Homework Assignment 4 – 600.455/655 Fall 2025

Name	Name
Email	Email
Other contact information (optional)	Other contact information (optional)
Signature (required) I/We have followed the rules in completing this assignment ______________________________	Signature (required) I/We have followed the rules in completing this assignment ______________________________

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone** or in **teams of two** and are not to discuss the problems with anyone other than the TAs or the instructor.
3. **IMPORTANT NOTE:** If you work in teams of two, you are **not** to split up the questions and each answer a subset individually. You are to work **together**. I encourage teaming on these problems because I believe that it encourages learning, not as a way to reduce the required work for students taking the course. By signing this sheet you are asserting that each of you has contributed equally to each answer and can individually explain the answer as well as if you had answered the question alone. I view this as a question of trust and ethics.
4. It is otherwise open book, notes, and web. But you should cite any references you consult.
5. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
6. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
7. Submit the assignment on GradeScope as a neat and legible PDF file. We will not insist on typesetting your answers, but we must be able to read them. We will not go to extraordinary lengths to decipher what you write. If the graders cannot make out an answer, the score will be 0.
8. Sign and hand in this page as the first sheet of your assignment. If you work with a partner, then you both should sign the sheet, but you should only submit one PDF file for both of you, using the GradeScope teaming feature. Indicate clearly who it is from.

This assignment has more than 100 points, but the most that will be applied to your grade is 100

Scenario: Robot Virtual Fixtures

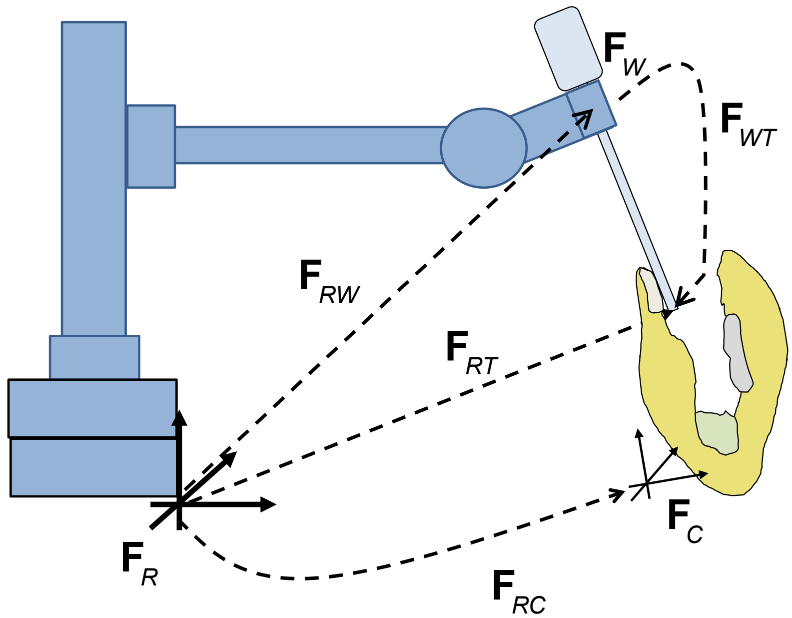

Fig. 1: Surgical robot, tool, and anatomy

Consider the situation shown in the figure above, in which a robot holds a surgical tool. The pose of the robot's wrist is given by $\mathbf{F}_{RW}(\vec{\mathbf{q}}) = [\mathbf{R}_{RW}(\vec{\mathbf{q}}), \vec{\mathbf{p}}_{RW}(\vec{\mathbf{q}})]$, where $\vec{\mathbf{q}}$ gives the positions of the robot's joints. The effect of a small incremental joint motion is $\mathbf{F}_{RW}(\vec{\mathbf{q}} + \Delta\mathbf{q}) = \mathbf{F}(\vec{\mathbf{q}})\Delta\mathbf{F}_{RW}(\vec{\mathbf{q}}, \Delta\vec{\mathbf{q}})$, where $\Delta\mathbf{F}_{RW}(\vec{\mathbf{q}}, \Delta\vec{\mathbf{q}}) \approx \Delta\mathbf{F}_{RW}(\vec{\eta}_{RW}) = [\mathbf{I} + sk(\vec{\alpha}_{RW}), \vec{\varepsilon}_{RW}]$ where $\vec{\eta}_{RW} = [\vec{\alpha}_{RW}, \vec{\varepsilon}_{RW}]^T = \mathbf{J}_{RW}(\vec{\mathbf{q}})\Delta\vec{\mathbf{q}}$.

