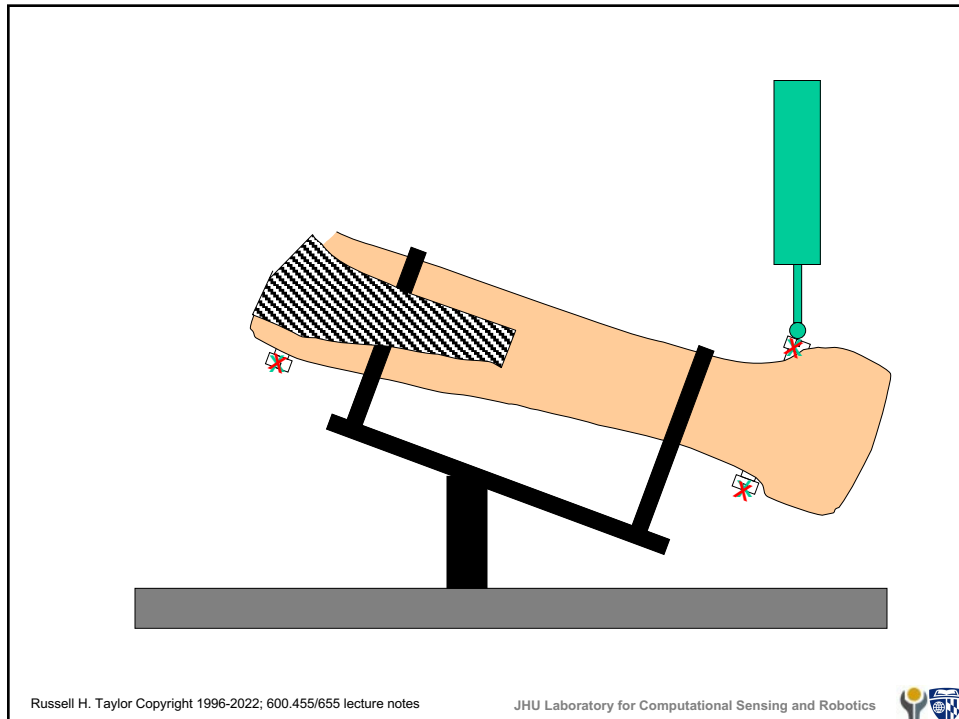# Cartesian Coordinates, Points, and Transformations

