

Homework Assignment 3 – 601.455/655 Fall 2025

Name	Name
Email	Email
Other contact information (optional)	Other contact information (optional)
Signature (required) I/We have followed the rules in completing this assignment	Signature (required) I/We have followed the rules in completing this assignment

- Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
- You are to work alone or in teams of two and are not to discuss the problems with anyone other than the TAs or the instructor.
- IMPORTANT NOTE:** If you work in teams of two, you are not to split up the questions and each answer a subset individually. You are to work together. I encourage teaming on these problems because I believe that it encourages learning, not as a way to reduce the required work for students taking the course. By signing this sheet you are asserting that each of you has contributed equally to each answer and can individually explain the answer as well as if you had answered the question alone. I view this as a question of trust and ethics.
- It is otherwise open book, notes, and web. But you should cite any references you consult.
- Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
- Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
- Submit the assignment on GradeScope as a neat and legible PDF file. We will not insist on typesetting your answers, but we must be able to read them. We will not go to extraordinary lengths to decipher what you write. If the graders cannot make out an answer, the score will be 0.
- Sign and hand in this page as the first sheet of your assignment. If you work with a partner, then you both should sign the sheet, but you should only submit one PDF file for both of you, using the GradeScope teaming feature. Indicate clearly who it is from.
- This assignment has more than 100 points, but the most that will be applied to your grade is 100.
- For this assignment only:** This is a design assignment in which you are asked to specify the design parameters and general workflow for a surgical navigation system. You are expected to provide a basic formulation, but you may use an AI-based computer algebra system to assist you in predicting the expected performance of the system. But you **must clearly disclose** which formulas were produced with the aid of the AI-based system.
- You should hand in **both** a pdf file with your answers **and** a small text file, as described in Question 2. We will use the auxiliary text file to run a program to verify your design.

Notes on this assignment

These notes are copied over from the previous assignments and are included here to provide a convenient reference.

Notational Notes

Generally, one would represent the covariance of a random variable X using a single subscript $\text{cov}(X) = \Sigma_X$ or $\text{cov}(X) = \mathbf{C}_X$ and the covariance of two random variables (X, Y) as $\text{cov}(X, Y) = \Sigma_{XY}$ or $\text{cov}(X, Y) = \mathbf{C}_{XY}$. Sometimes I will double the subscript for covariance of a single random variable, e.g., $\text{cov}(X) = \text{cov}(X, X) = \Sigma_{XX} = \mathbf{C}_{XX}$.

The questions below concern iterative updates of the distributions of random variables. For example, $\bar{\mathbf{x}}^{(k)} \sim N(\bar{\mu}_x^{(k)}, \mathbf{C}_{xx}^{(k)})$ would represent a multivariable Gaussian distribution for the random vector $\bar{\mathbf{x}}$. To reduce clutter, we will sometimes omit the vector symbol in subscripts. E.g., $\bar{\mathbf{x}}^{(k)} \sim N(\bar{\mu}_x^{(k)}, \mathbf{C}_{xx}^{(k)})$. We will omit the superscript for random variables representing quantities that are unique to an individual observation, and use the superscript (obs) for an observed value. For example, $\bar{\mathbf{y}}_k^{(obs)}$ would represent an observed value of random vector $\bar{\mathbf{y}}_k$ with $\bar{\mathbf{y}}_k \sim N(\bar{\mu}_{y,k}, \mathbf{C}_{yy,k})$.

Other Notes

Multivariable Gaussian: Given a d -dimensional Gaussian random vector $\bar{\mathbf{x}} \sim N(\mu_x, \mathbf{C}_{xx})$, then the probability that $\bar{\mathbf{x}}$ has any particular value $\bar{\mathbf{x}}_t$ is

$$\Pr(\bar{\mathbf{x}} = \bar{\mathbf{x}}_t) = \frac{1}{\sqrt{(2\pi)^d |\mathbf{C}_{xx}|}} \exp\left(-\frac{1}{2}(\bar{\mathbf{x}}_t - \bar{\mu}_x)^T \mathbf{C}_{xx}^{-1}(\bar{\mathbf{x}}_t - \bar{\mu}_x)\right)$$

Many more useful facts about multivariable Gaussians may be found online. The Wikipedia article at https://en.wikipedia.org/wiki/Multivariate_normal_distribution is one useful source.

