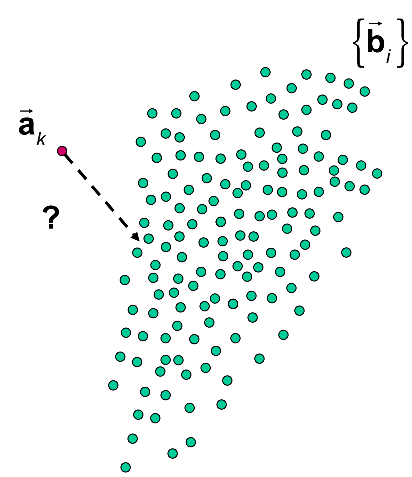
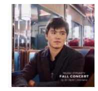
Finding point-pairs

- Given an **a**, find a corresponding **b** on the surface.
- One approach would be to search every possible triangle or surface point and then take the closest point.
- The key is to find a more efficient way to do this

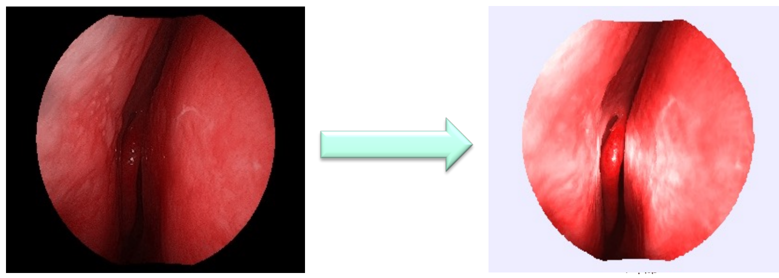
Suppose surface is represented by dense cloud of points



3D airway reconstruction during nasal endoscopic procedures without external tracking devices

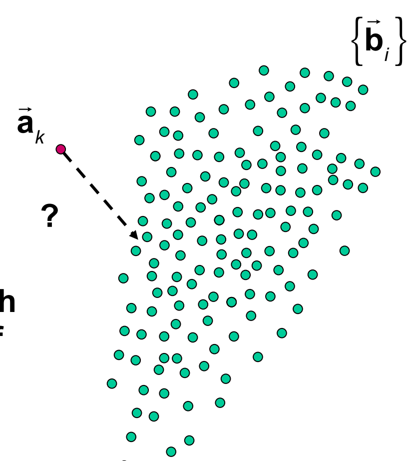


Xingtong Liu



Monoscopic Endoscope Video → Dense Point Cloud Reconstruction

Suppose surface is represented by dense cloud of points



How do we deal with the large number of possible matches?

Find Closest Point from Dense Cloud

- Basic approach is to divide space into regions. Suppose that we have one point $\mathbf{b}_k^*$ that is a possible match for a point $\mathbf{a}_k$. The distance $\Delta^* = \| \mathbf{b}_k^* - \mathbf{a}_k \|$ obviously acts as an upper bound on the distance of the closest point to the surface.
- Given a region **R** containing many possible points $\mathbf{b}_j$, if we can compute a lower bound Δ_L on the distance from **a** to any point in **R,** then we need only consider points inside **R** if $\Delta_L < \Delta^*$.

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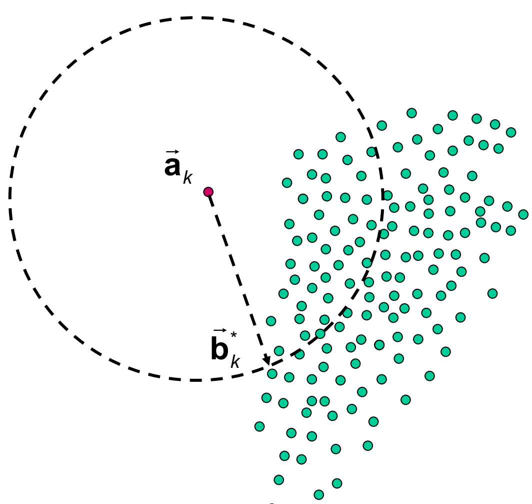
Given a match, is there anything closer?



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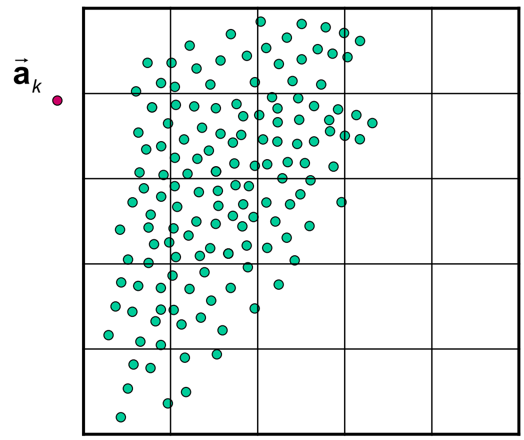
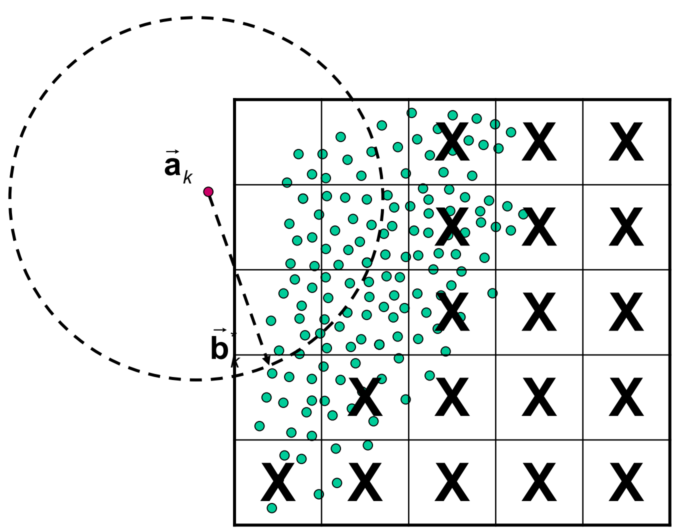
Divide cloud into cells



Can exclude everything outside circle



Find Closest Point from Dense Cloud

- There are many ways to implement this idea
  - Simply partitioning space into many buckets
  - Octrees, KD trees, covariance trees, etc.
- Basic idea also works with surface meshes, but need to find closest point on a triangle.

Approaches to closest triangle finding

1. (Simplest) Construct linear list of triangles and search sequentially for closest triangle to each point.
2. (Only slightly harder) Construct bounding spheres or bounding boxes around each triangle and use these to reduce the number of careful checks required.
3. (Faster if have lots of points) Construct hierarchical data structure to speed search.
4. (Better but harder) Rotate each level of the tree to align with data.

FindClosestPoint(a,[p,q,r])

Many approaches. One is to solve the system

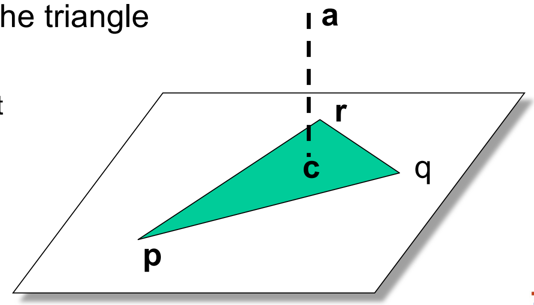
$$\mathbf{a}-\mathbf{p} \approx \lambda(\mathbf{q}-\mathbf{p})+\mu(\mathbf{r}-\mathbf{p})$$

in a least squares sense for λ and μ. Then compute

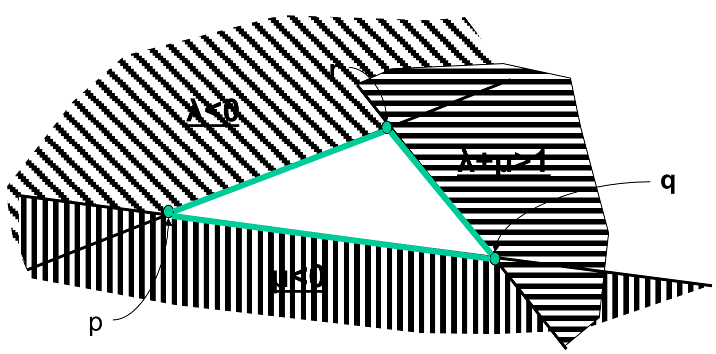
$$\mathbf{c} = \mathbf{p}+\lambda(\mathbf{q}-\mathbf{p})+\mu(\mathbf{r}-\mathbf{p})$$

If $\lambda \geq 0, \mu \geq 0, \lambda + \mu \leq 1$, then **c** lies within the triangle and is the closest point. Otherwise, you need to find a point on the border of the triangle

Hint: For efficiency, work out the least squares problem explicitly for λ, μ

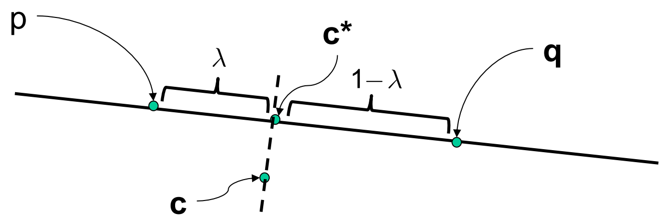


Finding closest point on triangle



Region	Closest point
λ<0	*ProjectOnSegment(***c,r,p***)*
μ<0	*ProjectOnSegment(***c,p,q***)*
λ+μ>1	*ProjectOnSegment(***c,q,r***)*

ProjectOnSegment(c,p,q)



$$\lambda = \frac{(\mathbf{c}-\mathbf{p})\bullet(\mathbf{q}-\mathbf{p})}{(\mathbf{q}-\mathbf{p})\bullet(\mathbf{q}-\mathbf{p})}$$

$$\lambda^{(seg)} = Max(0, Min(\lambda, 1))$$

$$\mathbf{c}^* = \mathbf{p} + \lambda^{(seg)} \times (\mathbf{q}-\mathbf{p})$$

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Simple Search with Bounding Boxes

// Triangle i has corners $[\vec{\mathbf{p}}_i, \vec{\mathbf{q}}_i, \vec{\mathbf{r}}_i]$

// Bounding box lower = $\vec{L}_i = [L_{xi}, L_{yi}, L_{zi}]^T$; upper = $\vec{U}_i = [U_{xi}, U_{yi}, U_{zi}]^T$

$bound = \infty$

for $i = 1$ to N do

{ if $(L_{xi} - bound \le a_x \le U_{xi} + bound)$ and $(L_{yi} - bound \le a_y \le U_{yi} + bound)$

and $(L_{zi} - bound \le a_z \le U_{zi} + bound)$ then

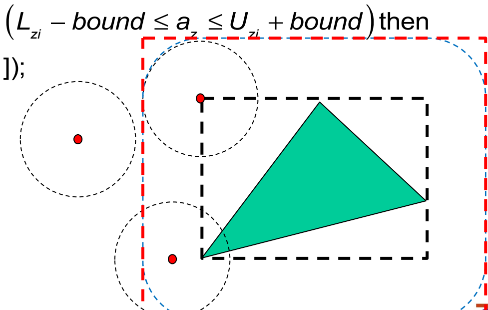
{ $\vec{\mathbf{h}} = \text{FindClosestPoint}(\vec{\mathbf{a}}, [\vec{\mathbf{p}}_i, \vec{\mathbf{q}}_i, \vec{\mathbf{r}}_i]);$

if $\|\vec{\mathbf{h}} - \vec{\mathbf{a}}\| < bound$ then

{ $\vec{\mathbf{c}} = \vec{\mathbf{h}};\ bound = \|\vec{\mathbf{h}} - \vec{\mathbf{a}}\|;$ };

};

};



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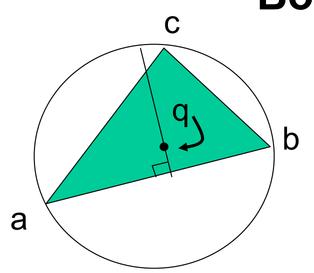
Bounding Sphere



Suppose you have a point $\vec{\mathbf{p}}$ and are trying to find the closest triangle $(\vec{\mathbf{a}}_k, \vec{\mathbf{b}}_k, \vec{\mathbf{c}}_k)$ to $\vec{\mathbf{p}}$. If you have already found a triangle $(\vec{\mathbf{a}}_j, \vec{\mathbf{b}}_j, \vec{\mathbf{c}}_j)$ with a point $\vec{\mathbf{r}}_j$ on it, when do you need to check carefully for some triangle k?

Answer: if $\vec{\mathbf{q}}_k$ is the center of a sphere of radius ρ_k enclosing $(\vec{\mathbf{a}}_k, \vec{\mathbf{b}}_k, \vec{\mathbf{c}}_k)$, then you only need to check carefully if $\left\|\vec{\mathbf{p}} - \vec{\mathbf{q}}_k\right\| - \rho_k < \left\|\vec{\mathbf{p}} - r_j\right\|$.