## CIS - 600.455/655
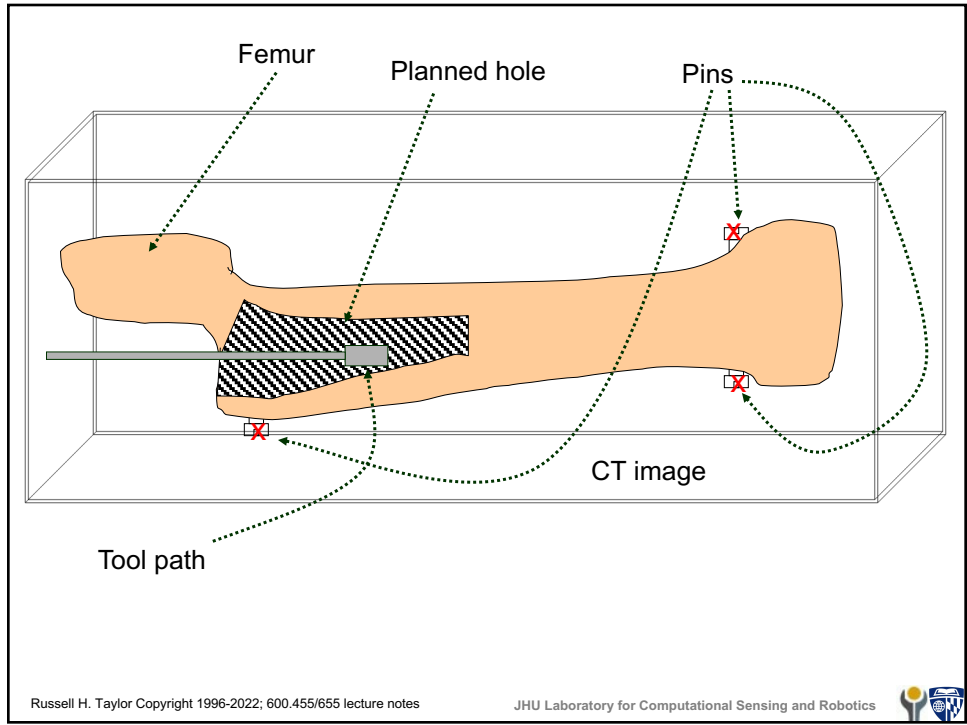
Russell Taylor
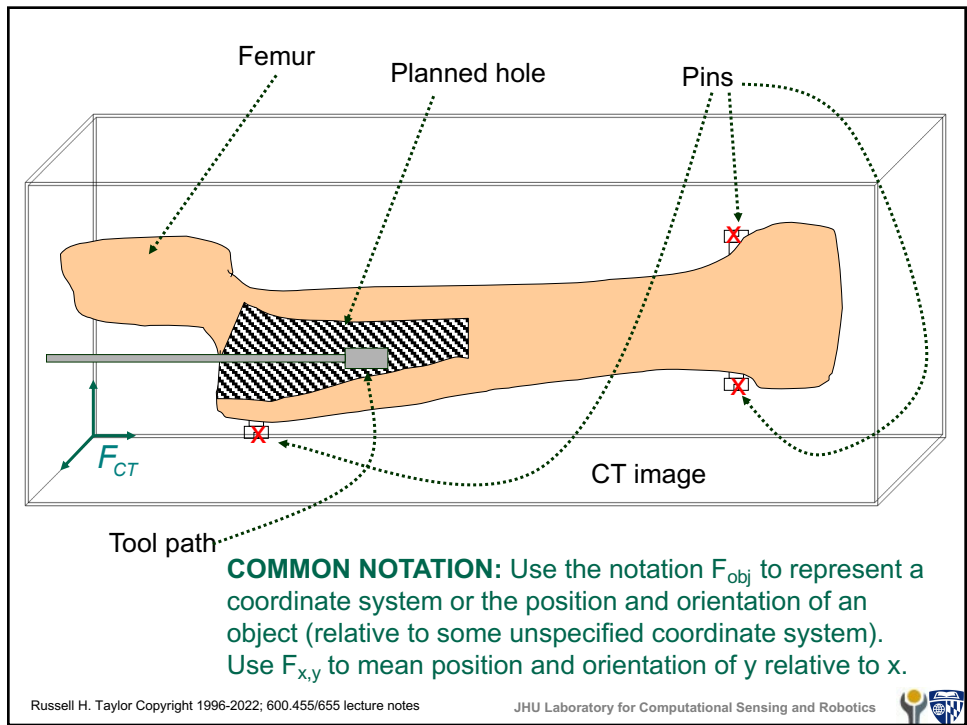
1

3

Femur    Planned hole    Pins

CT image

Tool path

4



Femur    Planned hole    Pins

$F_{CT}$

CT image

Tool path

**COMMON NOTATION:** Use the notation $F_{obj}$ to represent a coordinate system or the position and orientation of an object (relative to some unspecified coordinate system). Use $F_{x,y}$ to mean position and orientation of y relative to x.