Multiple observations: If we have an initial estimate for the distribution of a multivariable Gaussian random vector $\bar{\mathbf{x}} \sim N(\mu_x^{(0)}, \mathbf{C}_{xx}^{(0)})$ and make N observations $\bar{\mathbf{x}}_k^{(obs)}$ of $\bar{\mathbf{x}}$ where $\bar{\mathbf{x}}_k^{(obs)} \sim N(\bar{\mu}_x^{(k)}, \mathbf{C}_k)$, then our new estimate for the distribution of $\bar{\mathbf{x}}$ is given by

$$\bar{\mathbf{x}}^{(new)} \sim N(\bar{\mu}_x^{(new)}, \mathbf{C}_x^{(new)})$$

$$\mathbf{C}_x^{(new)} = \left(\sum_{0 \leq k \leq N} \mathbf{C}_k^{-1} \right)^{-1}$$

$$\bar{\mu}_x^{(new)} = \mathbf{C}_x^{(new)} \left(\sum_{0 \leq k \leq N} \mathbf{C}_k^{-1} \bar{\mu}_x^{(k)} \right)$$

This assignment will make extensive use of the rules for updating conditional multivariable Gaussian distributions. If

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 \end{bmatrix} \sim N \left(\begin{bmatrix} \bar{\mu}_1 \\ \bar{\mu}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \right) \text{ where } \mathbf{C}_{21} = \mathbf{C}_{12}^T$$

then

$$E(\bar{\mathbf{x}}_1 | \bar{\mathbf{x}}_2 = \bar{\mathbf{x}}_2^{(obs)}) = \bar{\mu}_1 + \mathbf{C}_{12} \mathbf{C}_{22}^{-1} (\bar{\mathbf{x}}_2^{(obs)} - \bar{\mu}_2)$$

$$\text{cov}(\bar{\mathbf{x}}_1 | \bar{\mathbf{x}}_2 = \bar{\mathbf{x}}_2^{(obs)}) = \mathbf{C}_{11} - \mathbf{C}_{12} \mathbf{C}_{22}^{-1} \mathbf{C}_{21}$$

Also, if $\bar{\mathbf{z}} = \bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2$ then

$$E(\bar{\mathbf{x}}_1 | \bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2 = \bar{\mathbf{z}}) = \bar{\mu}_1 + (\mathbf{C}_{11} + \mathbf{C}_{12}) (\mathbf{C}_{11} + \mathbf{C}_{12} + \mathbf{C}_{21} + \mathbf{C}_{22})^{-1} (\bar{\mathbf{z}} - \bar{\mu}_1 - \bar{\mu}_2)$$

$$\text{cov}(\bar{\mathbf{x}}_1 | \bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2 = \bar{\mathbf{z}}) = \mathbf{C}_{11} - (\mathbf{C}_{11} + \mathbf{C}_{12}) (\mathbf{C}_{11} + \mathbf{C}_{12} + \mathbf{C}_{21} + \mathbf{C}_{22})^{-1} (\mathbf{C}_{11} + \mathbf{C}_{21})$$

Further, if $\mathbf{C}_{12} = \mathbf{0}$ then $\text{cov}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2 | \bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2 = \bar{\mathbf{z}}) = \text{cov}(\bar{\mathbf{x}}_1) + \text{cov}(\bar{\mathbf{x}}_2)$. Similarly, if $\bar{\mathbf{z}} = \mathbf{A}\bar{\mathbf{x}}_1$ then $E(\bar{\mathbf{z}}) = \mathbf{A}\bar{\mu}_1$ and $\text{cov}(\bar{\mathbf{z}}) = \mathbf{A}\mathbf{C}_{11}\mathbf{A}^T$.