The surgical tool coordinate system is located at the end of the surgical tool, which is attached to the robot's wrist to that the pose of the tool relative to the robot is $\mathbf{F}_{WT}$. The tool has a long shaft with radius ρ_{tool} that can be inserted into the patient's body. The robot is equipped with a force sensor that is capable of measuring forces and torques $\vec{\zeta}_W = [\vec{\phi}_W^T, \vec{\tau}_W^T]^T$ in the coordinate system $\mathbf{F}_W$ of the robots wrist. **NOTE:** Force-torque vectors like $\vec{\zeta}_W$ are often referred to as "wrenches".

The midlevel controller for the robot executes the following algorithm every Δt seconds:

Step 1: Read the robot state $[\vec{\mathbf{q}}, \dot{\vec{\mathbf{q}}}, \vec{\zeta}_W]$ and perform basic safety checks. Output the previous velocity command $\dot{\vec{\mathbf{q}}}_{cmd}(t - \Delta t)$. Compute $\mathbf{J}_W(\vec{\mathbf{q}})$.

Step 2: $t \leftarrow t + \Delta t$; Define $S(t) = [\vec{\mathbf{q}}, \dot{\vec{\mathbf{q}}}, \mathbf{F}_W, \mathbf{J}_W, \vec{\zeta}_W]$

Step 3: Formulate and solve an optimization problem

$$\Delta \vec{\mathbf{q}}_{cmd} = \arg\min \sum_i w_i E_i(S(t), \Delta \vec{\mathbf{q}}_{cmd}, \vec{\eta}_{cmd}, \mathbf{G}_i, \cdots, \kappa_{i,j}, \cdots)^2$$

Subject to $\mathbf{A}_q \Delta \vec{\mathbf{q}}_{cmd} \leq \vec{\mathbf{b}}_q$ and other equality and inequality constraints

where $\mathbf{G}_i, \kappa_{i,j}$ are data and parameters associated with the problem and may vary with $S(t)$ and the w_i are weights associated with how important each term of the objective function is to the desired behavior. If only one term is present, then $w_1 = 1$. It may also be convenient to compute a Jacobean $\mathbf{J}_{RT}(\vec{\mathbf{q}})$ and add equality constraints such as $\vec{\eta}_{RT}^{cmd} = \mathbf{J}_{RT}(\vec{\mathbf{q}}) \Delta \vec{\mathbf{q}}_{cmd}$ and inequality constraints expressed in terms of $\vec{\eta}_{WT}^{cmd}$. Also, although the form of the objective function terms $E_i(\cdots)$ described above is very general, they typically take a form that looks like

$$E_i(\cdots)^2 = \left\| \vec{\eta}_{something}^{desired} - \mathbf{J}_{something}(\vec{\mathbf{q}}) \Delta \vec{\mathbf{q}}_{cmd} \right\|^2$$

where "something" represents some coordinate system and $\vec{\eta}_{something}^{desired}$ is some expression representing a desired incremental motion.

Step 4: $\dot{\vec{\mathbf{q}}}_{cmd}(t) \leftarrow \Delta \vec{\mathbf{q}}_{cmd} / \Delta t$

Step 5: Go to sleep until the next time interval.

The mechanical design of the robot has the following limits: $\vec{\mathbf{q}}_{\min} \leq \vec{\mathbf{q}} \leq \vec{\mathbf{q}}_{\max}$, $\left|\dot{\vec{\mathbf{q}}}\right| \leq \dot{\vec{\mathbf{q}}}_{\max}$, $\left|\ddot{\vec{\mathbf{q}}}\right| \leq \ddot{\vec{\mathbf{q}}}_{\max}$, where $\dot{\vec{\mathbf{q}}}$ and $\ddot{\vec{\mathbf{q}}}$ are the joint velocity and acceleration, respectively.