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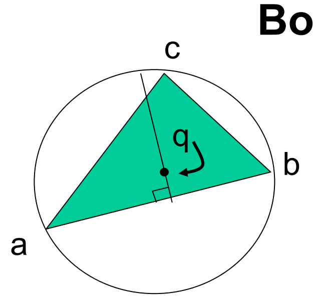
Bounding Sphere



Assume edge $(\vec{\mathbf{a}}, \vec{\mathbf{b}})$ is the longest.
Then the center $\vec{\mathbf{q}}$ of the sphere will obey

$$\left(\vec{\mathbf{b}} - \vec{\mathbf{q}}\right) \cdot \left(\vec{\mathbf{b}} - \vec{\mathbf{q}}\right) = \left(\vec{\mathbf{a}} - \vec{\mathbf{q}}\right) \cdot \left(\vec{\mathbf{a}} - \vec{\mathbf{q}}\right)$$
$$\left(\vec{\mathbf{c}} - \vec{\mathbf{q}}\right) \cdot \left(\vec{\mathbf{c}} - \vec{\mathbf{q}}\right) \le \left(\vec{\mathbf{a}} - \vec{\mathbf{q}}\right) \cdot \left(\vec{\mathbf{a}} - \vec{\mathbf{q}}\right)$$
$$\left(\vec{\mathbf{b}} - \vec{\mathbf{a}}\right) \times \left(\vec{\mathbf{c}} - \vec{\mathbf{a}}\right) \cdot \left(\vec{\mathbf{q}} - \vec{\mathbf{a}}\right) = 0$$

Simple approach: Try $\vec{\mathbf{q}} = \left(\vec{\mathbf{a}} + \vec{\mathbf{b}}\right) / 2$.
If inequality holds, then done.
Else solve the system to get $\vec{\mathbf{q}}$ (next page).
The radius $\rho = \left\|\vec{\mathbf{q}} - \vec{\mathbf{a}}\right\|$.

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Bounding Sphere

Assume edge $(\vec{\mathbf{a}},\vec{\mathbf{b}})$ is the longest side of triangle.

Compute $\vec{\mathbf{f}} = \left(\vec{\mathbf{a}} + \vec{\mathbf{b}}\right)/2.$

Define

$$\vec{\mathbf{u}} = \vec{\mathbf{a}} - \vec{\mathbf{f}}; \vec{\mathbf{v}} = \vec{\mathbf{c}} - \vec{\mathbf{f}}$$

$$\vec{\mathbf{d}} = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \times \vec{\mathbf{u}}$$

Then the sphere center $\vec{\mathbf{q}}$ lies somewhere along the line

$$\vec{\mathbf{q}} = \vec{\mathbf{f}} + \lambda\vec{\mathbf{d}}$$

with $(\lambda\vec{\mathbf{d}} - \vec{\mathbf{v}})^2 \leq (\lambda\vec{\mathbf{d}} - \vec{\mathbf{u}})^2$. Simplifying gives us

$$\lambda \geq \frac{\vec{\mathbf{v}}^2 - \vec{\mathbf{u}}^2}{2\vec{\mathbf{d}} \bullet \left(\vec{\mathbf{v}} - \vec{\mathbf{u}}\right)} = \gamma$$

If $\gamma \leq 0$, then just pick λ=0. Else pick λ=γ.

Simple Search with Bounding Spheres

// Triangle i has corners $[\vec{\mathbf{p}}_i, \vec{\mathbf{q}}_i, \vec{\mathbf{r}}_i]$

// Surrounding sphere i has radius ρ_i center $\vec{\mathbf{q}}_i$

$bound = \infty$;

for i=1 to N do

{ if $\left\|\vec{\mathbf{q}}_i - \vec{\mathbf{a}}\right\| - \rho_i \leq bound$ then

  { $\vec{\mathbf{h}} = \text{FindClosestPoint}(\vec{\mathbf{a}}, [\vec{\mathbf{p}}_i, \vec{\mathbf{q}}_i, \vec{\mathbf{r}}_i])$;

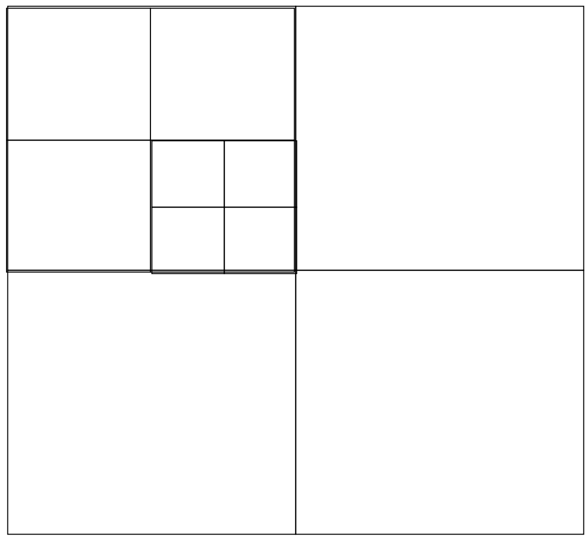
   if $\left\|\vec{\mathbf{h}} - \vec{\mathbf{a}}\right\| < bound$ then

    { $\vec{\mathbf{c}} = \vec{\mathbf{h}}$; $bound = \left\|\vec{\mathbf{h}} - \vec{\mathbf{a}}\right\|$; };

  };

};

Hierarchical cellular decompositions



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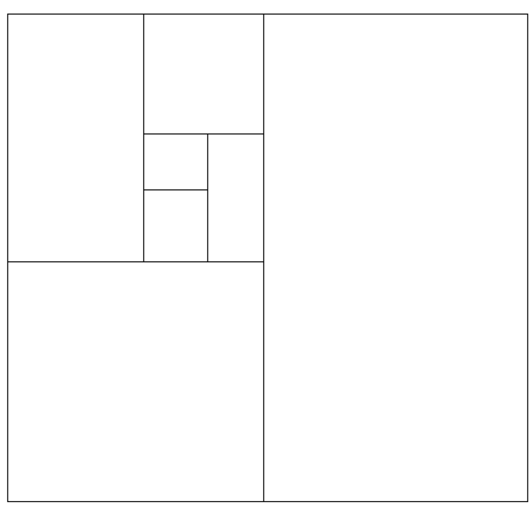
Hierarchical cellular decompositions



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Constructing tree of bounding spheres

```
class BoundingSphere {
  public:
    Vec3 Center;          // Coordinates of center
    double Radius;        // radius of sphere
    Thing* Object;        // some reference to the thing
                          // bounded
    };
```


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Constructing octree of bounding spheres

```
class BoundingBoxTreeNode {
   Vec3 Center;          // splitting point
   Vec3 UB;              //  corners of box
   Vec3 LB;
   int HaveSubtrees;
   int nSpheres;
   double MaxRadius;           // maximum radius of sphere in box
   BoundingBoxTreeNode* SubTrees[2][2][2];
   BoundingSphere** Spheres;
        :
        :
   BoundingBoxTreeNode(BoundingSphere** BS, int nS);
   ConstructSubtrees();
   void FindClosestPoint(Vec3 v, double& bound, Vec3& closest);
   };
```


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Constructing octree of bounding spheres

```
BoundingBoxTreeNode(BoundingSphere** BS, int nS)
{   Spheres = BS; nSpheres = nS;
    Center = Centroid(Spheres, nSpheres);
            // This will be the splitting point
            //  Centroid is efficient to compute
            //   But other choices are possible
    MaxRadius = FindMaxRadius(Spheres,nSpheres);
    UB = FindMaxCoordinates(Spheres,nSpheres);
    LB = FindMinCoordinates(Spheres,nSpheres);
    ConstructSubtrees();
    };
```

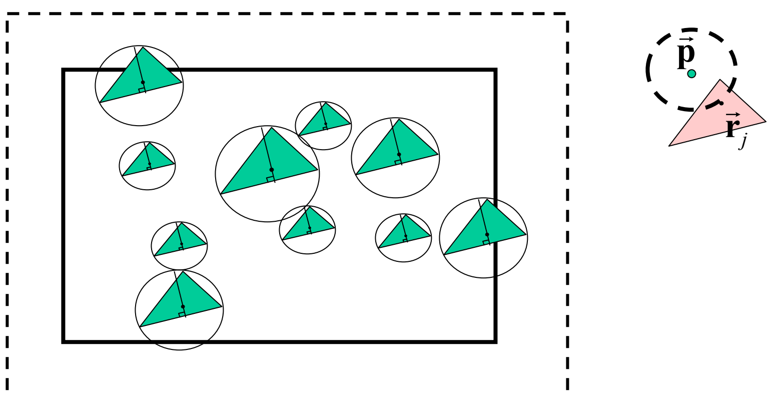

Constructing octree of bounding spheres

```
ConstructSubtrees()
{  if (nSpheres<= minCount || length(UB-LB)<=minDiag)
       { HaveSubtrees=0; return; };
  HaveSubtrees = 1;
  int nnn,npn,npp,nnp,pnn,ppn,ppp,pnp;
         // number of spheres in each subtree
   SplitSort(Center, Spheres, nnn,npn,npp,nnp,pnn,ppn,ppp,pnp);
  Subtrees[0][0][0] = BoundingBoxTree(Spheres[0],nnn);
  Subtrees[0][1][0] = BoundingBoxTree(Spheres[nnn],npn);
  Subtrees[0][1][1] = BoundingBoxTree(Spheres[nnn+npn],npp);
                    :
                    :
  }
```

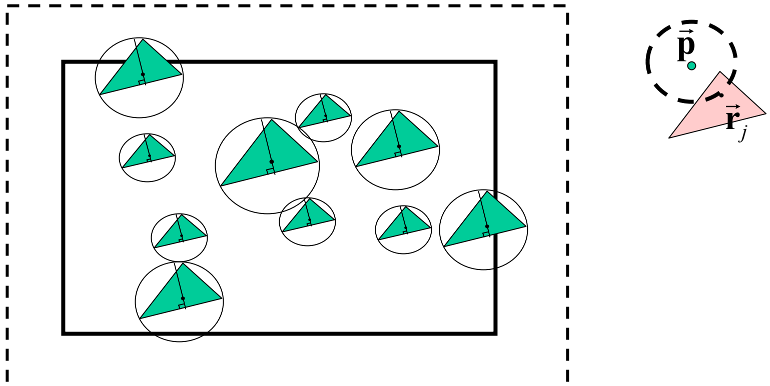
Constructing octree of bounding spheres

```
SplitSort(Vec3 SplittingPoint, BoundingSphere** Spheres,
            int& nnn, int& npn, … ,int& pnp)
{  // reorder Spheres(…) into eight buckets according to
   // comparison of coordinates of Sphere(k)->Center
   // with coordinates of splitting point.  E.g., first bucket has
   //       Sphere(k)->Center.x < SplittingPoint.x
   //       Sphere(k)->Center.y < SplittingPoint.y
   //       Sphere(k)->Center.z < SplittingPoint.z
   // This can be done "in place" by suitable exchanges.
   // Set nnn = number of spheres with all coordinates less than
   // splitting point, etc.
}
```


Searching an octree of bounding spheres



Searching an octree of bounding spheres



Searching an octree of bounding spheres

```
void BoundingBoxTreeNode::FindClosestPoint
                        (Vec3 v, double& bound, Vec3& closest)
{  double dist = bound + MaxRadius;
   if (v.x > UB.x+dist) return;  if (v.y > UB.y+dist) return;
       …. ;  if (v.z < LB.z-dist) return;
   if (HaveSubtrees)
       { Subtrees[0][0][0].FindClosestPoint(v,bound,closest);
            :
         Subtrees[1][1][1].FindClosestPoint(v,bound,closest);
       }
  else
    for (int i=0;i<nSpheres;I++)
         UpdateClosest(Spheres[i],v,bound,closest);

};
```

Searching an octree of bounding spheres

```
void UpdateClosest(BoundingSphere* S,
                   Vec3 v, double& bound, Vec3& closest)
{  double dist = v-S->Center;;
   if (dist - S->Radius > bound) return;
  Vec3 cp  = ClosestPointTo(*S->Object,v);
  dist = LengthOf(cp-v);
  if (dist<bound) { bound = dist; closest=cp;};
};
```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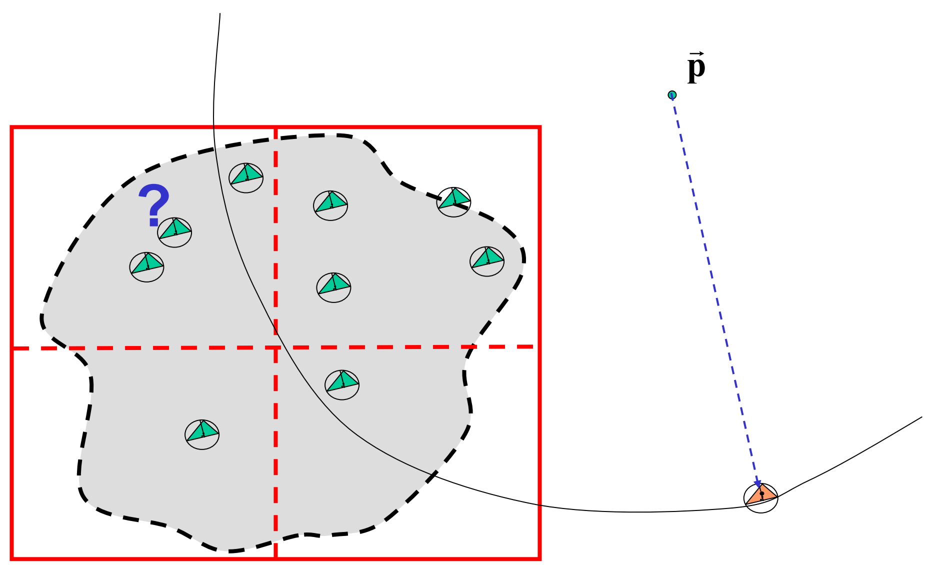

