of equations and inequalities that will express what can be known about $\vec{\alpha}_{GC}$ and $\vec{\varepsilon}_{GC}$. Here, your answer should be expressed in terms of $f(\bullet)$. **Note:** Typical examples of $f(\vec{\mathbf{v}})$ include the Euclidean norm $f(\vec{\mathbf{v}}) = \sqrt{\vec{\mathbf{v}} \bullet \vec{\mathbf{v}}}$, the L1 norm $f(\vec{\mathbf{v}}) = \left\|\vec{\mathbf{v}}\right\|_1 = |v_x| + |v_y| + |v_z|$, the infinity norm $\left\|\vec{\mathbf{v}}\right\|_\infty = \max(|v_x|, |v_y|, |v_z|)$, and the Mahalonobis norm.

H. (5 points) How can you simplify this relationship if $f(\vec{\mathbf{v}})$ is the Euclidean norm?

I. (10 points) Your answer to Question 1.H can be rewritten as $\vec{\eta}_{GC}^T \bullet \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \bullet \vec{\eta}_{GC}$ where $\vec{\eta}_{GC} = \begin{bmatrix} \vec{\alpha}_{GC} \\ \vec{\varepsilon}_{GC} \end{bmatrix}$ and A_k, B_k, C_k, D_k are 3x3 matrices. What are the values of A_k, B_k, C_k, D_k?

J. (10 points) How would your answer to 1.G change if there is also some segmentation error, so that $\vec{\mathbf{a}}_k^{\,*} = \vec{\mathbf{a}}_k + \Delta\vec{\mathbf{a}}_k$ and $0 \le f(\Delta\vec{\mathbf{a}}_k) \le \xi_k$?

K. (10 points) Now, let's treat $\vec{\eta}_{GC} \sim N(\vec{\mathbf{0}}, \mathbf{C}_{GC})$ as a zero mean Gaussian random 6 vector with covariance $\mathbf{C}_{GC}$ and $\Delta\vec{\mathbf{p}}_{tip} = \vec{\varepsilon}_{tip} \sim N(\vec{\mathbf{0}}, \mathbf{C}_{tip})$ as a zero mean Gaussian random 3 vector with covariance $\mathbf{C}_{tip}$. $\mathbf{C}_{CG}$ is a 6x6 symmetric matrix with 3x3 submatrices

$$\mathbf{C}_{CG} = \begin{bmatrix} \mathbf{C}_{aa} & \mathbf{C}_{ae} \\ \mathbf{C}_{ae} & \mathbf{C}_{ee} \end{bmatrix}$$

Given values for $\mathbf{F}_{GC}$ and $\vec{\mathbf{p}}_{tip}$ subject to errors $\mathbf{F}_{GC}^* = \mathbf{F}_{GC}\Delta\mathbf{F}_{GC} \approx \mathbf{F}_{GC} \bullet [\mathbf{I} + sk(\vec{\alpha}_{GC}), \vec{\varepsilon}_{GC}]$ and $\vec{\mathbf{p}}_{tip}^* = \vec{\mathbf{p}}_{tip} + \vec{\varepsilon}_{tip}$, your answer to Question 1.F should be a linear expression for the error $\Delta\vec{\mathbf{p}}_{Ct} = \vec{\varepsilon}_{Ct} \sim N(\vec{\mathbf{0}}, \mathbf{C}_{Ct})$. Give a formula for $\mathbf{C}_{Ct}$.

Question 2

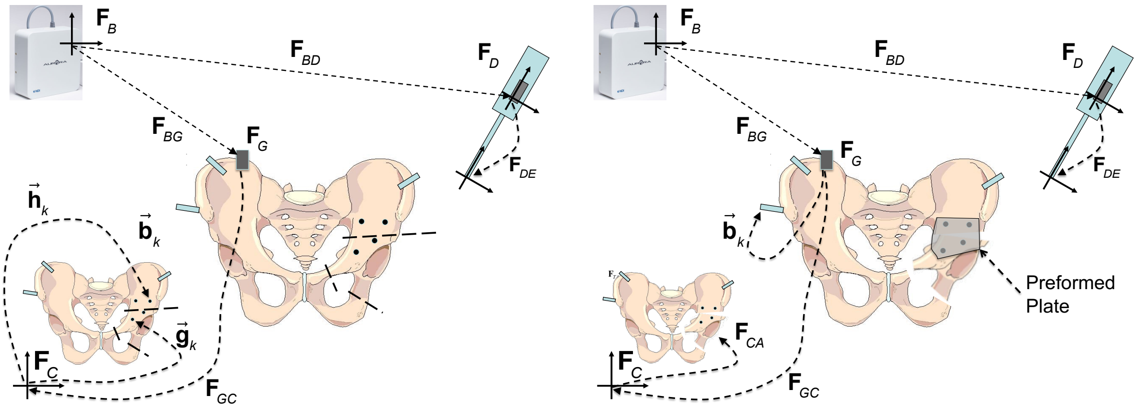

Fig. 1: Osteotomy with preformed plates. (Left) surgical plan and holes drilled; (Right) acetabular fragment relocated and affixed to predrilled and preformed plates.