Presurgical planning has been done based on a segmented CT image, and a registration step has been performed such that a position $\vec{\mathbf{p}}_{CT}$ in CT coordinates corresponds to a position $\mathbf{F}_{RC}(t)\vec{\mathbf{p}}_{CT}$ in robot coordinates. However, $\mathbf{F}_{RC}$ is based on a tracked reference body that may vary slowly during the procedure. Four anatomic structures, labeled as A, B, C, and D, have been identified. A fifth label "E", indicates free space. Initially, the pose of the surgical tool relative to the robot is $\mathbf{F}_{RT}^{init}$, and a safe starting pose $\mathbf{F}_{CT}^{start}$ for the tool relative to $\mathbf{F}_C$ has been determined.

The system has software that can return properties associated with every 3D voxel $\vec{\mathbf{c}}$ in the CT image. These properties can be retrieved via a function $propvalue \leftarrow GetMap(\vec{\mathbf{c}}, "name")$. The *Get* function is capable of interpolating values appropriately if $\vec{\mathbf{c}}$ is a real vector. An additional function $PutMap(\vec{\mathbf{c}}, "name", value)$ is available to store additional properties for subsequent retrieval. The following properties have been computed and stored by the planning software:

- Tissue_Type – returns A, B, C, D, or E
- *SDF_A, SDF_B, SDF_C, SDF_D* – The signed distance field values for anatomic structures A, B, C, D
- *Grad_A, Grad_B, Grad_C, Grad_D* – The SDF gradients for anatomic structures A, B, C, D
- Closest_A, Closest_B, Closest_C, Closest_D – the coordinates of the closest point to A, B, C, D

The software also includes a function $\vec{\mathbf{g}} \leftarrow GetGrad(\vec{\mathbf{c}}, SDF_X)$ for computing the gradient at a point $\vec{\mathbf{c}}$ in a signed distance field X.