Chains of inferences: Given $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ are multivariable Gaussian random vectors with

$$\vec{\mathbf{a}} = \mathbf{M}_{ab} \vec{\mathbf{b}} + \vec{\mathbf{a}}_0, \quad \mathbf{C}_{a|b} = \text{cov}(\vec{\mathbf{a}} | \vec{\mathbf{b}})$$

$$\vec{\mathbf{b}} = \mathbf{M}_{bc} \vec{\mathbf{c}} + \vec{\mathbf{b}}_0, \quad \mathbf{C}_{b|c} = \text{cov}(\vec{\mathbf{b}} | \vec{\mathbf{c}})$$

Then

$$E(\vec{\mathbf{a}} | \vec{\mathbf{c}}) = \mathbf{M}_{ab} \mathbf{M}_{bc} E(\vec{\mathbf{c}}) + \mathbf{M}_{ab} \vec{\mathbf{b}}_0 + \vec{\mathbf{a}}_0$$

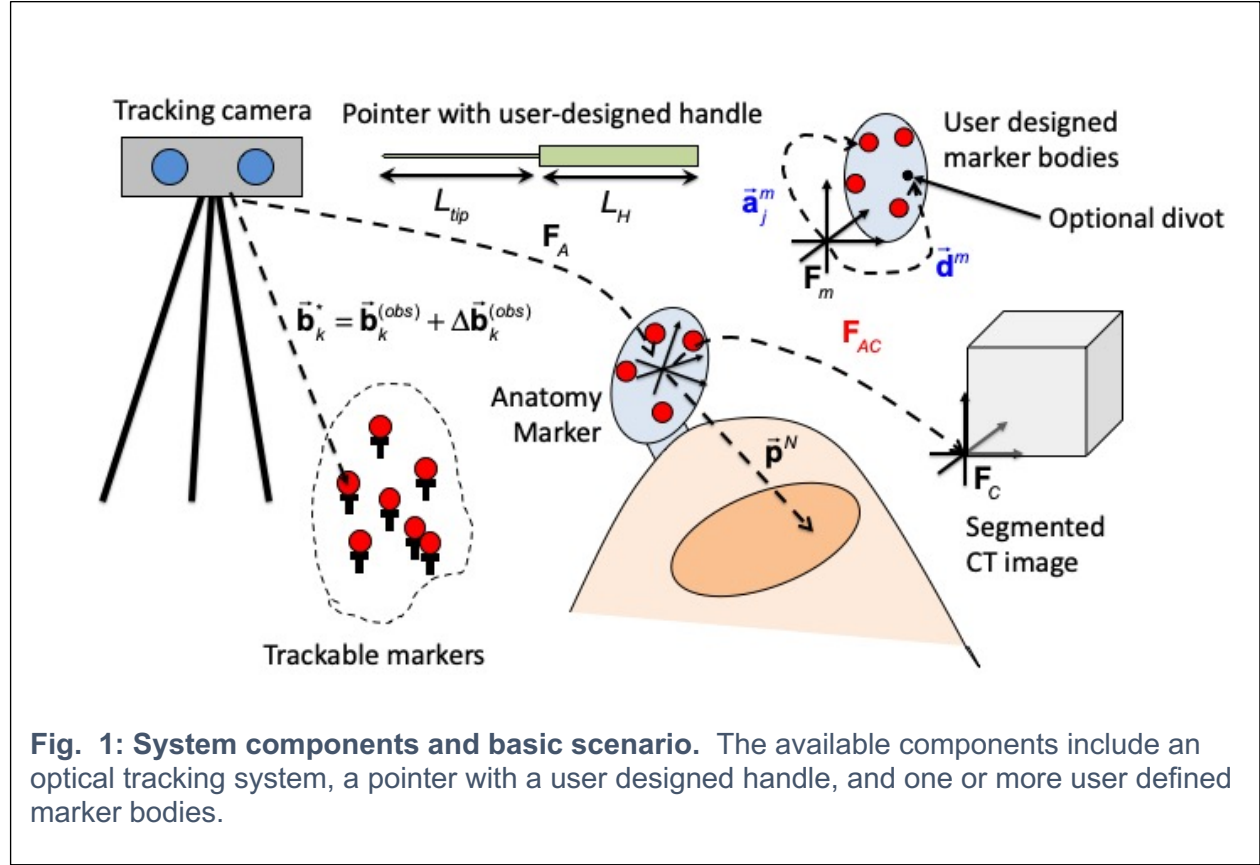
$$\text{cov}(\vec{\mathbf{a}} | \vec{\mathbf{c}}) = \mathbf{C}_{a|b} + \mathbf{M}_{ab} \mathbf{C}_{b|c} \mathbf{M}_{ab}^T$$

Projections: Given a unit vector $\vec{\mathbf{d}}$ and random vector $\bar{\mathbf{x}} \sim N(\bar{\mu}, \mathbf{C})$ then $\vec{\mathbf{d}} \cdot \bar{\mathbf{x}} \sim N(\vec{\mathbf{d}} \cdot \bar{\mu}, \vec{\mathbf{d}}^T \mathbf{C} \vec{\mathbf{d}})$.