5

2
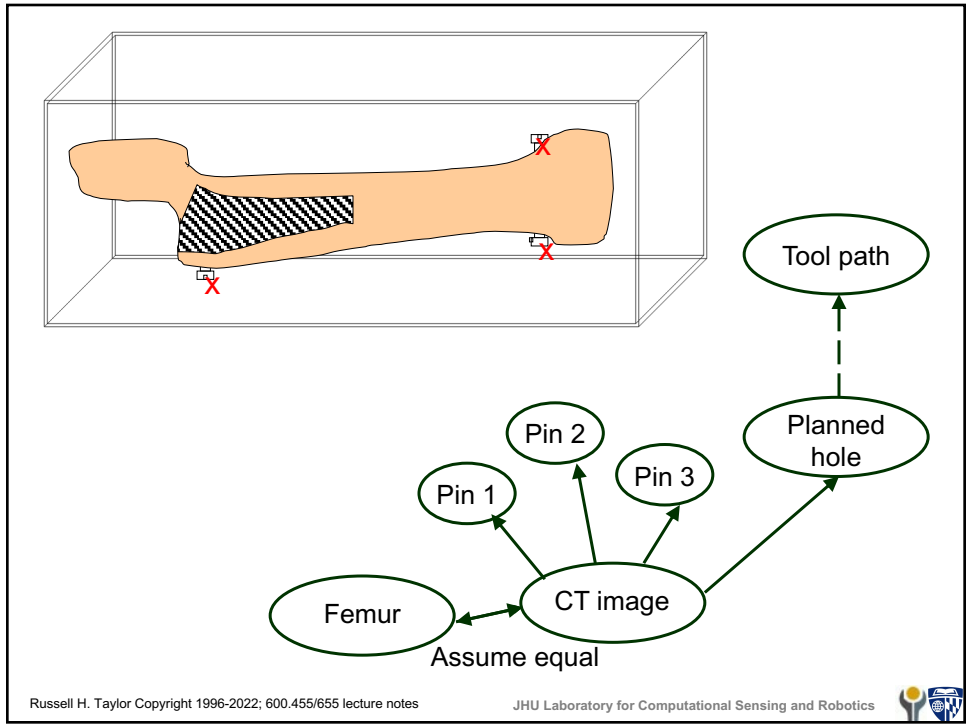
6
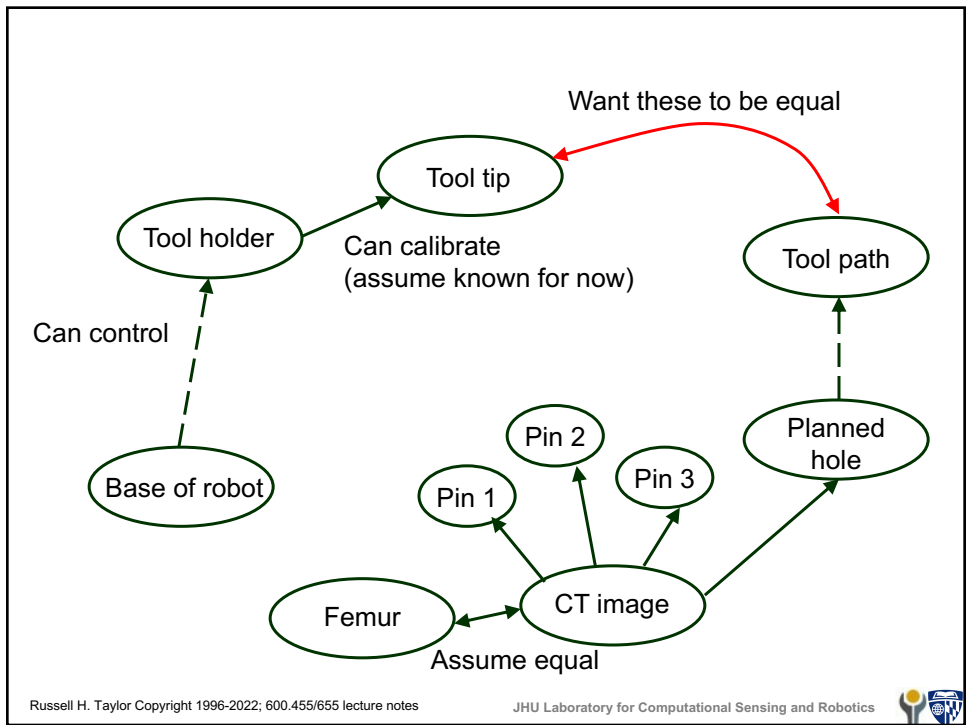
7

More correctly, this would be $\mathbf{F}_{\text{Base,Wrist}}$

Tool tip

Tool holder  $\mathbf{F}_{\text{WT}}$

$\mathbf{F}_{\text{Tip}} = \mathbf{F}_{\text{Wrist}} \bullet \mathbf{F}_{\text{WT}}$

**I**

Target

$\mathbf{F}_{\text{Wrist}}$

$\mathbf{F}_{\text{Target}}$

Base of robot

Question: What value of $\mathbf{F}_{\text{Wrist}}$ will make $\mathbf{F}_{\text{Tip}} = \mathbf{F}_{\text{Target}}$ ?