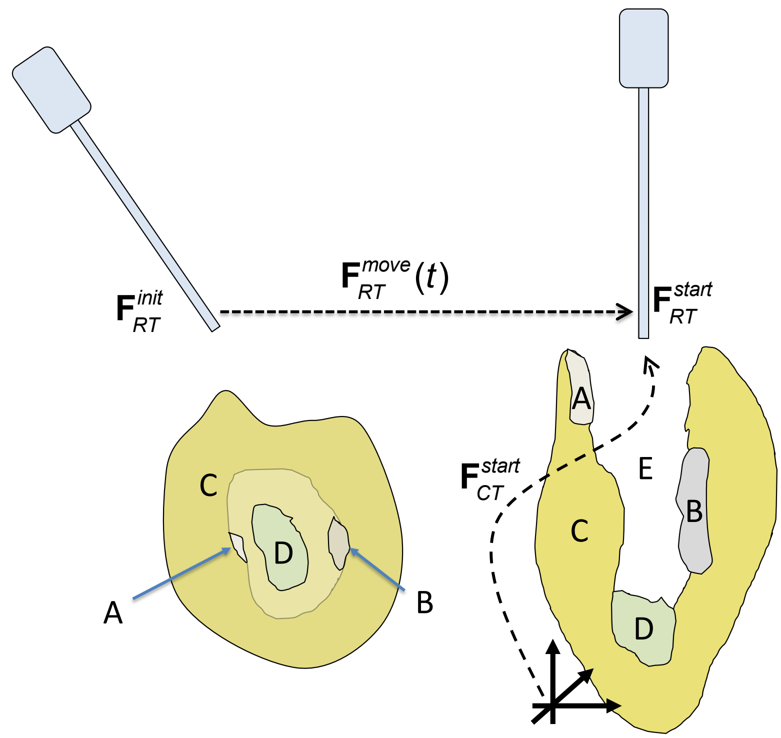

Fig. 2: Labeled anatomy and approach path

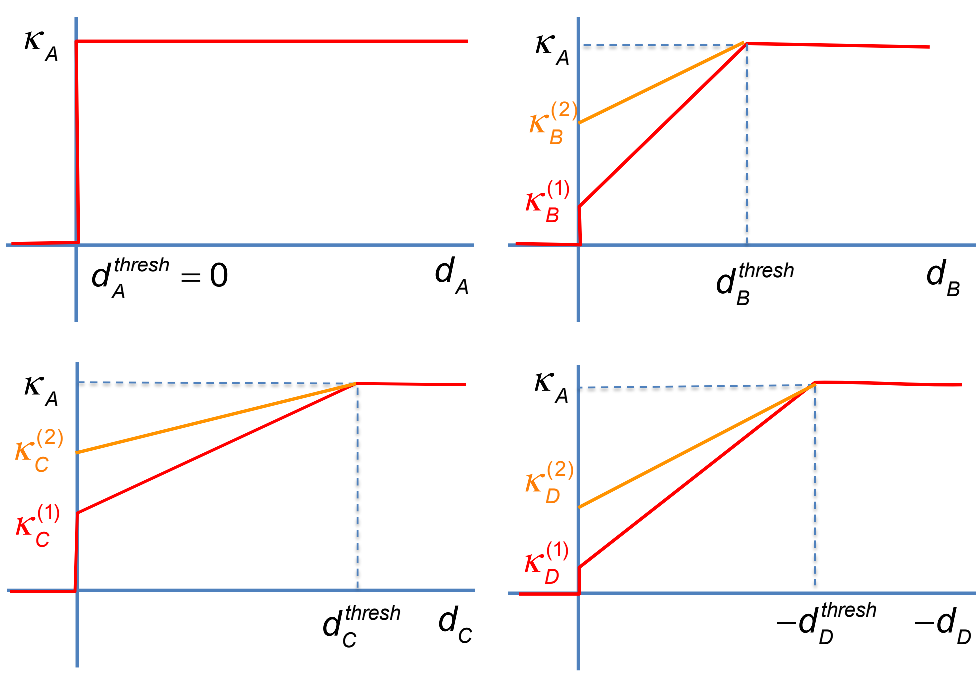

Fig. 3: Distance-based virtual fixture gains (explained below)

Questions

NOTE: For the purposes of this assignment, you may assume that $\mathbf{F}^*_{RC} = \mathbf{F}_{RC}\Delta\mathbf{F}_{RC}(\vec{\eta}_{RC}) \approx \mathbf{F}_{RC} \bullet [\mathbf{I} + sk(\vec{\alpha}_{RC}), \vec{\varepsilon}_{RC}]$ and that $\mathbf{F}_{RC}(t)$ changes only slowly for all questions. Also, you may find it useful to consult the notes on the admittance-style SDF-based virtual fixtures discussed in class, which are relevant for many of the questions.