Also, $\vec{\mathbf{d}} \cdot \bar{\mathbf{x}} \leq \lambda_{\max}$ where $\lambda_{\max}$ is the largest eigen value of $\mathbf{C}$.

Estimation: Given multivariate Gaussian random variables $\vec{\mathbf{X}}$ and $\vec{\mathbf{B}}$ with $\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{B}}$ and $\vec{\mathbf{B}} \sim N(\bar{\mu}_B, \mathbf{C}_B)$, then $\vec{\mathbf{X}} \sim N \left(\mathbf{A}^+ \bar{\mu}_B, \mathbf{A}^+ \mathbf{C}_B (\mathbf{A}^+)^T \right)$ where $\mathbf{A}^+ = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ is the Moore-Penrose pseudo-inverse of $\mathbf{A}$.

Scenario



Consider a surgical navigation with the components shown in Fig. 1, above. These include:

- A tripod-mounted tracking camera mounted on a tripod capable of tracking multiple spherical markers, which may be mounted on various objects. For each marker k located at position $\vec{b}_k^* = \vec{b}_k^{(obs)} + \Delta\vec{b}_k^{(obs)}$, $\vec{b}_k^{(obs)}$ is the value reported by the tracking system to the computer and $\Delta\vec{b}_k^{(obs)}$ is a multivariate Gaussian with $\Delta\vec{b}_k^{(obs)} \sim N(\vec{0}, \mathbf{C}_b)$ and $\text{cov}(\Delta\vec{b}_i^{(obs)}, \Delta\vec{b}_j^{(obs)}) = \mathbf{0}_{3 \times 3}$ for $i \neq j$.
- One or more user-defined marker bodies, which can be of different sizes and designs. Each marker may contain several marker spheres and an optional small “divot” that can be used for pivot calibrations or similar purposes. The markers and divot positions ($\vec{a}_j^m$ and $\vec{d}^m$) relative to the marker body coordinate system are assumed to be known to high precision.
- A pointer with a user-designed handle, which may contain one or more tracking spheres at known positions $\vec{a}_j^{(ptr)}$ in the pointer coordinate system. The pointer shaft is a thin rod with a sharp tip located at position $[0, 0, -L_{tip}]^T$ in pointer body coordinates, and $z_h \geq 0$ for every point $[x_h, y_h, z_h]^T$ on the pointer handle. Nothing may be attached to the pointer shaft other than the pointer handle. The handle itself can be any shape with overall length L_H .

- Software is available to locate tracking spheres in the CT and to segment the surface of the anatomy, as well as internal structures. If $\vec{\mathbf{s}}_k$ is the reported position of any sphere in CT coordinates, then the actual position of the sphere will be $\vec{\mathbf{s}}_k^* = \vec{\mathbf{s}}_k + \Delta\vec{\mathbf{s}}_k$ with $\vec{\mathbf{s}}_k \sim N(\vec{\mathbf{0}}, \mathbf{C}_s)$ where $\mathbf{C}_s = \text{diag}([\sigma_s^2, \sigma_s^2, \sigma_s^2])$.

Workflow

The approximate workflow for the system is as follows:

Step 1. Attach a marker body with coordinate system $\mathbf{F}_A$ to the anatomy so that for every point in the volume of interest for navigation whose position is $\vec{\mathbf{p}}^N$ relative to $\mathbf{F}_A$ has

$$\|\vec{\mathbf{p}}^N\| \leq D_{\max}.$$

Step 2. A CT scan of the patient is performed and the image is segmented to compute the positions $\vec{\mathbf{s}}_j^A$ of the spheres in the CT volume corresponding to corresponding to $\vec{\mathbf{a}}_j^A$ on reference body A. Use these values to compute a transformation $\mathbf{F}_{AC}$ so that

$$\mathbf{F}_{AC} \vec{\mathbf{a}}_j^A = \vec{\mathbf{s}}_j^A$$

Step 3. In the operating room, the following steps are performed repeatedly

- The tracking camera locates all visible marker spheres.
- The positions $\vec{\mathbf{b}}_j^A$ of the marker spheres on marker body A are used to compute a transformation $\mathbf{F}_A$ such that $\mathbf{F}_A \vec{\mathbf{a}}_j^A \approx \vec{\mathbf{b}}_j^A$ and a transformation $\mathbf{F}_{AC}$ so that $\vec{\mathbf{s}}_j^A \approx \mathbf{F}_{AC} \vec{\mathbf{a}}_j^A$. (Note the use of the “ $\approx$ ” to indicate that these are approximate (most likely) solutions.