Answer:

$$\mathbf{F}_{\text{Wrist}} = \mathbf{F}_{\text{Target}} \bullet \mathbf{I} \bullet \mathbf{F}_{\text{WT}}^{-1}$$

$$= \mathbf{F}_{\text{Target}} \bullet \mathbf{F}_{\text{WT}}^{-1}$$

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Tool tip

Tool holder  $\mathbf{F}_{\text{WT}}$

$\mathbf{F}_{\text{Tip}} = \mathbf{F}_{\text{Wrist}} \bullet \mathbf{F}_{\text{WT}}$

**I**

Target

$\mathbf{F}_{\text{Wrist}}$

$\mathbf{F}_{\text{Target}}$

**Notational Note**

We use the notation $\mathbf{A} \bullet \mathbf{B}$ to represent composition or transformation. Where the context is clear, we may also use $\mathbf{AB}$ for the same thing.

Question: What value of $\mathbf{F}_{\text{Wrist}}$ will make $\mathbf{F}_{\text{Tip}} = \mathbf{F}_{\text{Target}}$ ?
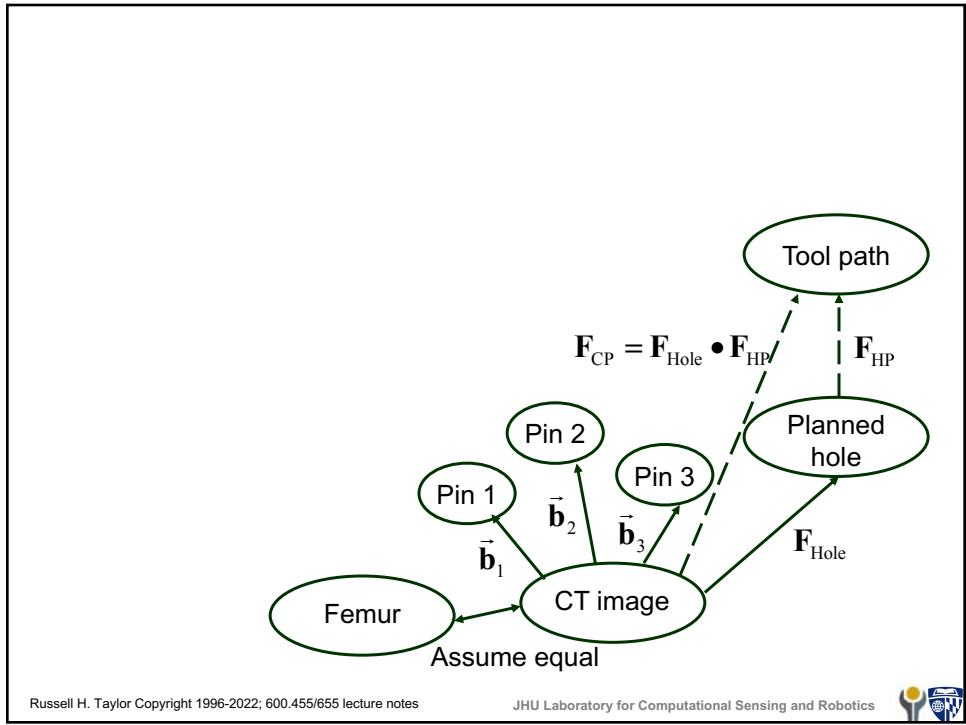
Answer:

$$\mathbf{F}_{\text{Wrist}} = \mathbf{F}_{\text{Target}} \bullet \mathbf{I} \bullet \mathbf{F}_{\text{WT}}^{-1}$$