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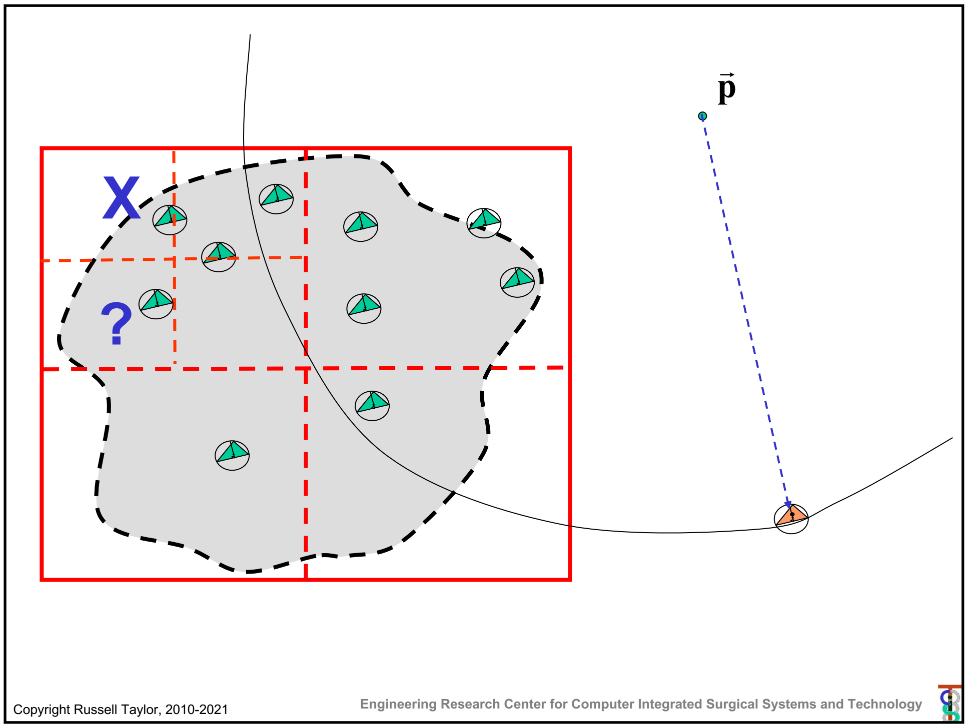
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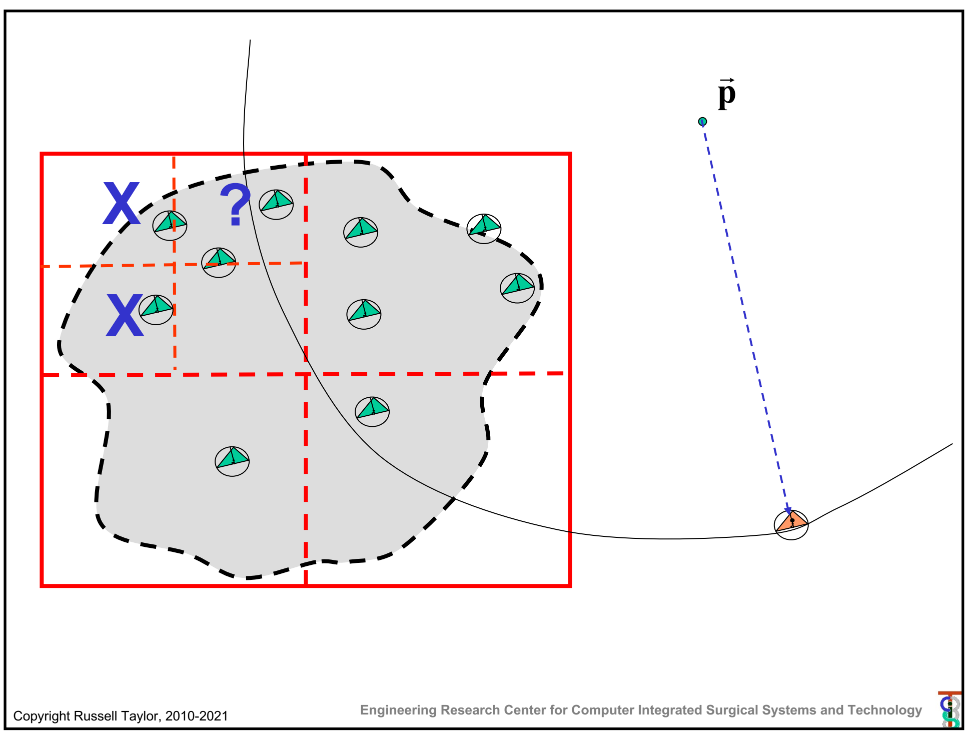
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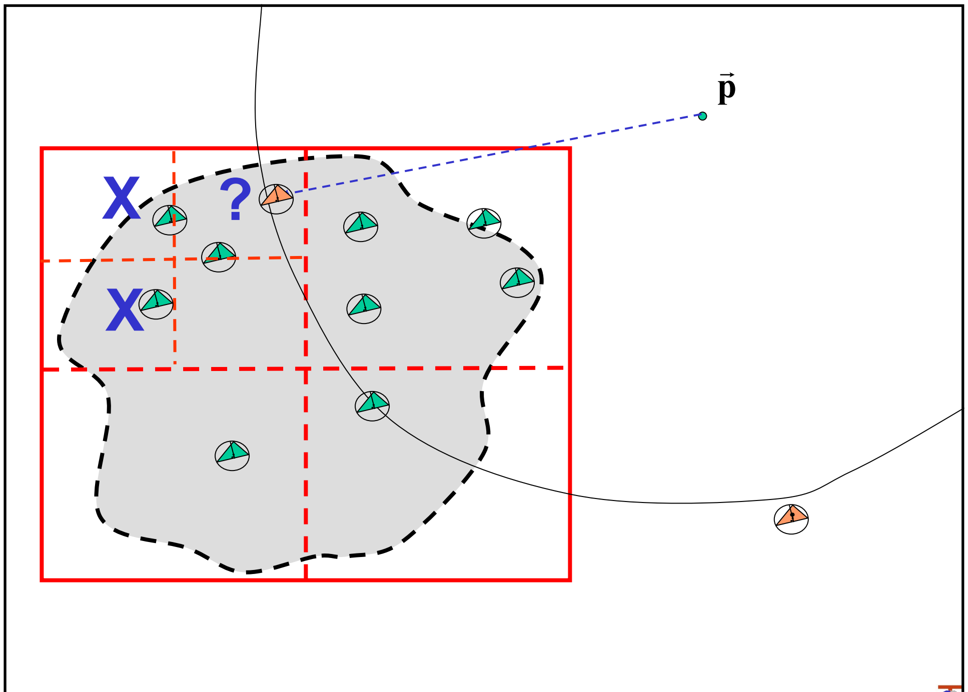

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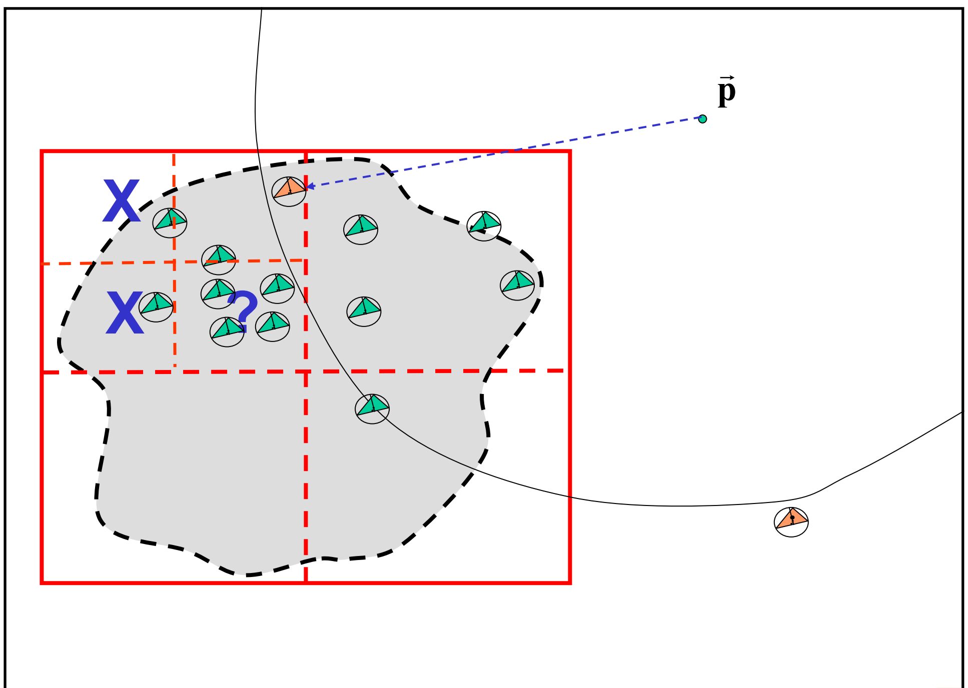

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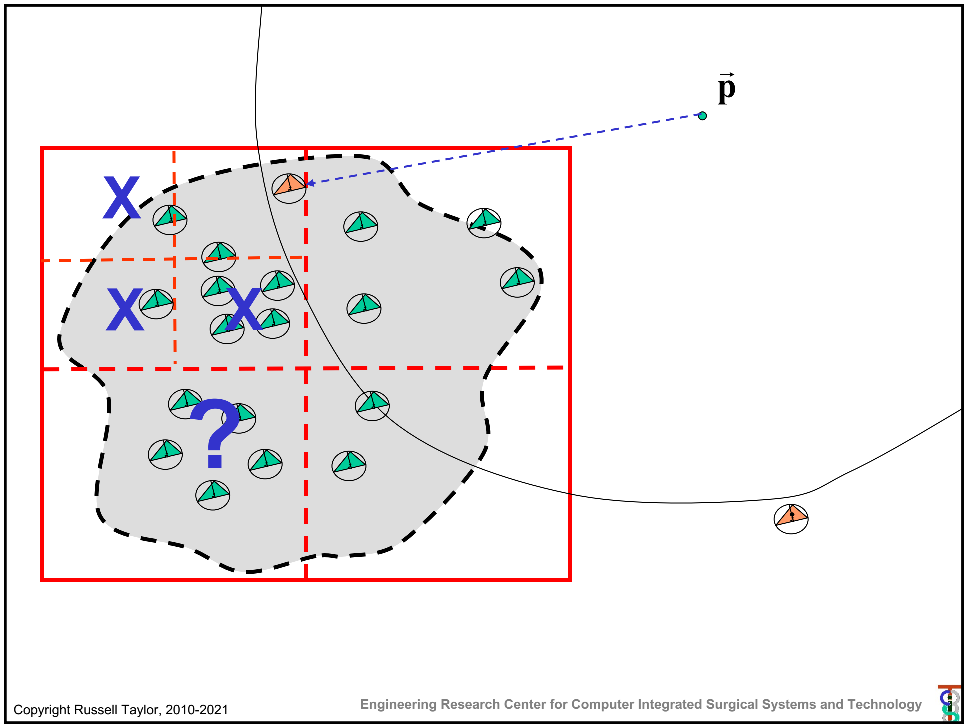

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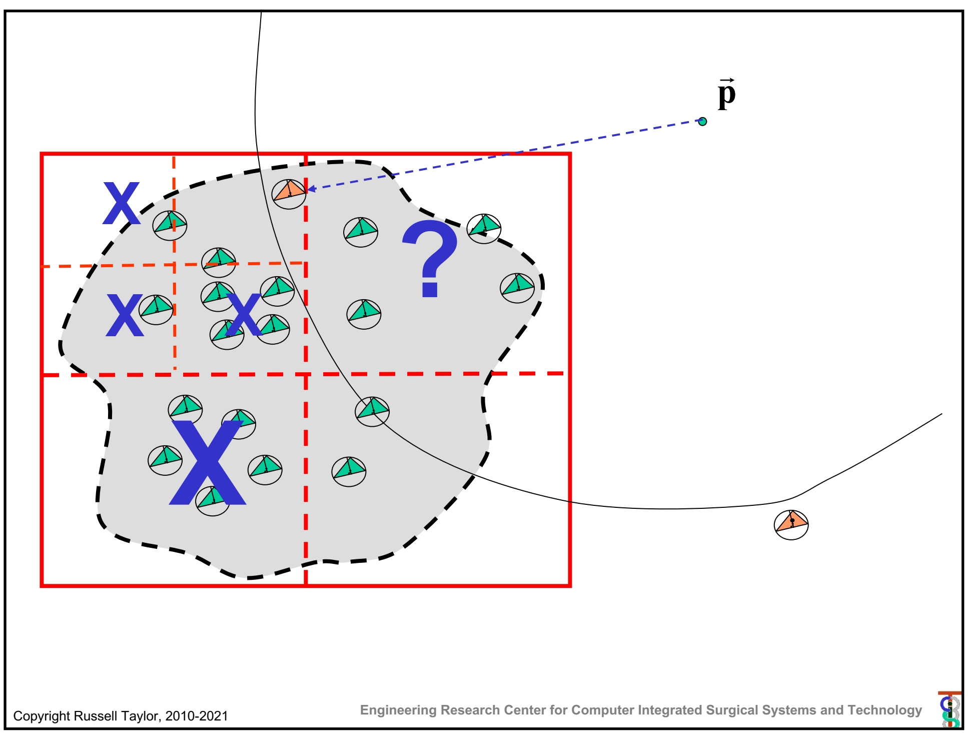