Note: There was a typo here that made this step confusing. The point is that $\vec{\mathbf{b}}_j^A = \mathbf{F}_A \vec{\mathbf{a}}_j^A = \mathbf{F}_A \mathbf{F}_{AC}^{-1} \vec{\mathbf{s}}_j^A$ refer to the same expression. So $\vec{\mathbf{s}}_j^A \approx \mathbf{F}_{AC} \vec{\mathbf{a}}_j^A$. The typo had $\vec{\mathbf{b}}_j^A \approx \mathbf{F}_{AC} \vec{\mathbf{s}}_j^A$.

- Use the observed positions $\vec{\mathbf{b}}_k^{(ptr)}$ to compute the transformation $\mathbf{F}_{ptr}$ such that

$$\mathbf{F}_{ptr} \vec{\mathbf{a}}_k^{(ptr)} \approx \vec{\mathbf{b}}_k^{(ptr)}$$

- Compute the most likely position $\vec{\mathbf{p}}_{nav}$ in CT coordinates corresponding to the position of the pointer tip, where $\vec{\mathbf{p}}_{nav}^* = \vec{\mathbf{p}}_{nav} + \Delta\vec{\mathbf{p}}_{nav}$ and $\|\Delta\vec{\mathbf{p}}_{nav}\| \sim N(0, \sigma_{nav})$
- Use this value to update a navigational display.

Design Problem

Your design tasks are

- Specify the number and positions $\vec{\mathbf{a}}_k^{ptr}$ of the markers that are to be placed on the pointer handle.

- Specify the number and positions $\vec{\mathbf{a}}_k^A$ of the markers that are to be placed on the anatomy marker body.
- Ensure that the required navigational accuracy is obtained

In designing the handle and anatomy marker body, it is important to keep their overall dimensions as small as possible, consistent with the accuracy requirements.

Question 1 (Preliminaries)

NOTE: This question involves symbolic reasoning. You may use an AI-based mathematical system in answering these questions, but you need to clearly state how you formulated the questions(s) provided to the AI system and make your reasoning plain. Also, you should check the output of any AI system carefully. I used ChatGPT in answering these questions and had to go back repeatedly to guide the system to find correct formulas. I then redid the answers in the old-fashioned way. This proved less frustrating and also let me do some significant simplification.