Consider now the periacetabular osteotomy strategy shown in Fig. 1. The goal is to free the acetabulum and reposition it to a pose $\mathbf{F}_{CA}$ relative to the rest of the pelvis. In other words, and point on the acetabulum that has a position $\vec{\mathbf{p}}_{A,j}$ in the original CT coordinates will have a new position $\mathbf{F}_{CA}\vec{\mathbf{p}}_{A,j}$ after relocation. In this case, the osteotomy plan includes the places that the pelvis must be cut to free the acetabulum from the rest of the pelvis. The planning software determines where holes should be drilled (before any cuts are made) into the acetabular fragment at positions $\vec{\mathbf{g}}_k$ in CT coordinates, It also determines where other holes are to be drilled into the other part of the pelvis at positions $\vec{\mathbf{h}}_k$ in CT coordinates. The directions of the holes in the CT coordinate system are $\vec{\mathbf{n}}_{g,k}$ and $\vec{\mathbf{n}}_{h,k}$, respectively. The plan assumes that a custom plate is available with holes drilled at predrilled positions $\vec{\mathbf{q}}_k$ and $\vec{\mathbf{r}}_k$ relative to the plate coordinate system corresponding to the acetabular and main pelvis holes, respectively. For convenience, the plate coordinate system has been chosen so that $\vec{\mathbf{h}}_k = \vec{\mathbf{r}}_k$. The pointer of Question 1 has been upgraded to a drill or awl that can drill holes in bone.

In surgery, a registration step is performed. After registration, surgical navigation is performed to assist the surgeon in drilling the holes in the desired places in the pelvis. Then, navigation is used to assist the surgeon in cutting the acetabulum free from the pelvis. Then the plate is attached to the acetabulum fragment with screws. Finally, the fragment is manipulated so that the plate holes are aligned with the holes in the main part of the pelvis, and the plate is attached with screws.

A. (5 points) Suppose that the planning workflow comprises two steps: 1) determine the positions $\vec{\mathbf{g}}_k$ and $\vec{\mathbf{h}}_k$ relative to CT coordinates of the holes to be drilled in the pelvis before the osteotomy is performed; and 2) determine the positions of plate holes $\vec{\mathbf{q}}_k$ and $\vec{\mathbf{r}}_k$ relative to plate coordinates. By design, the planning system has chosen an arbitrary plate coordinate system so that $\vec{\mathbf{h}}_k = \vec{\mathbf{r}}_k$. Give a formula for the location of the plate holes $\vec{\mathbf{q}}_k$ in order to realign the acetabular component to pose $\mathbf{F}_{CA}$ in CT coordinates.

B. (5 points) Suppose that the planning system has also defined a preferred direction $\vec{\mathbf{d}}_{Ck}$ relative to CT coordinates for drilling hole k, where $\left\|\vec{\mathbf{d}}_{Ck}\right\| = 1$. Suppose that the direction of the drill shaft relative to the drilling tool coordinate system is $\vec{\mathbf{d}}_{Dt} = \mathbf{R}_{DE}\vec{\mathbf{z}}$. What is the angle between the desired and actual alignment of the drill axis?

C. (10 points) Assume that the tracking system is now very accurate, but that there has been some other source of registration error, so that $\mathbf{F}_{GC}^{*} = \mathbf{F}_{GC}\Delta\mathbf{F}_{GC}$. How accurately can the surgical plan be carried out? I.e., what can you say about the accuracy with which the surgeon achieve the desired alignment? I.e., give expressions for $\vec{\alpha}_{CA}$ and $\vec{\varepsilon}_{CA}$ in terms of $\vec{\alpha}_{GC}$ and $\vec{\varepsilon}_{GC}$, where $\Delta\mathbf{F}_{CA} = \left(\mathbf{F}_{GC}\mathbf{F}_{CA}\right)^{-1}\mathbf{F}_{GC}^{*}\mathbf{F}_{CA} \approx [\mathbf{I} + sk(\vec{\alpha}_{CA}), \vec{\varepsilon}_{CA}]$. Express your answer is normalized linear form.

D. (10 points) Suppose the registration is now error-free, but that the holes are slightly over-sized, so that if each screw has radius r, then each hole has radius $r+\rho$. Can you say anything about how this will affect the accuracy with which the acetabulum can be aligned? **Hints:** Let $\mathbf{F}_{AP}^{*} = \mathbf{F}_{AP}\Delta\mathbf{F}_{AP}$ represent the actual coordinate transformation between the acetabular fragment and the plate, where $\mathbf{F}_{AP}$ is the nominal transformation. and $\mathbf{F}_{PC}^{*} = \Delta\mathbf{F}_{PC}\mathbf{F}_{PC}$ represent the actual transformation between the plate and the rest of the pelvis. Consider the effect of the hole clearance on the worst-case accuracy of these transformations, then put them together to form a set of constraint equations.