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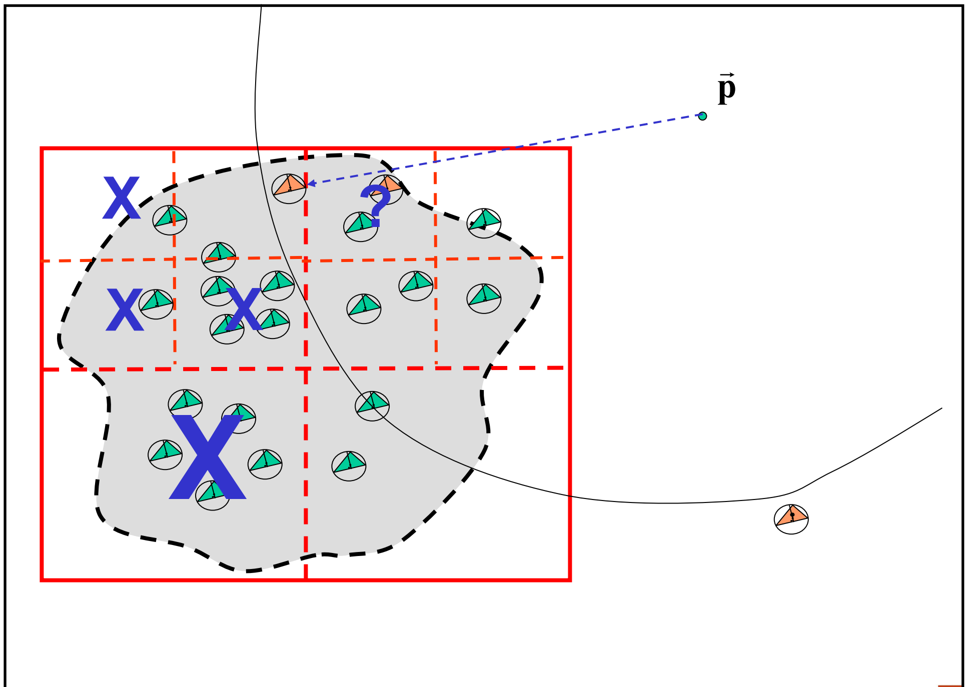

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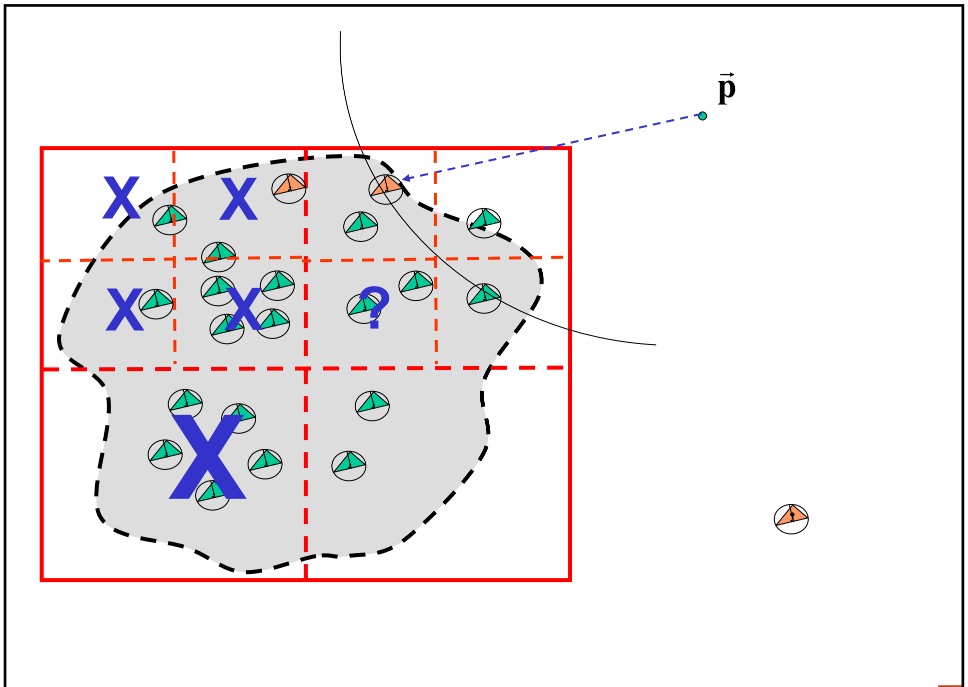

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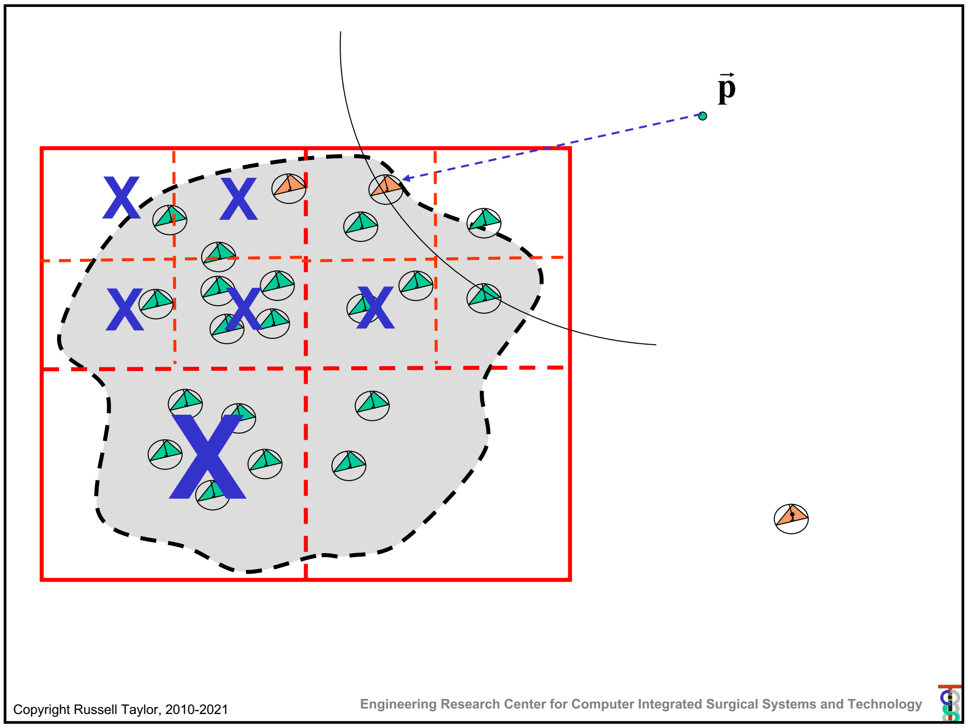
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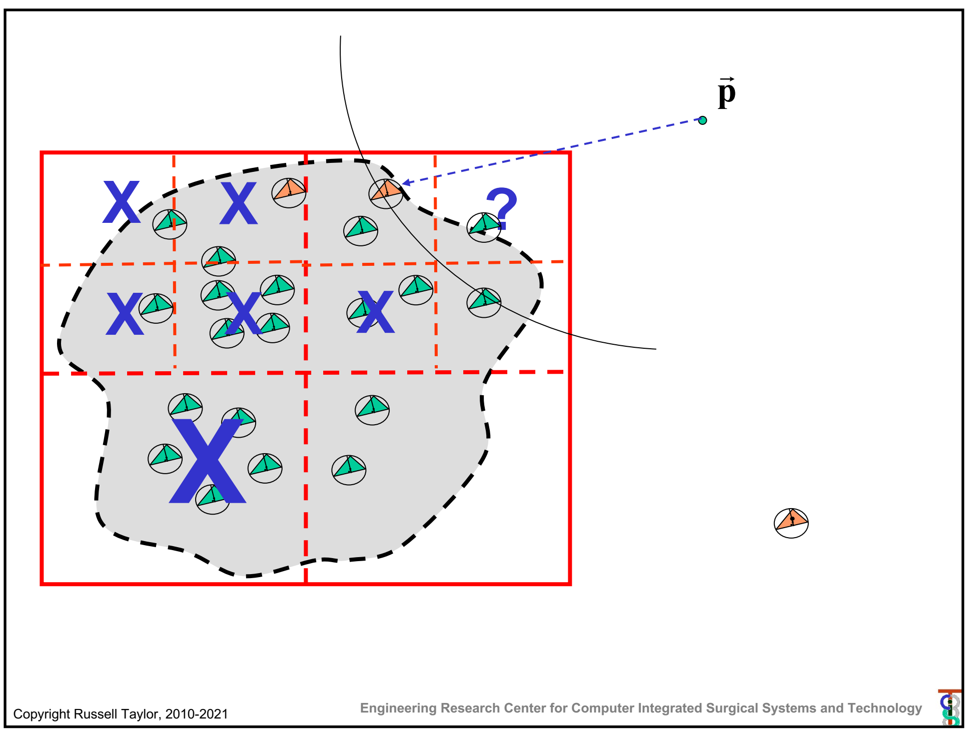
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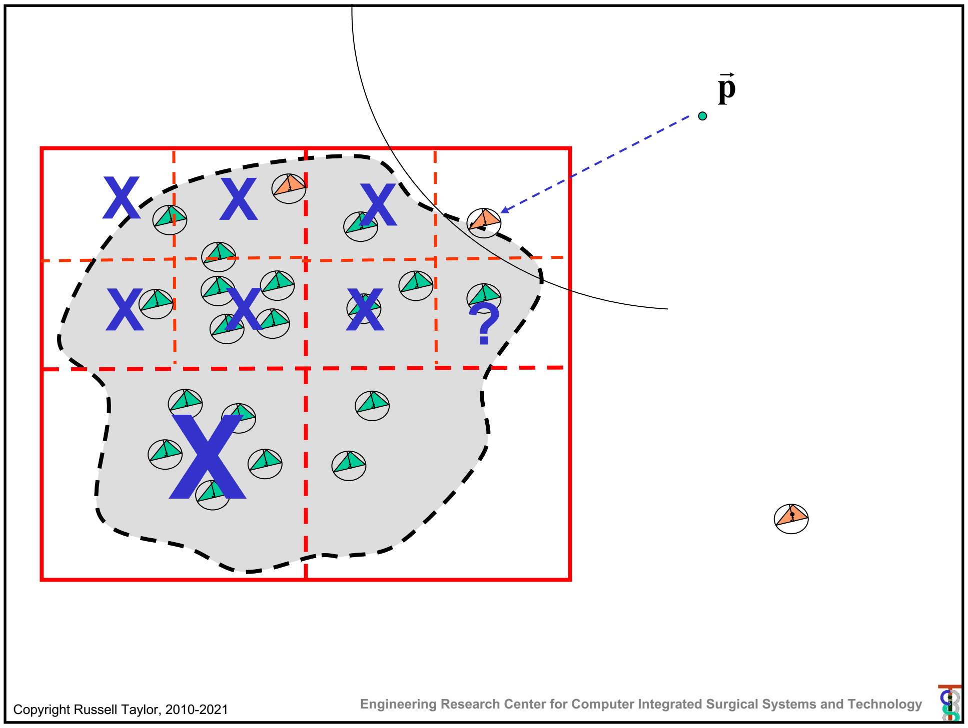
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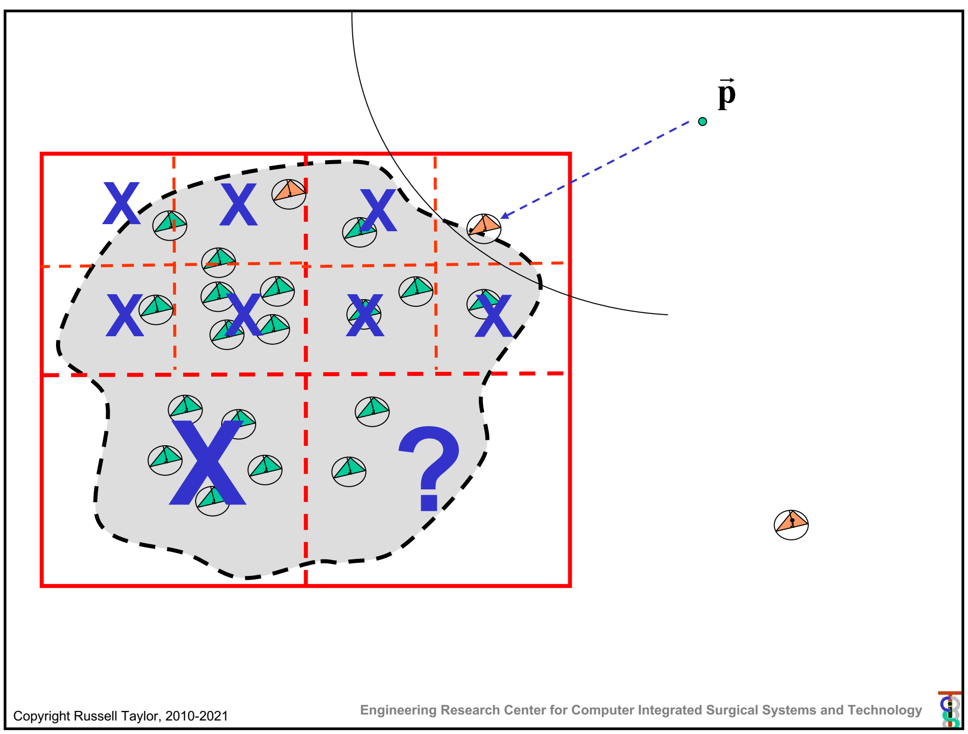
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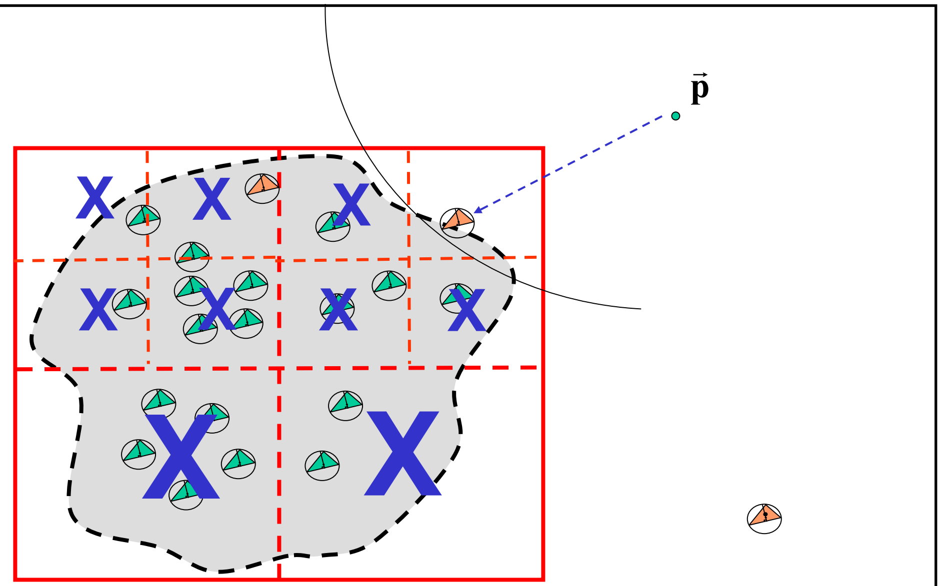