- A. (4 points) Give an expression for $\bar{\mathbf{p}}_{nav}$ in terms of $\mathbf{F}_{ptr}, \mathbf{F}_A, \mathbf{F}_{AC}$
- B. (4 points) Give a linearized expression for $\Delta \bar{\mathbf{p}}_{nav} = \mathbf{p}_{nav}^* - \bar{\mathbf{p}}_{nav}$ where $\Delta \mathbf{F}_{ptr} = [\mathbf{I} + sk(\bar{\alpha}_{ptr}), \bar{\epsilon}_{ptr}]$, $\Delta \mathbf{F}_A = [\mathbf{I} + sk(\bar{\alpha}_A), \bar{\epsilon}_A]$, and $\Delta \mathbf{F}_{AC} = [\mathbf{I} + sk(\bar{\alpha}_{AC}), \bar{\epsilon}_{AC}]$. Express your answer in normalized form as $\Delta \bar{\mathbf{p}}_{nav} = \mathbf{M}_{ptr} \bar{\eta}_{ptr} + \mathbf{M}_A \bar{\eta}_A + \mathbf{M}_{AC} \bar{\eta}_{AC}$. I.e., give expressions for $\mathbf{M}_{ptr}, \mathbf{M}_A, \mathbf{M}_{AC}$, where $\bar{\eta} = [\bar{\alpha}^T, \bar{\epsilon}^T]^T$.
- C. (4 points) Produce a linearized system of equations for estimating $\Delta \mathbf{F}_{ptr} \approx [\mathbf{I} + sk(\bar{\alpha}_{ptr}), \bar{\epsilon}_{ptr}]$ in terms of $\mathbf{F}_{ptr} = [\mathbf{R}_{ptr}, \bar{\mathbf{p}}_{ptr}]$ and the $[\bar{\mathbf{a}}_k^{ptr}, \Delta \bar{\mathbf{b}}_k^{ptr}]$ pairs. The system will have the general form $\mathbf{H}^{(ptr)} \bar{\eta}_{ptr} = \bar{\mathbf{B}}^{(ptr)}$ where $\bar{\eta}_{ptr} = [\bar{\alpha}_{ptr}^T, \bar{\epsilon}_{ptr}^T]^T$; $\mathbf{H}^{(ptr)}$ and $\bar{\mathbf{B}}^{(ptr)}$ have rows of the form $\mathbf{H}_k^{(ptr)} \bar{\eta} \approx \Delta \bar{\mathbf{b}}_k^{ptr} = \bar{\mathbf{b}}_k^{(obs)} - \mathbf{F}_{ptr} \bar{\mathbf{a}}_k^{ptr}$ corresponding to $\mathbf{F}_{ptr}$ and each $[\bar{\mathbf{a}}_k^{ptr}, \Delta \bar{\mathbf{b}}_k^{ptr}]$ pair.
- D. (4 points) If we treat $\bar{\eta}_{ptr} \sim N(\bar{\mu}_{ptr}, \mathbf{C}_{ptr})$ as a random Gaussian 6-vector, produce formulas for $\bar{\mu}_{ptr} = Ex(\bar{\eta}_{ptr})$ and $cov(\bar{\eta}_{ptr})$ based on your answer to Question 1C.
- E. (4 points) Produce a linearized formula for $\Delta \bar{\mathbf{q}} = \mathbf{F}_{ptr} \Delta \mathbf{F}_{ptr} \bar{\mathbf{p}}_{tip} - \mathbf{F}_{ptr} \bar{\mathbf{p}}_{tip}$ in terms of $\bar{\mathbf{p}}_{tip}, \mathbf{R}_{ptr}, \bar{\alpha}_{ptr}, \bar{\epsilon}_{ptr}$. Express your answer in normalized form as $\Delta \bar{\mathbf{q}} = \mathbf{H}_q \bar{\eta}_{ptr}$, where $\mathbf{H}_q$ has elements based on $\mathbf{F}_{ptr}$ and $\bar{\mathbf{p}}_{tip}$.
- F. (4 points) Produce an estimate for the covariance $\mathbf{C}_q = cov(\Delta \bar{\mathbf{q}})$ based on your answers to Questions 1D and 1E.
- G. (4 points) Following the process outlined in Questions 1C and 1D, provide an estimate for $\mathbf{C}_A = cov(\bar{\eta}_A)$ where $\bar{\eta}_A = [\bar{\alpha}_A^T, \bar{\epsilon}_A^T]^T$ and $\mathbf{C}_b = cov(\Delta \bar{\mathbf{b}}_k)$.
- H. (4 points) Following the steps outlined in Questions 1C and 1D, provide an estimate for $\mathbf{C}_{AC} = cov(\bar{\eta}_{AC})$ where $\bar{\eta}_{AC} = [\bar{\alpha}_{AC}^T, \bar{\epsilon}_{AC}^T]^T$ and $\mathbf{C}_{\Delta \bar{\mathbf{s}}} = cov(\bar{\sigma}_{\Delta \bar{\mathbf{s}}})$ in terms of $\mathbf{F}_{AC}$ and the $[\bar{\mathbf{a}}_k^A, \bar{\mathbf{s}}_k]$ pairs.
- I. (10 points) Now, provide a formula for $\mathbf{C}_{nav} = cov(\Delta \bar{\mathbf{p}}_{nav})$ in terms of $\mathbf{C}_{ptr}, \mathbf{C}_A, \mathbf{C}_{AC}$ (**Note:** Fixed typo. This was $\mathbf{C}_{ptr}, \mathbf{C}_A, \mathbf{C}_b$).