1. (5 points) Explain why it is desirable to output $\vec{\mathbf{q}}_{cmd}(t)$ in Step 1 of the midlevel control loop rather than in Step 4.
2. (5 points) Describe constraints to be added to Step 3 of the midlevel control loop to guarantee that the joint position, velocity, and acceleration constraints will not be violated.
3. (10 points) Give an expression for $\vec{\eta}_{RT}(\vec{\mathbf{q}}, \Delta\vec{\mathbf{q}}) = [\vec{\alpha}_{RT}, \vec{\varepsilon}_{RT}]$ and a corresponding Jacobean $\mathbf{J}_{RT}(\vec{\mathbf{q}}, \Delta\vec{\mathbf{q}})$ in terms of $\vec{\eta}_{RW}$ and other quantities in the problem scenario, such that $\mathbf{F}_{RT}(\vec{\mathbf{q}} + \Delta\vec{\mathbf{q}}) = \mathbf{F}(\vec{\mathbf{q}})\Delta\mathbf{F}_{RT}(\vec{\eta}_{RT})$ and $\vec{\eta}_{RT} \approx \mathbf{J}_{RT}(\vec{\mathbf{q}})\Delta\vec{\mathbf{q}}$. **Strong Hint (you will need this later on):** Find the formula for $\mathbf{J}_{WT}(\vec{\mathbf{q}})$ such that $\vec{\eta}_{RT} = \mathbf{J}_{WT}(\vec{\mathbf{q}})\vec{\eta}_{RW}$. Then $\mathbf{J}_{RT}(\vec{\mathbf{q}}) = \mathbf{J}_{WT}(\vec{\mathbf{q}})\mathbf{J}_{RW}(\vec{\mathbf{q}})$.
4. (5 points) Suppose one wishes to limit the rate of change of the tool pose such that $\left|\vec{\eta}_{RT}\right| \le \vec{\eta}^{\max}_{RT}$. Describe a set of linear constraints that would accomplish this in terms of $\vec{\eta}_{RW}$ and $\mathbf{F}_{RW}(\vec{\mathbf{q}})$.
5. (10 points) Suppose that the robot is located at an initial pose $\mathbf{F}^{init}_{RW}$, so that the tool is located at $\mathbf{F}^{init}_{RT} = \mathbf{F}^{init}_{RW}\mathbf{F}_{WT}$. The desired start pose $\mathbf{F}^{start}_{RT}$ is (of course) $\mathbf{F}^{start}_{RT} = \mathbf{F}_{RC}\mathbf{F}^{start}_{CT}$ and $\mathbf{F}^{init}_{CT} = \mathbf{F}^{-1}_{RC}\mathbf{F}^{init}_{RT}$ Provide details for Step 3 of the midlevel controller that will cause the tool pose $\mathbf{F}^{move}_{RT}(t)$ to move in a nearly uniform motion in CT coordinates. The desired motion will be

$$\mathbf{F}^{move}_{RT}(t) \approx \left[Rot\left(\vec{\mathbf{a}}^{move}_{RT}(t), \theta^{move}_{RT}(t)\right), \vec{\mathbf{p}}^{move}_{RT}(t) \right]$$

   where $\left\|\dot{\vec{\mathbf{p}}}^{move}_{RT}(t)\right\| \approx v_{move} \le v_{\max}$, $\left|\dot{\theta}^{move}_{RT}(t)\right| \approx \omega_{move} \le \omega_{\max}$ and $\vec{\mathbf{a}}^{move}_{RT}(t)$, $\dot{\theta}^{move}_{RT}(t)$, and $\dot{\vec{\mathbf{p}}}^{move}_{RT}(t)$ all change only slowly. N**OTE:** Your answer should include formulas and appropriate pseudocode, possibly including conditional statements. **HINT:** Remember that $\mathbf{F}_{RC}$ may be changing over time.
6. (5 points) Suppose that the robot is kinematically deficient (e.g., it has only 5 actuated joints). In this case, the robot might never be able to reach its specified destination pose. How would you modify your answer to Question 4 to ensure that the robot gets as close as possible to its intended pose but that the motion terminates when this state is reached.
7. (5 points) Suppose that the robot is kinematically redundant (e.g., it has 7 or 8 joints). How would you modify your answer to Question 4 to ensure that the robot's joints stay as close to their midpoint as possible while (more importantly) staying as close as possible to the desired straight-line path?
8. (15 points) Suppose that hand-over-hand cooperative control of the robot is desired. Describe an implementation of midlevel controller Step 3 that will yield the behavior $\dot{\vec{\alpha}}_{RW} = \mathbf{K}^{rot}_{3x3}\vec{\tau}_W / \Delta t$ and $\dot{\vec{\varepsilon}}_{RW} = \mathbf{K}^{vel}_{3x3}\vec{\phi}_W / \Delta t$, where Δt is the sample interval of the midlevel controller, subject to the constraints on joint position, velocity, and acceleration constraints from Question 2. **NOTE:** Your answer should include formulas and appropriate pseudocode, possibly including conditional statements. Your answer should be expressed in terms of a combined admittance law

$$h\,\dot{\vec{\eta}}_{RW} = \mathbf{K}_{RW}\vec{\zeta}_W / \Delta t = \begin{bmatrix} \mathbf{K}^{rot}_{3x3} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^{vel}_{3x3} \end{bmatrix} \begin{bmatrix} \vec{\tau}_W \\ \vec{\phi}_W \end{bmatrix} / \Delta t$$