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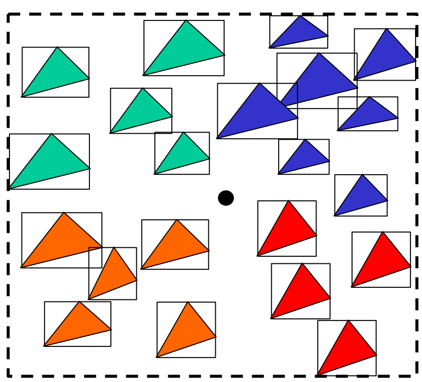
Constructing octree of bounded things



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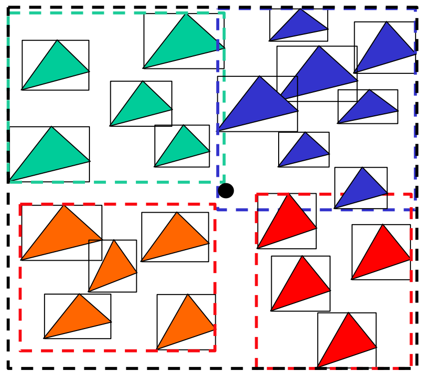
Constructing octree of bounded things



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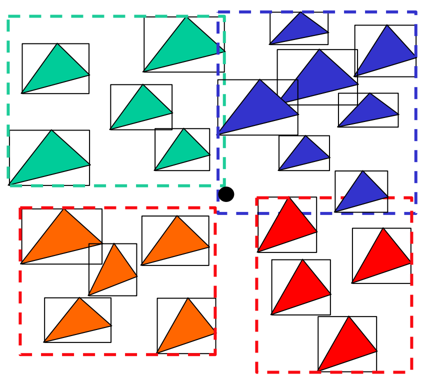
Constructing octree of bounded things



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Constructing octree of bounded things

```
class BoundingBoxTreeNode {
    Vec3 Center;          // splitting point
    Vec3 UB;              //  corners of box
    Vec3 LB;
    int HaveSubtrees;
    int nThings;
    BoundingBoxTreeNode* SubTrees[2][2][2];
    Thing** Things;
        :
        :
    BoundingBoxTreeNode(Thing** BS, int nS);
    ConstructSubtrees();
    void FindClosestPoint(Vec3 v, double& bound, Vec3& closest);
    };
```


Properties of "Things"

```
Class Thing
{ public:
          :
  vec3 SortPoint();
          // returns a point that can be used to sort the object
  vec3 ClosestPointTo(vec3 p);
          // returns point in this thing closest to p
  [vec3,vec3] EnlargeBounds(frame F,vec3 LB, vec3 UB);
          // Given frame F, and corners LB and UB of bounding box
          // around some other things, returns a the corners of a bounding
          // box that includes this Thing2 as well,
          // where Thing2=F.Inverse()*this thing
  [vec3,vec3] BoundingBox(F);
          { return EnlargeBounds(F,[∞, ∞, ∞],[-∞,-∞,-∞]);};

   int MayBeInBounds(Frame F, vec3 LB, vec3 UB);
          // returns 1 if any part of this F.Inverse()*this thing could be
          // in the bounding box with corners LB and UB
}
```

Triangle Things

```
Class Triangle : public Thing
{vec3 Corners[3];  // vertices of triangle
            :
 vec3 SortPoint() { return Mean(Corners);}; // or use Corner[0]
 [vec3,vec3] EnlargeBounds(frame F,vec3 LB, vec3 UB)
          { vec3 FiC[3]=F.inverse()*Corners;
            for (int I=0;I<3;I++)
                   { LB.x = min(LB.x,FiC[i].x); UB.x = max(UB.x,FiC[i].x);
                     LB.y = min(LB.y,FiC[i].y); UB.y = max(UB.y,FiC[i].y);
                     LB.z = min(LB.y,FiC[i].z); UB.z = max(UB.y,FiC[i].z);
                   };
          return [LB, UB];
          };
  [vec3,vec3] BoundingBox(F)
          { return EnlargeBounds(F,[∞, ∞, ∞],[-∞,-∞,-∞]);};
  int MayBeInBounds(Frame F, vec3 LB, vec3 UB)
          { vec3 FiC[3]=F.inverse()*Corners;
            for (int k=0;k<3; k++) if (InBounds(FiC[k],LB,UB)) return 1;
            return 0;}
}
```


Constructing octree of bounded things

```
BoundingBoxTreeNode(Thing** BS, int nS)
{   Things = BS; nThings = nS;
    UB = FindMaxCoordinates(Things,nThings);
    LB = FindMinCoordinates(Things,nThings);
    Center = LB+(UB-LB)/2.0;    // Splitting point
                                // Not necessarily the best
                                // Alternatives would be centroid or
                                // the median of the SortPoint()'s.
    ConstructSubtrees();
    };
```

Constructing octree of bounded things

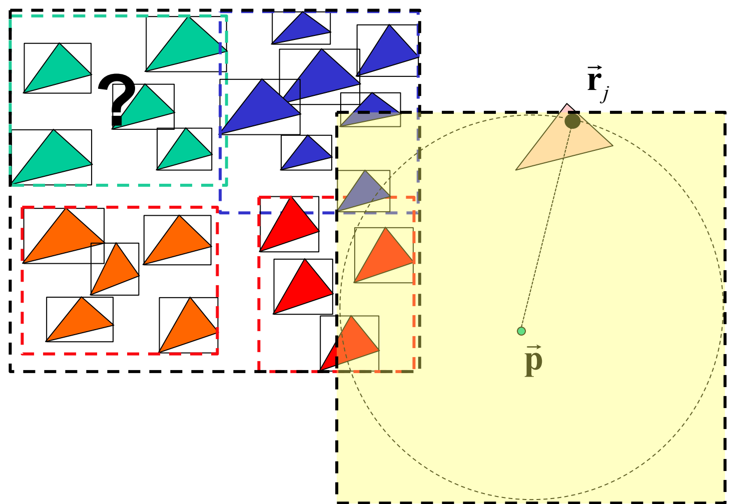
```
ConstructSubtrees()
{  if (nThings<= minCount || length(UB-LB)<=minDiag)
      { HaveSubtrees=0; return; };
  HaveSubtrees = 1;
  int nnn,npn,npp,nnp,pnn,ppn,ppp,pnp;
        // number of things in each subtree
  SplitSort(Center, Things, nnn,npn,npp,nnp,pnn,ppn,ppp,pnp);
  Subtrees[0][0][0] = BoundingBoxTree(Things[0],nnn);
  Subtrees[0][1][0] = BoundingBoxTree(Things[nnn],npn);
  Subtrees[0][1][1] = BoundingBoxTree(Things[nnn+npn],npp);
                   :
                   :
  }
```


Constructing octree of bounded things

```
SplitSort(Vec3& SplittingPoint, BoundingThing** Things,
            int& nnn, int& npn, … ,int& pnp)
{  // reorder Spheres(…) into eight buckets according to
   // comparison of coordinates of Thing(k)->SortPoint()
   // with coordinates of splitting point.  E.g., first bucket has
   //       Thing(k)->Center.x < SplittingPoint.x
   //       Thing(k)->Center.y < SplittingPoint.y
   //       Thing(k)->Center.z < SplittingPoint.z
   // This can be done "in place" by suitable exchanges.
   // Set nnn = number of spheres with all coordinates less than
   // splitting point, etc.

   // If desired, may be modified to simultaneously find a good
   // value for SplittingPoint (e.g., median)
}
```

Searching octree of bounded things



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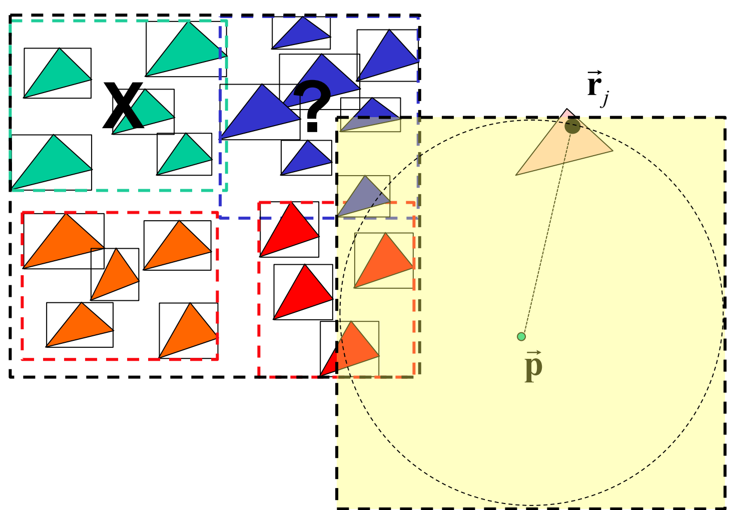
Searching octree of bounded things



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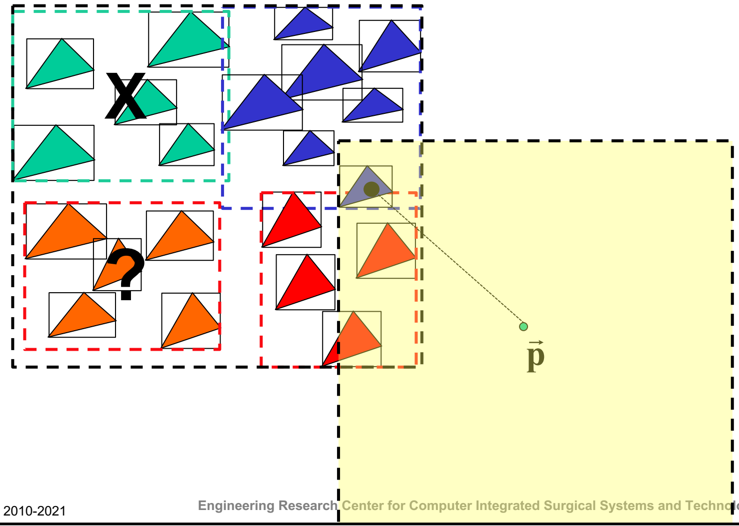
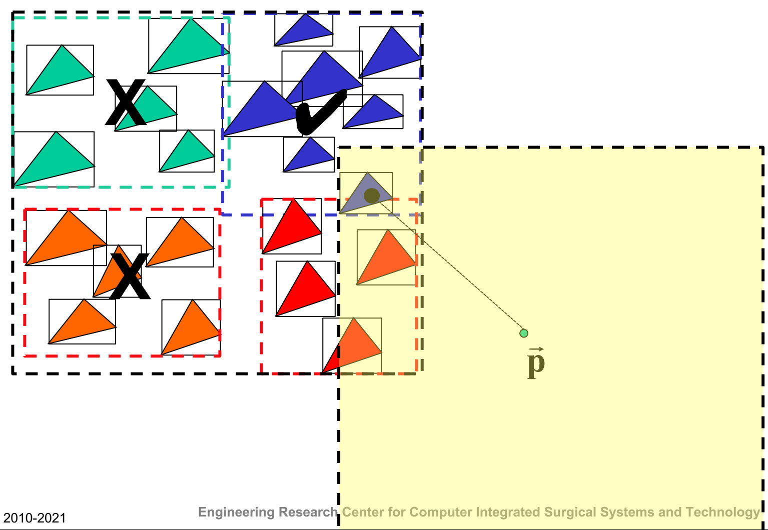
Searching octree of bounded things