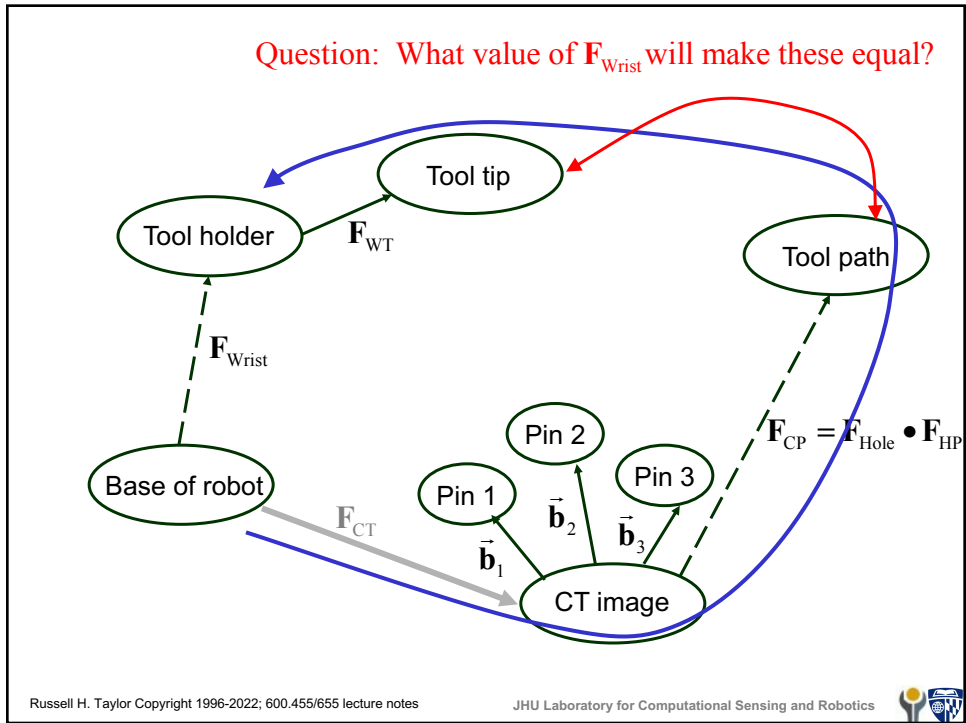$$= \mathbf{F}_{\text{Target}} \bullet \mathbf{F}_{\text{WT}}^{-1}$$

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$$\mathbf{F}_{CP} = \mathbf{F}_{Hole} \bullet \mathbf{F}_{HP}$$

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Question: What value of $\mathbf{F}_{Wrist}$ will make these equal?

$$\mathbf{F}_{CP} = \mathbf{F}_{Hole} \bullet \mathbf{F}_{HP}$$

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**But**: We must find $F_{CT}$ … Let's review some math

Tool tip

Tool holder $\quad \mathbf{F}_{WT}$

$\mathbf{F}_{Wrist} = \mathbf{F}_{CT} \bullet \mathbf{F}_{CP} \bullet \mathbf{I} \bullet \mathbf{F}_{WT}^{-1}$

$\quad\quad\quad = \mathbf{F}_{CT} \bullet \mathbf{F}_{CP} \bullet \mathbf{F}_{WT}^{-1}$