This form will allow for more sophisticated admittance behavior that may be desired in subsequent questions. Note also that a commanded incremental motion $\dot{\vec{\eta}}^{cmd}_{RW}\Delta t$ corresponds to a commanded pose change $\Delta\mathbf{F}^{cmd}_{RW} = [\mathbf{I} + sk(\vec{\alpha}^{cmd}_{RW}), \vec{\varepsilon}^{cmd}_{RW}]$. Also, note that we have described an admittance controller with admittance gains $\mathbf{K}_{RW}$.

9. (15 points) The cooperative control scheme described in Question 8 provides more or less unconstrained motion of the robot, subject only to joint constraints. However, we also want to constrain the tool motion with respect to anatomic structures A, B, and C. Fig. 3 describes a set of admittance gain parameters associated with anatomic structures A, B, C, and D. Here, κ_X is a 3x3 admittance matrix associated with anatomic structure X and $d_X(t)$ is the distance of $\vec{\mathbf{p}}_{RT}(t)$ of the tool tip from structure X. Further, $\left\|\mathbf{K}^{vel}\vec{\phi}_W\right\| = \kappa_A\left\|\vec{\phi}_W\right\|$ for unconstrained admittance control. The desired behavior is that the motion $\dot{\vec{\mathbf{p}}}_{CT}(t)$ should be constrained is as follows:

   - If $d_X(t) \le d^{thresh}_X$ then the maximum admittance gain in the $-\nabla d_X$ direction is an interpolated value of $\kappa^{(1)}_X$. The maximum gain in other directions is an interpolated value of $\kappa^{(2)}_X$.
   - If $d_X(t) < 0$, then we also must require $\dot{\vec{\mathbf{p}}}^{cmd}_{CT} \bullet \nabla d_X > 0$.

   Describe how you would modify the mid-level controller to enforce these constraints concurrently for structures A, B, and C.

10. (15 points) How would you modify your answer to Question 9 in order to enforce the constraints of Question 9 for the portions of the tool shaft within some distance L_{insert} of the tool tip? Here, you do not need to repeat all the steps of Question. 9. Just explain what you would add.

11. (15 points) Suppose not that $\mathbf{F}^*_{RC}$ is fixed, but that the computer's value $\mathbf{F}_{RC}$ has some error, so that $\mathbf{F}^*_{RC} = \mathbf{F}_{RC}\Delta\mathbf{F}_{RC}(\vec{\eta}_{RC})$, where $\mathrm{cov}(\vec{\eta}_{RC}) = \mathbf{C}_{RC}$. How might you modify your answer to Question 9 in order to ensure that any part of the tool tip does not touch anatomic structures A,B, and C?

12. (15 points) Suppose that the tool is equipped with a proximity sensor that can return the distance δ to the closest point on the anatomy, so long as $d_X(\vec{\mathbf{p}}_{CT}) > d^{thresh}$ for X=A,B,C,D. The sensor cannot know which anatomic object it is seeing, and it is subject to some error, with $\mathrm{var}(\delta) = \sigma^2_\delta$, For present purposes, we can assume that the robot is very accurate, so that $\mathrm{cov}(\vec{\mathbf{p}}_{CT}) \approx \mathbf{0}_{3x3}$. Describe how you might use this sensor to improve your value for $\mathbf{F}_{RC}$, I.e., update your estimated value for $\mathbf{F}_{RC}$ and the value for $\mathrm{cov}(\vec{\eta}_{RC})$.

13. (15 points) Now suppose that $\mathrm{cov}(\vec{\eta}_{RC})$ has been reduced to a negligible value, and that the tool is now approaching the anatomic structure D. At this point, the goal is to use the tool to perform an ablative procedure inside D without violating the boundaries of structure D, except to move in and out of open space E. Describe how to modify your answer to Question 9 in order to accomplish this.