55

Searching octree of bounded things



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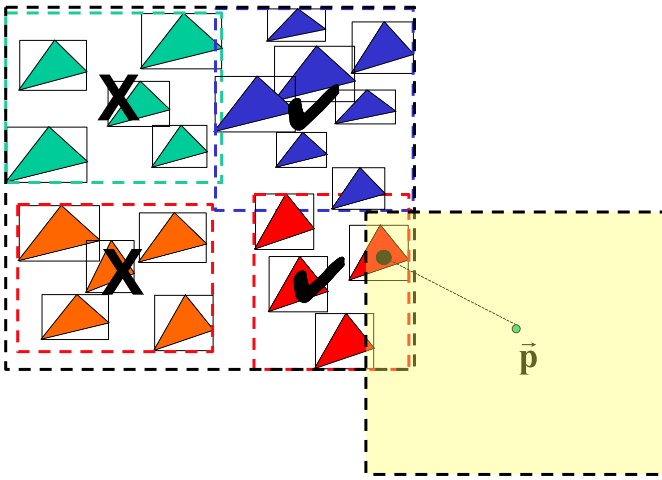
Searching octree of bounded things



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Searching octree of bounded things

```
void BoundingBoxTreeNode::FindClosestPoint
                        (Vec3 v, double& bound, Vec3& closest)
{   if ((v.x > UB.x+bound) || (v.x<LB.x-bound)) return;
    if ((v.y > UB.y+bound) || (v.y<LB.y-bound)) return;
    if ((v.z > UB.z+bound) || (v.z<LB.z-bound)) return;
    if (HaveSubtrees)
       { Subtrees[0][0][0].FindClosestPoint(v,bound,closest);
            :
         Subtrees[1][1][1].FindClosestPoint(v,bound,closest);
       }
   else
     for (int i=0;i<nThings;I++)
         UpdateClosest(Things[i],v,bound,closest);

};
```


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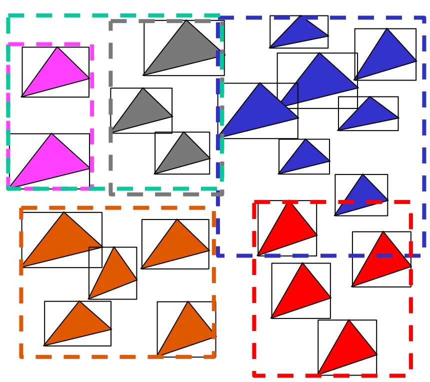
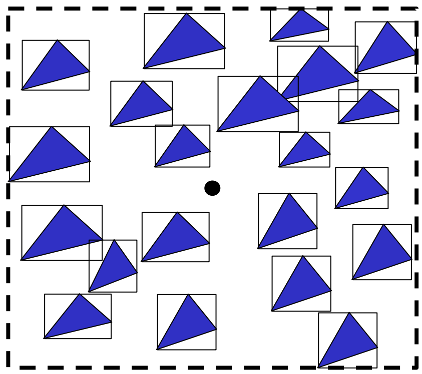
Searching octree of bounded things

```
void UpdateClosest(Thing* Thing,
                          Vec3 v, double& bound, Vec3& closest)
{ Vec3 cp  = Thing->ClosestPointTo(v);
  dist = LengthOf(cp-v);
  if (dist<bound) { bound = dist; closest=cp;};
};
```


Constructing KD tree of bounded things



Constructing KD tree of bounded things



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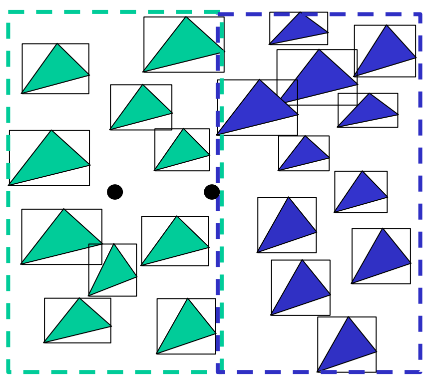
Constructing KD tree of bounded things



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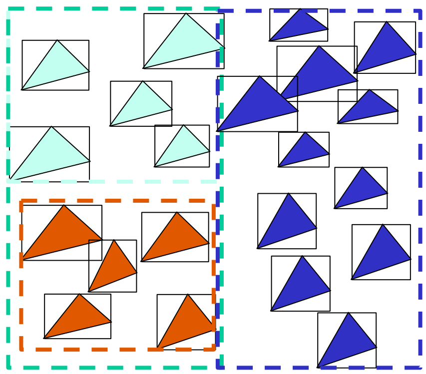
Constructing KD tree of bounded things



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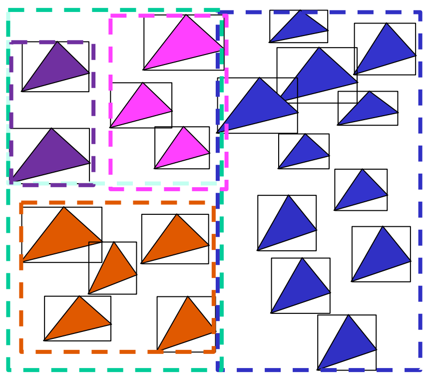
Constructing KD tree of bounded things



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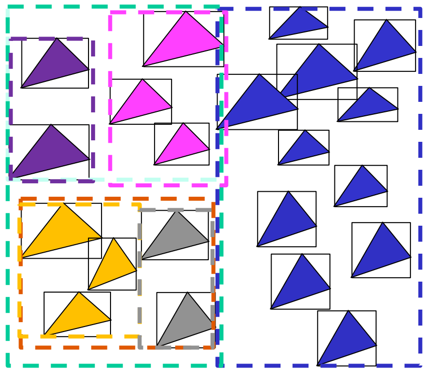
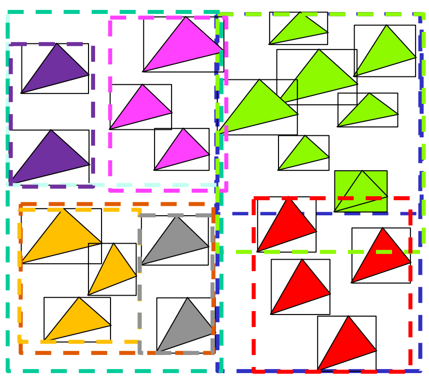
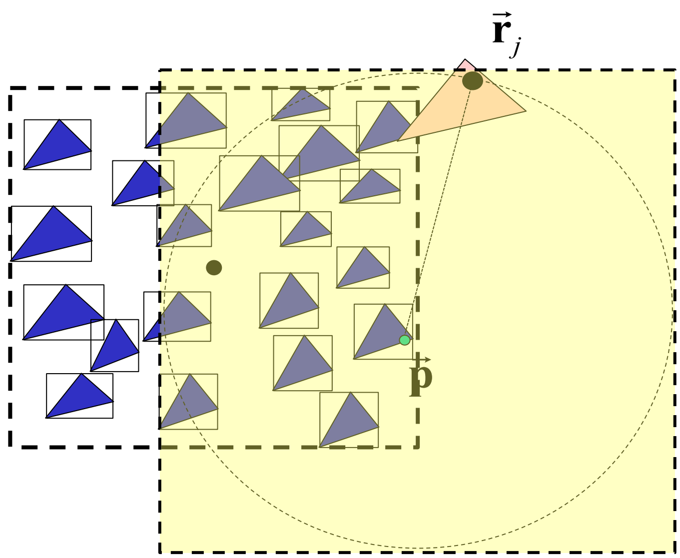
Constructing KD tree of bounded things



Constructing KD tree of bounded things



Searching KD Tree of Bounded Things



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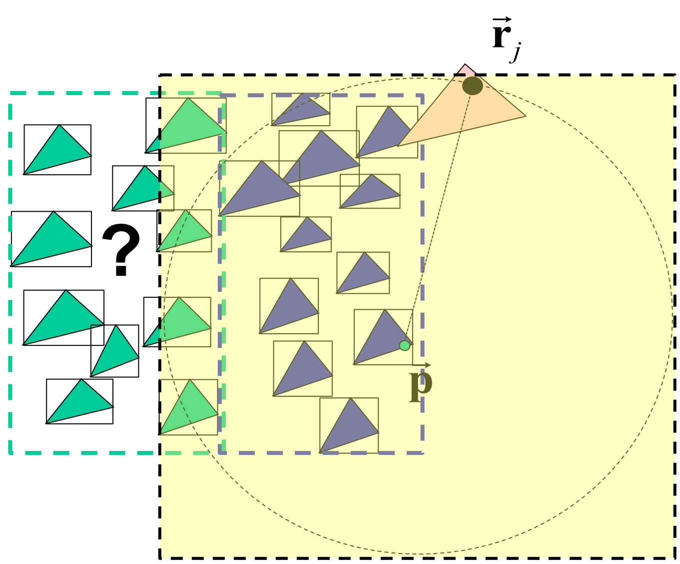
Searching KD Tree of Bounded Things



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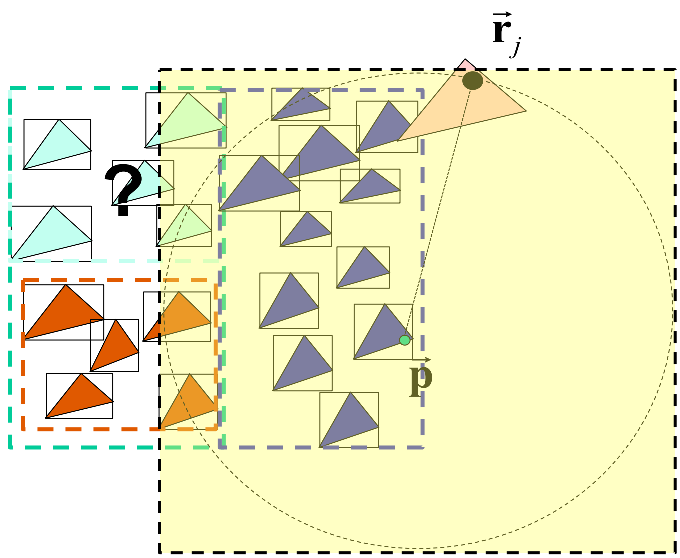
Searching KD Tree of Bounded Things



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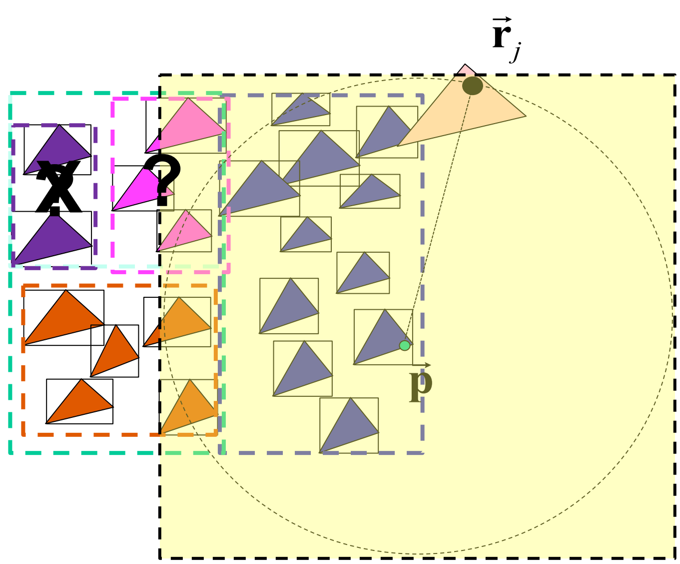
Searching KD Tree of Bounded Things



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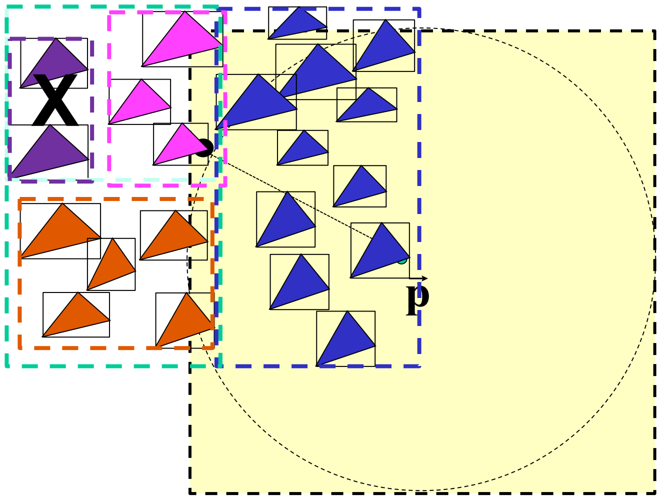
Searching KD Tree of Bounded Things