$\mathbf{F}_{Wrist}$

Tool path

Pin 2

Base of robot

Pin 1 $\quad \vec{\mathbf{b}}_2 \quad$ Pin 3

$\mathbf{F}_{CP} = \mathbf{F}_{Hole} \bullet \mathbf{F}_{HP}$

$\mathbf{F}_{CT}$

$\vec{\mathbf{b}}_1 \quad \vec{\mathbf{b}}_3$

CT image

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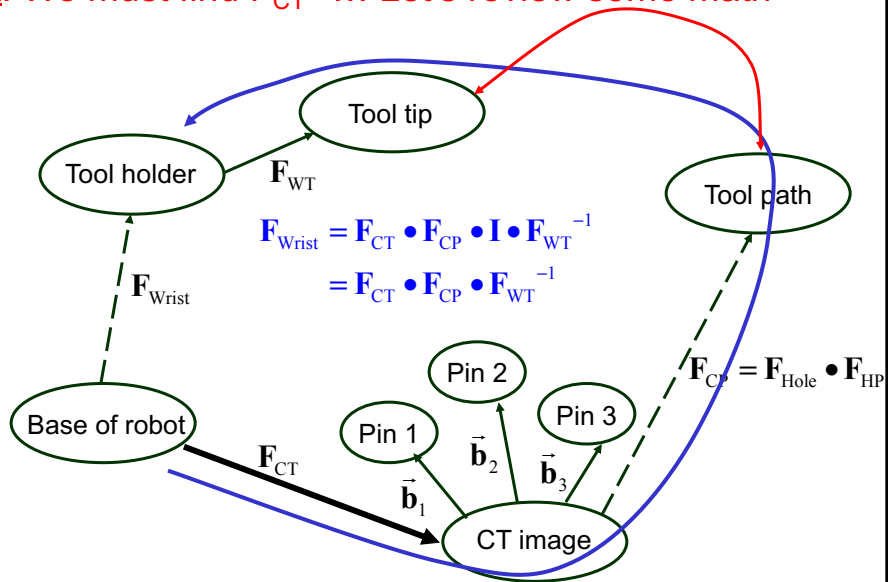
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# Coordinate Frame Transformation
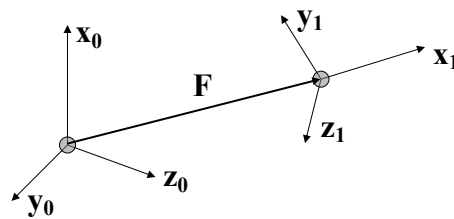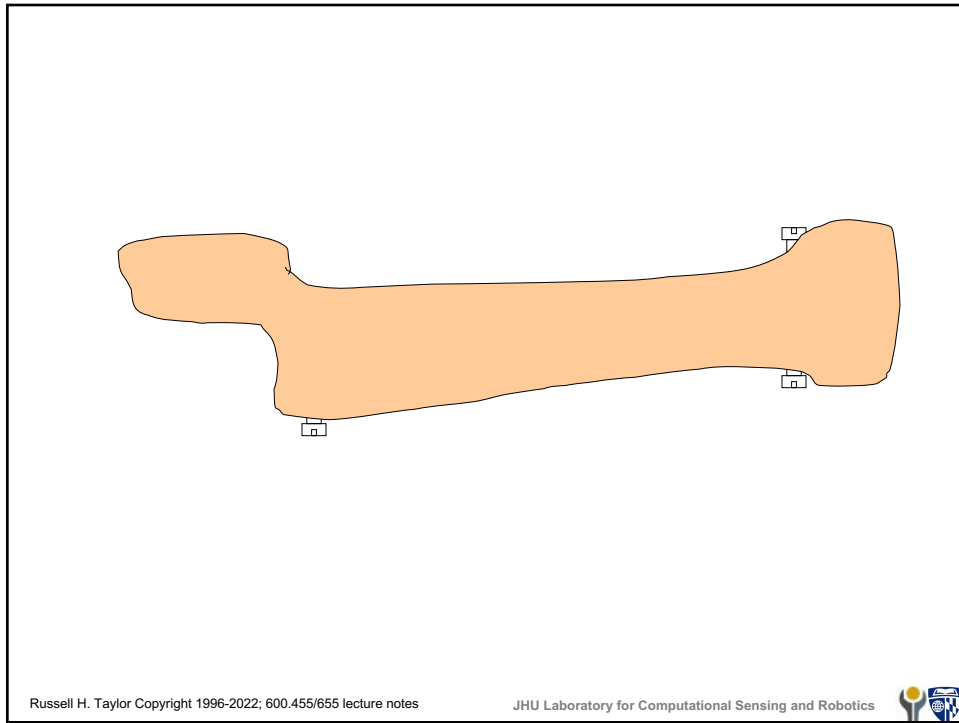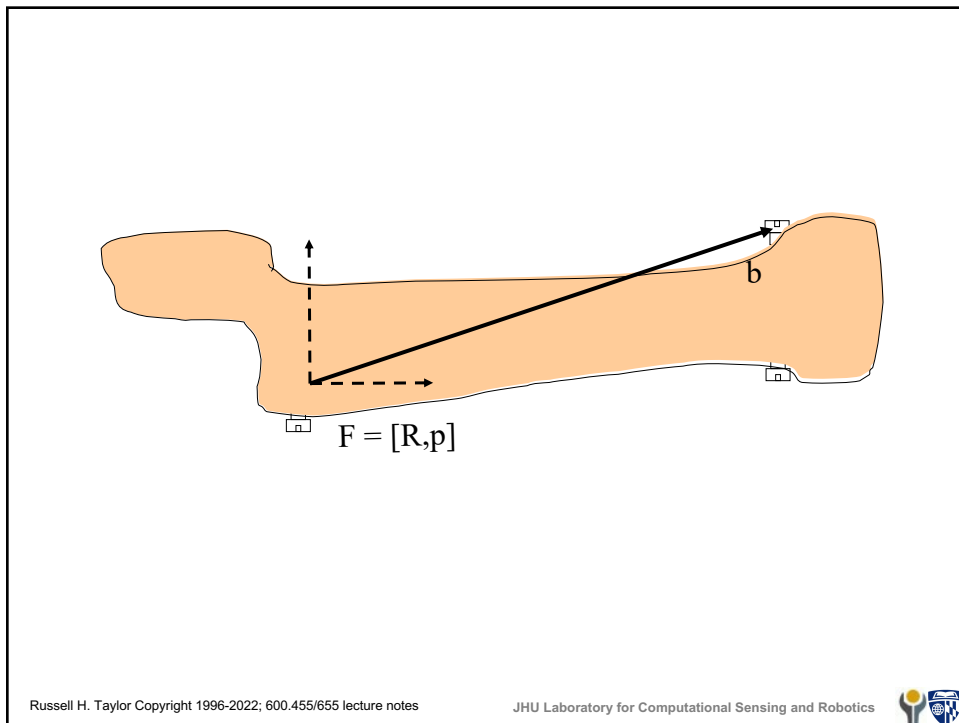
$$\mathbf{F} = [\mathbf{R}, \mathbf{p}]$$