- J. (10 points) Now, provide formulas for $\mathbf{C}_{ptr}$, $\mathbf{C}_A$, $\mathbf{C}_{AC}$ if $\mathbf{C}_b = \text{diag}([\sigma_b^2, \sigma_b^2, \sigma_b^2])$. Note that this will produce significant simplification. **Hint:** You should work the algebra out and do some appropriate simplifications in terms of the R's. Question 2 makes this assumption about $\mathbf{C}_b$ (i.e., the errors are uniform, Gaussian, and independent.). So getting a simpler formula will help you there.

Question 2 (Design)

You are given the following specifications:

Tracking system:

- Update rate: 120 Hz
- Accuracy: $\sigma_b = 0.15$ mm

$$\mathbf{C}_b = \text{diag}([\sigma_b^2, \sigma_b^2, \sigma_b^2])$$

for $\vec{b}$ in the viewing pyramid shown in Fig. 2.

Pointer:

- $L_{tip} = 100$ mm

Navigation system:

- Required navigational accuracy
 $\sigma_{nav} \leq 2$ mm,

Segmentation software:

- $\sigma_s = 0.1$ mm

Navigation requirement:

- $\sigma_{nav} \leq 2$ mm for $D_{nav} = \text{dist of tool from } \vec{p}_A \leq 100$ mm
- $\sigma_q^{\max} = 0.5$ mm

In the design questions below, you may use a computer program to evaluate the formulas you developed in Question 1, but you should otherwise describe your design process, and **you must clearly disclose any use that you make of any AI tools or other aids that you use to write any programs that you use**. If you have questions, you can ask me or the TA over Piazza or after class.

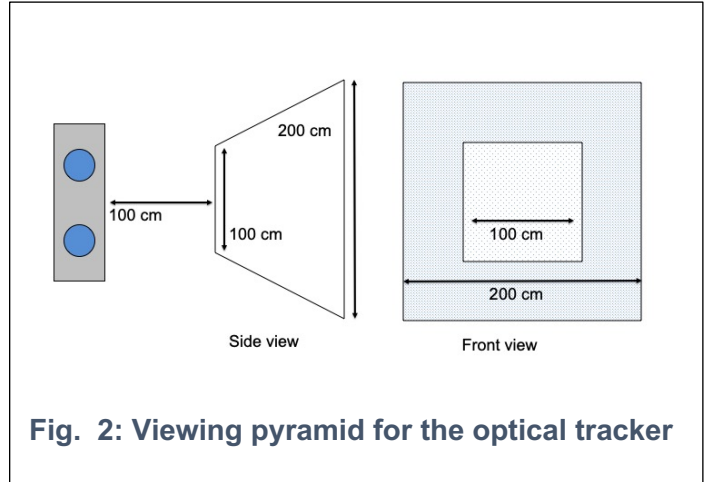
One possibility is based on the use a “spreadsheet” approach in which you use simple programs to compute numerical values for the covariances based on your marker positions and then fiddle with the designs and $\vec{a}_k^{ptr}$ and $\vec{a}_k^A$ values until you have a satisfactory solution.

In the questions below, you may assume that

$$\mathbf{F}_{ptr} = \text{Frame}\left(\text{Rot}(\vec{x}, \pi/4) \text{Rot}(\vec{y}, -\pi/6) \text{Rot}(\vec{z}, -\pi/6), [1500, 200, 300]^T\right)$$

$$\mathbf{F}_A = \text{Frame}\left(\text{Rot}(\vec{y}, \pi/6), \text{Rot}(\vec{z}, \pi/6), \mathbf{F}_{ptr} \cdot [50, 50, -50]\right)$$

when finding the eigen values of the covariance matrices for putting into your pdf answers. However, in addition we want you to submit an additional text file containing your design parameters in the following format:



$$\begin{array}{c}
A, \mathbf{a}_{1,x}^A, \mathbf{a}_{1,y}^A, \mathbf{a}_{1,z}^A \\
\vdots \\
A, \mathbf{a}_{N_A,1,x}^A, \mathbf{a}_{N_A,1,y}^A, \mathbf{a}_{N_A,1,z}^A \\
P, \mathbf{a}_{1,x}^{ptr}, \mathbf{a}_{1,y}^{ptr}, \mathbf{a}_{1,z}^{ptr} \\
\vdots \\
P, \mathbf{a}_{N_{ptr},x}^{ptr}, \mathbf{a}_{N_{ptr},y}^{ptr}, \mathbf{a}_{N_{ptr},y}^{ptr}
\end{array}$$

Please also include this information in your written report as well.