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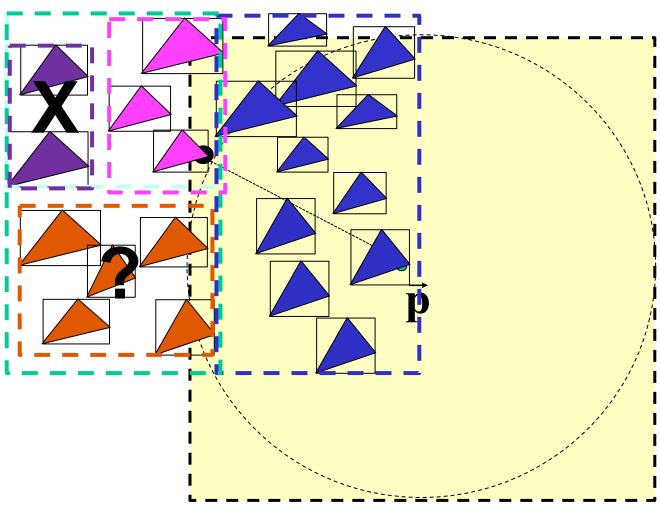
Searching KD Tree of Bounded Things



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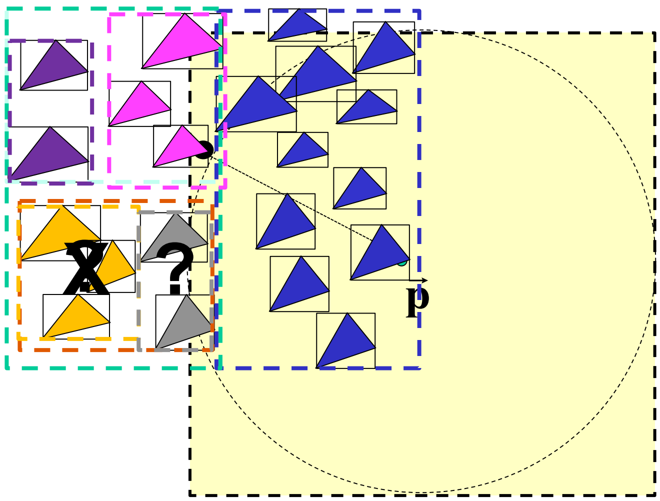
Searching KD Tree of Bounded Things



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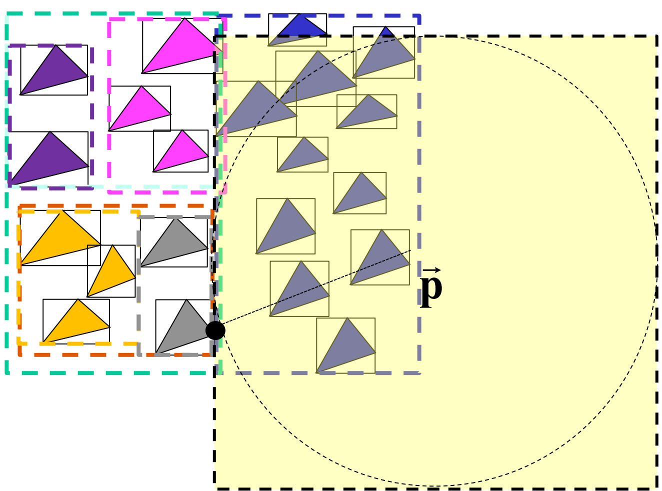
Searching KD Tree of Bounded Things



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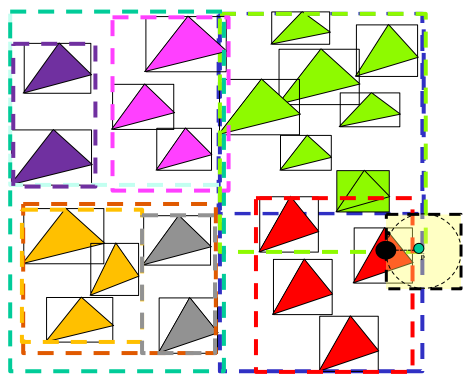
Searching KD Tree of Bounded Things



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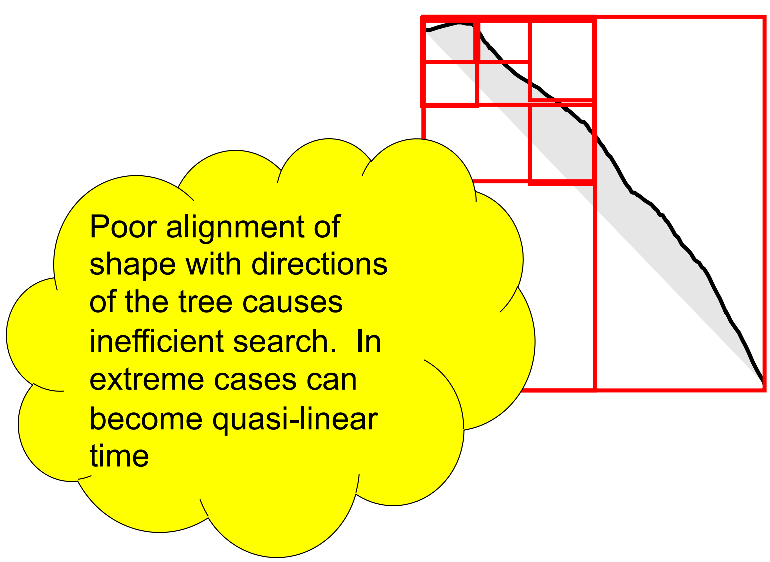
Possible pathology with KD trees and Octrees



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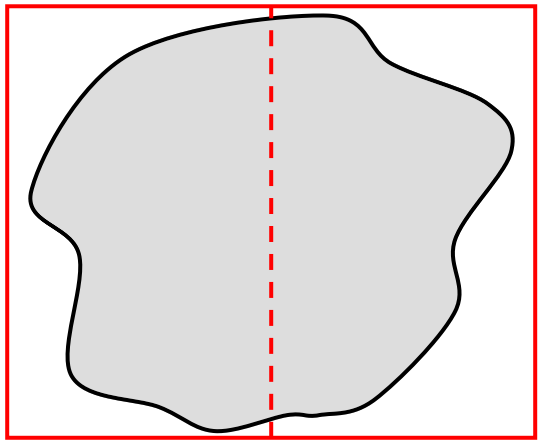
Solution: Covariance Trees*



* Referred to by my former student Seth Billings as Principal Direction Trees

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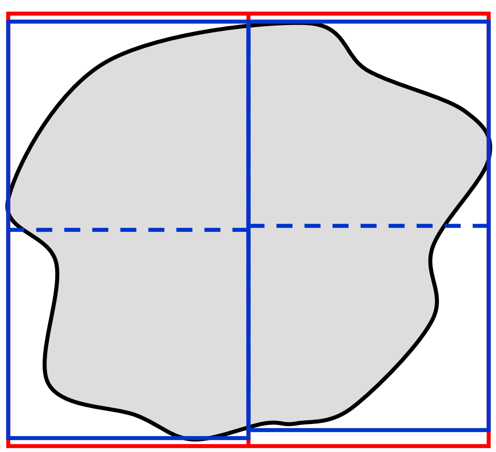
Covariance Trees



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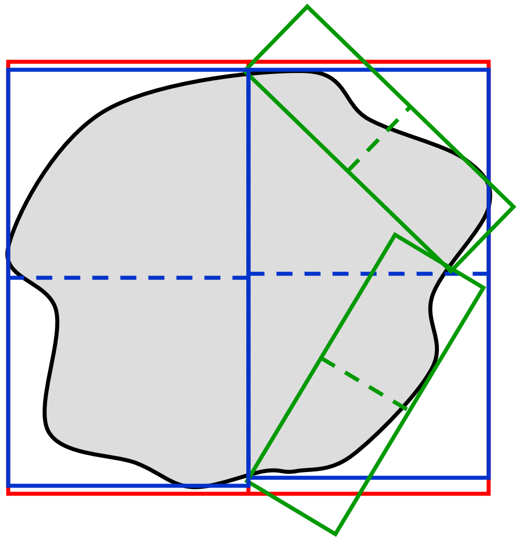
Covariance Trees



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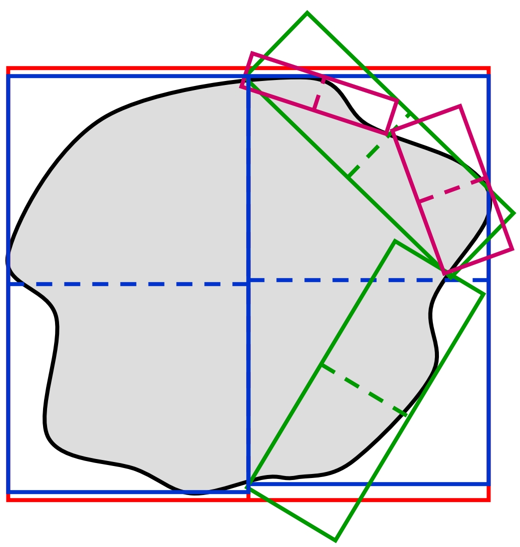
Covariance Trees



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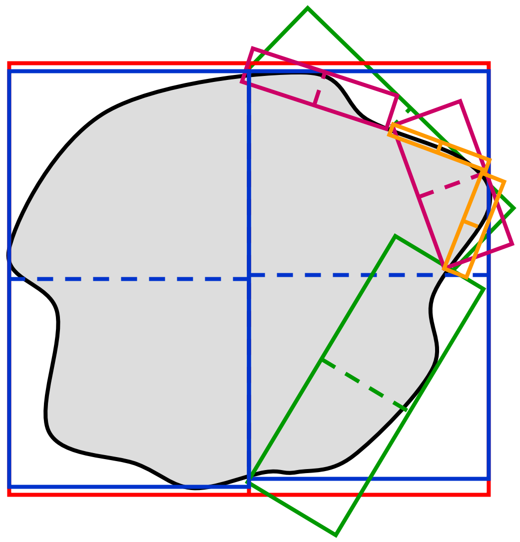
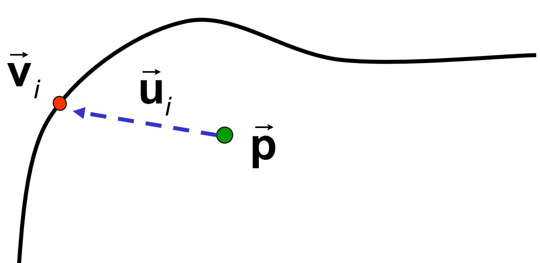
Covariance Trees



Covariance Tree Construction

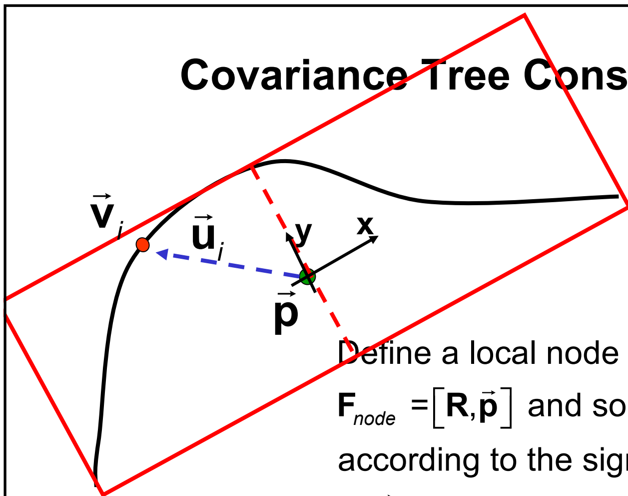


Given surface sample of N points $\{\vec{\mathbf{v}}_i\}$

Compute centroid $\vec{\mathbf{p}} = \frac{1}{N}\sum_{i=1}^{N} \vec{\mathbf{v}}_i$

Compute residual vectors $\vec{\mathbf{u}}_i = \vec{\mathbf{v}}_i - \vec{\mathbf{p}}$

Covariance Tree Construction



Define a local node coordinate system $\mathbf{F}_{node} = [\mathbf{R}, \vec{\mathbf{p}}]$ and sort the surface points according to the sign of the x component of $\vec{\mathbf{b}}_i = \mathbf{R}^{-1} \bullet \vec{\mathbf{u}}_i$. Compute bounding box

$$\vec{\mathbf{b}}^{\min} \leq \mathbf{R}^{-1} \bullet \vec{\mathbf{u}}_i \leq \vec{\mathbf{b}}^{\max}$$

Assign these points to "left" and "right" subtree nodes.

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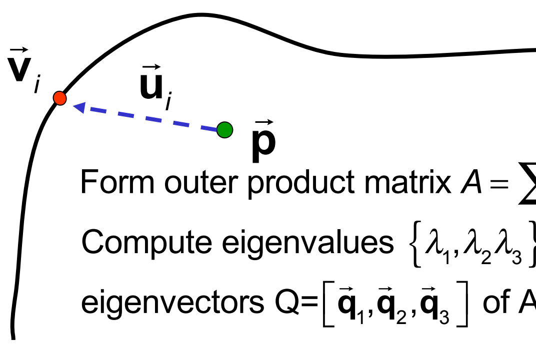
Covariance Tree Construction



Form outer product matrix $A = \sum_i \vec{\mathbf{u}}_i \vec{\mathbf{u}}_i^T$

Compute eigenvalues $\{\lambda_1, \lambda_2 \lambda_3\}$ and eigenvectors Q=$[\vec{\mathbf{q}}_1, \vec{\mathbf{q}}_2, \vec{\mathbf{q}}_3]$ of A

Find a rotation $\mathbf{R}$ such that $\mathbf{R}_x$ is the eigenvector corresponding to the largest eigenvalue.