$\mathbf{x}_0$ $\quad$ $\mathbf{y}_1$

$\mathbf{F}$ $\quad$ $\mathbf{x}_1$

$\mathbf{z}_1$

$\mathbf{z}_0$

$\mathbf{y}_0$

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**Slide acknowledgment: Sarah Graham and Andy Bzostek**

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$$F = [R,p]$$

b

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$F = [R,p]$

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$\mathbf{F = [\ I,0]}$

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$$\vec{b}$$

$$\vec{v}_R = R \bullet \vec{b}$$

$$F = [R,0]$$

$$b$$

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$$\vec{v}_R = R \bullet \vec{b}$$

$$\vec{p}$$

$$F = [R,p]$$

$$\vec{p}$$

$$b$$

$$\vec{v} = \vec{v}_R + \vec{p}$$
$$= R \bullet \vec{b} + \vec{p}$$

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# Coordinate Frames

$$\vec{\mathbf{v}} = \mathbf{F} \bullet \vec{\mathbf{b}}$$
$$= [\mathbf{R}, \bar{\mathbf{p}}] \bullet \vec{\mathbf{b}}$$
$$= \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}$$

b

$$F = [R,p]$$

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# Forward and Inverse Frame Transformations

### Forward

$$\mathbf{F} = [\mathbf{R}, \vec{\mathbf{p}}]$$

$$\vec{\mathbf{v}} = \mathbf{F} \bullet \vec{\mathbf{b}}$$
$$= [\mathbf{R}, \vec{\mathbf{p}}] \bullet \vec{\mathbf{b}}$$
$$= \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}$$

### Inverse

$$\mathbf{F}^{-1}\vec{\mathbf{v}} = \vec{\mathbf{b}}$$
$$\vec{\mathbf{b}} = \mathbf{R}^{-1} \bullet (\vec{\mathbf{v}} - \vec{\mathbf{p}})$$
$$= \mathbf{R}^{-1} \bullet \vec{\mathbf{v}} - \mathbf{R}^{-1} \bullet \vec{\mathbf{p}}$$

$$\mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1} \bullet \vec{\mathbf{p}}]$$