Note that nominal values of the $\vec{\mathbf{s}}_k$ can be computed from $\vec{\mathbf{s}}_k = \mathbf{F}_{AC}^{-1} \vec{\mathbf{a}}_k^A = \vec{\mathbf{a}}_k^A$ for $\mathbf{F}_{AC} = \mathbf{I}$.

For the purposes of running test cases in Question 2, we will use $\mathbf{F}_{AC} = \mathbf{I}$ and $\mathbf{F}_{AC}^* = \Delta \mathbf{F}_{AC}$. This would correspond to a situation in which markers visible in both CT and visually are placed on the patient anatomy during scanning and remain there (or are replaced in the same place) during the procedure. I asked for more general formulas in Question 1 to cover some other possible use cases.

We will run a small program to evaluate the covariance matrices for a number of angles using your design parameters and will be examining the largest eigen value $\lambda_{nav}^{\max}$ of $\mathbf{C}_{nav}$ to ensure that $\lambda_{nav}^{\max} \leq \sigma_{nav}^2$ and the largest eigen value $\lambda_q^{\max}$ of $\mathbf{C}_q$ to ensure that $\lambda_q^{\max} \leq \sigma_q^{\max}$. You may consider doing the same thing in your design process.

- A. (15 points) Provide a sketch of your design for the probe handle, specify the positions $\vec{\mathbf{a}}_k^{ptr}$ for each marker in probe coordinates, and provide numerical values for the eigen values of $\mathbf{C}_{ptr} = \text{cov}(\vec{\eta}_{ptr})$ and $\mathbf{C}_q = \text{cov}(\mathbf{F}_{ptr}^* \vec{\mathbf{p}}_{tip} - \mathbf{F}_{ptr} \vec{\mathbf{p}}_{tip})$. **Note:** You will also compute numerical values for the covariance matrices, of course, but we don't need to see them.
- B. (15 points) Provide a sketch of your design for the anatomy marker body, specify the positions of the $\vec{\mathbf{a}}_k^A$, and provide a numerical value for the eigen values $\mathbf{C}_A$.
- C. (15 points) Provide numerical values for the eigen values of $\mathbf{C}_{AC}$.
- D. (15 points) Provide numerical values for the eigen values of $\mathbf{C}_{nav} = \text{cov}(\Delta \vec{\mathbf{p}}_{nav})$.

Question 3

Suppose that we know that $\Delta \vec{\mathbf{b}}_k^{(obs)} = \Delta \mathbf{b}_k^{sys} + \Delta \mathbf{b}_k^{random}$, where $\Delta \vec{\mathbf{b}}_k^{(sys)} \sim N(\vec{\mu}_b^{sys}, \mathbf{C}_b^{(sys)})$ and

$\Delta \vec{\mathbf{b}}_k^{(random)} \sim N(\vec{\mathbf{0}}, \mathbf{C}_b^{(random)})$ where $\mathbf{C}_b^{(sys)} = diag([\sigma_{sys}^2, \sigma_{sys}^2, \sigma_{sys}^2])$ and

$\mathbf{C}_b^{(random)} = diag([\sigma_{random}^2, \sigma_{random}^2, \sigma_{random}^2])$. Also, we have learned that the navigation loop needs to run at 30 Hz.

- A. (6 points) Describe a method to produce an improved estimate $\vec{\mathbf{b}}_k^{(new)}$ of $\vec{\mathbf{b}}_k^*$ during the navigation cycle. $\Delta \vec{\mathbf{b}}_k^{(new)} \sim N(diag([\sigma_{new}^2, \sigma_{new}^2, \sigma_{new}^2]))$.
- B. (6 points) Provide a value for σ_{new}^2 based on your answer to Question 3A.
- C. (6 points) Suppose that we know that $\sigma_{sys}^2 \ll \sigma_{random}^2$ but we do not know the value of $\vec{\mu}_b^{sys}$. How can we use this information to further improve our estimate of $\vec{\mathbf{b}}_k^{(new)}$ while still providing navigation updates at 30Hz?