(Depending on algorithm used, Q will be a rotation matrix, so all you may have to do is rotate Q)

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Constructing Cov Tree of Objects

```
class CovTreeNode {
    Frame F;            // splitting point
    Vec3 UB;                 //  corners of box
    Vec3 LB;
    int HaveSubtrees;
    int nThings;
    CovTreeNode* SubTrees[2];
    Thing** Things;
        :
        :
    CovTreeNode(Thing** Ts, int nT);
    ConstructSubtrees();
    void FindClosestPoint(Vec3 v, double& bound, Vec3& closest);
    };
```


Constructing Cov Tree of Things

```
CovTreeNode(Thing** Ts, int nT)
{   Things = Ts; nThings = nT;
    F = ComputeCovFrame(Things,nThings);
    [UB,LB] = FindBoundingBox(F,Things,nThings);
    ConstructSubtrees();
    };

[vec3 UB,vec3 LB]=FindBoundingBox(F,Things,nThings)
{ UB = LB = F.inverse()*(Things[0]->SortPoint());
   for (int k=0;k<nThings;k++)
    { [LB,UB] = Things[k]->EnlargeBounds(F,LB,UB);
    };
  return [UB,LB];
};
```

Constructing Cov Tree of Things

```
Frame F = FindCovFrame(Thing** Ts, int nT)
{   [vec3 Points, int nP] = ExtractPoints(Ts,nT);
          // may extract nT sort points or perhaps
          // all corner points if things are triangles
    return FindCovFrame(Points,nP);
    };

Frame F = FindCovFrame(vec3* Ps, int nP)
{   vec3 C = Centroid(Ps,nP);
    Matrix A = 0;
    for (i=0;i<nP;i++) A+=OuterProduct(Ps[i],Ps[i]);
    R = CorrespondingRotationMatrix(A);  // see notes
    return Frame(R,C);
    };
```


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Constructing Cov Tree of Things

```
ConstructSubtrees()
{  if (nThings<= minCount || length(UB-LB)<=minDiag)
        { HaveSubtrees=0; return; };
  HaveSubtrees = 1;
  int nSplit;
  nSplit = SplitSort(F,things);
  Subtrees[0] = CovarianceTreeNode(Things[0],nSplit);
  Subtrees[1] = CovarianceTreeNode(Things[nSplit],nThings-nSplit);
}
```


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Constructing Cov Tree of Things

```
Int nSplit = SplitSort(Frame F, Thing** Ts,int nT)
{  // find an integer nSplit and reorder Things(…)  so that
   //      F.inverse()*(Thing[k]->SortPoint()).x <0 if and only if k<nSplit
   // This can be done "in place" by suitable exchanges.
   return nSplit;
}
```


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Covariance tree search

Given

- node with associated $\mathbf{F}_{node}$ and surface sample points $\vec{\mathbf{s}}_i$.
- sample point $\vec{\mathbf{a}}$, previous closest point $\vec{\mathbf{c}}$, $dist = \left\| \vec{\mathbf{a}} - \vec{\mathbf{c}} \right\|$

Transform $\vec{\mathbf{a}}$ into local coordinate system $\vec{\mathbf{b}} = \mathbf{F}_{node}^{-1}\vec{\mathbf{a}}$

Check to see if the point $\vec{\mathbf{b}}$ is inside an enlarged bounding box $\vec{\mathbf{b}}^{\min} - dist \leq \vec{\mathbf{b}} \leq \vec{\mathbf{b}}^{\max} + dist$. If not, then quit.

Otherwise, if no subnodes, do exhaustive search for closest.

Otherwise, search left and right subtrees.

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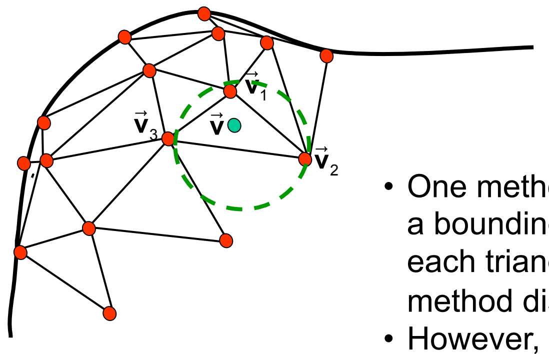
Searching a Covariance Tree of Things

```
void CovarianceTreeNode::FindClosestPoint
                    (Vec3 v, double& bound, Vec3& closest)
{   vLocal=F.Inverse()*v; // transform v to local coordinate system
    if (vLocal.x > UB.x+bound) return;
    if (vLocal.y > UB.y+bound) return;
      // similar checks on remaining bounds go here …. ;
    if (vLocal.z < LB.z-bound) return;
   if (HaveSubtrees)
      { Subtrees[0].FindClosestPoint(v,bound,closest);
        Subtrees[1].FindClosestPoint(v,bound,closest);
      }
   else
    for (int i=0;i<nThings;I++)
         UpdateClosest(Things[i],v,bound,closest);
};
```


Searching a Covariance Tree of Things

```
void UpdateClosest(Thing* T, Vec3 v, double& bound, Vec3& closest)
{  // here can include filter if have a bounding sphere to check
  Vec3 cp  = T->ClosestPointTo(v);
  dist = LengthOf(cp-v);
  if (dist<bound) { bound = dist; closest=cp;};
};
```

Covariance Trees for Triangle Meshes



- One method is simply to place a bounding sphere around each triangle, and then use the method discussed previously
- However, this may be inconvenient if the mesh is deforming

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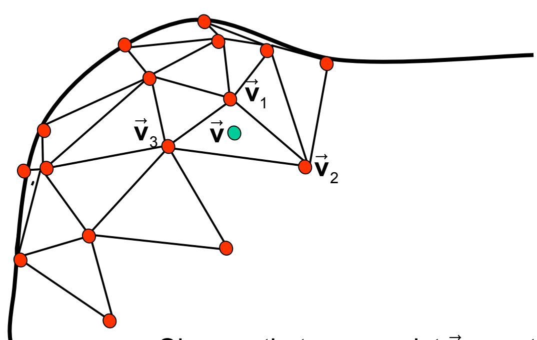
Covariance Trees for Triangle Meshes



Observe that every point $\vec{\mathbf{v}}$ on a triangle $\left[\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\right]$ can be expressed as a convex linear combination $\vec{\mathbf{v}} = \lambda_1 \vec{\mathbf{v}}_1 + \lambda_2 \vec{\mathbf{v}}_2 + \lambda_3 \vec{\mathbf{v}}_3$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Therefore, if $\left[\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\right]$ are in some bounding box, then $\vec{\mathbf{v}}$ will also be in that bounding box

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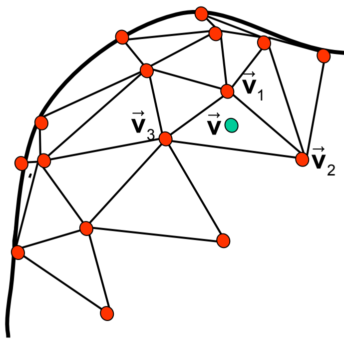
Covariance Trees for Triangle Meshes



- Select one point on the triangle to use as the “sort” point for selection of left/right subtrees.
- Good choices are centroid of triangle or just one of the vertices.
- However use <u>all</u> vertices of each triangle in determining the size of bounding boxes.
- Note this would work equally well for octrees.

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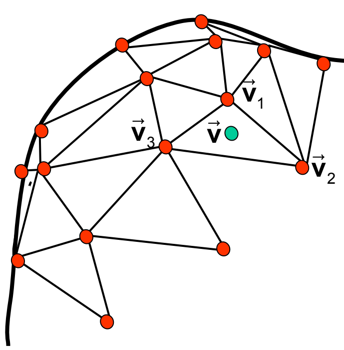
Covariance Trees for Triangle Meshes



- Select one point on the triangle to use as the “sort” point for selection of left/right subtrees.
- Good choices are centroid of triangle or just one of the vertices.
- However use <u>all</u> vertices of each triangle in determining the size of bounding boxes.
- Note this would work equally well for octrees.

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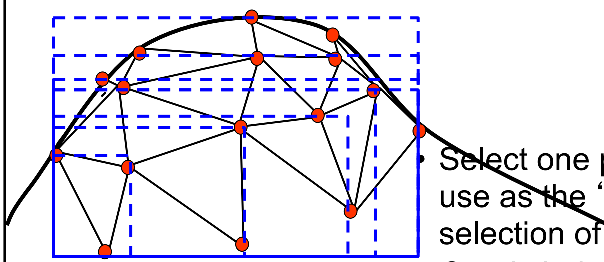
Covariance Trees for Triangle Meshes



- Select one point on the triangle to use as the “sort” point for selection of left/right subtrees.
- Good choices are centroid of triangle or just one of the vertices.
- However use all vertices of each triangle in determining the size of bounding boxes.
- Note this would work equally well for octrees.

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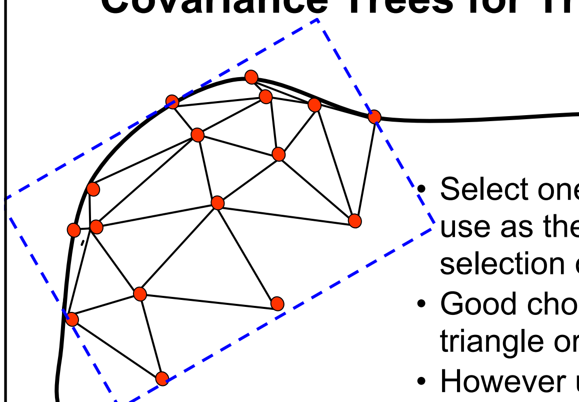
Covariance Trees for Triangle Meshes



- Select one point on the triangle to use as the “sort” point for selection of left/right subtrees.
- Good choices are centroid of triangle or just one of the vertices.
- However use all vertices of each triangle in determining the size of bounding boxes.
- Note this would work equally well for octrees.

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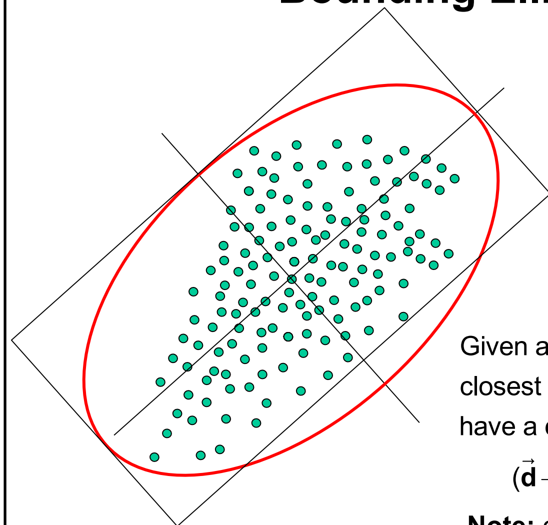
An Alternative to Bounding Boxes: Bounding Ellipsoids



Compute

$$\vec{\mathbf{p}}_C = \frac{1}{N}\sum_N \vec{\mathbf{v}}_i$$

$$\vec{\mathbf{u}}_i = \vec{\mathbf{v}}_i - \vec{\mathbf{p}}_C$$

$$\mathbf{A} = \sum_i \vec{\mathbf{u}}_i \vec{\mathbf{u}}_i^T = \mathbf{Q}\Lambda\mathbf{Q}^T$$

$$\Lambda = diag(\vec{\lambda})$$

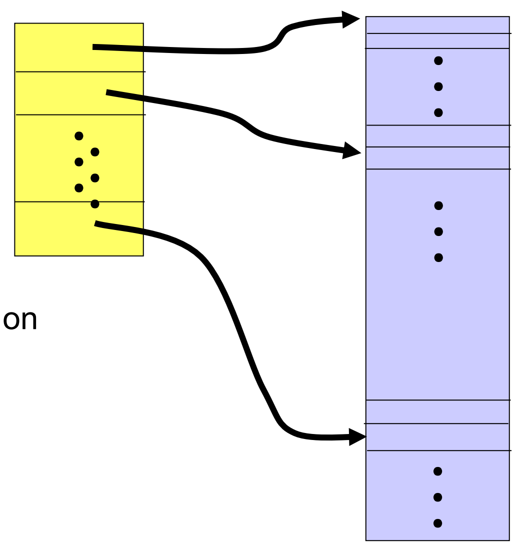
$$\rho^2 = \max_i \vec{\mathbf{u}}_i^T \mathbf{A} \vec{\mathbf{u}}_i$$

Given a search point $\vec{\mathbf{d}}$ and previous closest distance δ, the ellipsiod may have a closer point if

$$(\vec{\mathbf{d}} - \vec{\mathbf{p}}_C)^T \mathbf{A} (\vec{\mathbf{d}} - \vec{\mathbf{p}}_C) < \rho^2 + (\delta \max_k \lambda_k)^2$$

Note: can probably get a tighter bound, but this will work

Simple spatial sort



Point list

Index based on coordinates