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## Composition

Assume $\mathbf{F}_1 = [\mathbf{R}_1, \vec{\mathbf{p}}_1], \quad \mathbf{F}_2 = [\mathbf{R}_2, \vec{\mathbf{p}}_2]$

Then

$$
\begin{aligned}
\mathbf{F}_1 \bullet \mathbf{F}_2 \bullet \vec{\mathbf{b}} &= \mathbf{F}_1 \bullet (\mathbf{F}_2 \bullet \vec{\mathbf{b}}) \\
&= \mathbf{F}_1 \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\
&= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\
&= \mathbf{R}_1 \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) + \vec{\mathbf{p}}_1 \\
&= \mathbf{R}_1 \bullet \mathbf{R}_2 \bullet \vec{\mathbf{b}} + \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1 \\
&= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1] \bullet \vec{\mathbf{b}}
\end{aligned}
$$

So

$$
\begin{aligned}
\mathbf{F}_1 \bullet \mathbf{F}_2 &= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet [\mathbf{R}_2, \vec{\mathbf{p}}_2] \\
&= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1]
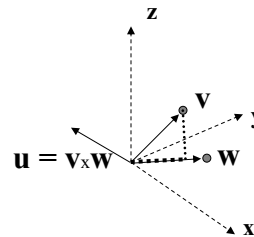\end{aligned}
$$

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## Vectors

$$\vec{\mathbf{v}}_{col} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\vec{\mathbf{v}}_{row} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}$$



$\mathbf{u} = \mathbf{v}_x\mathbf{w}$

length: $\left\| \vec{\mathbf{v}} \right\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

dot product: $a = \vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = \left( v_x w_x + v_y w_y + v_z w_z \right) = \left\| \vec{\mathbf{v}} \right\| \left\| \vec{\mathbf{w}} \right\| \cos\theta$

cross product: $\vec{\mathbf{u}} = \vec{\mathbf{v}} \times \vec{\mathbf{w}} = \begin{bmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{bmatrix}, \left\| \mathbf{u} \right\| = \left\| \mathbf{v} \right\| \left\| \mathbf{w} \right\| \sin\theta$

**Slide acknowledgment: Sarah Graham and Andy Bzostek**

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## Matrix representation of cross product operator

Define

$$\hat{\vec{\mathbf{a}}} \overset{\Delta}{=} skew(\mathbf{a}) \overset{\Delta}{=} \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Then

$$\vec{\mathbf{a}} \times \vec{\mathbf{v}} = skew(\vec{\mathbf{a}}) \bullet \vec{\mathbf{v}}$$

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## Rotations: Some Notation

$$Rot(\vec{\mathbf{a}}, \alpha) = \text{ Rotation by angle } \alpha \text{ about axis } \vec{\mathbf{a}}$$
$$\mathbf{R}_{\vec{\mathbf{a}}}(\alpha) = \text{ Rotation by angle } \alpha \text{ about axis } \vec{\mathbf{a}}$$
$$\mathbf{R}(\vec{\mathbf{a}}) = Rot(\vec{\mathbf{a}}, \|\vec{\mathbf{a}}\|)$$

$$\mathbf{R}_{\mathbf{xyz}}(\alpha, \beta, \gamma) = \mathbf{R}(\vec{\mathbf{x}}, \alpha) \bullet \mathbf{R}(\vec{\mathbf{y}}, \beta) \bullet \mathbf{R}(\vec{\mathbf{z}}, \gamma)$$
$$\mathbf{R}_{\mathbf{zyz}}(\alpha, \beta, \gamma) = \mathbf{R}(\vec{\mathbf{z}}, \alpha) \bullet \mathbf{R}(\vec{\mathbf{y}}, \beta) \bullet \mathbf{R}(\vec{\mathbf{z}}, \gamma)$$

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## Rotations: A few useful facts

$$Rot(s\vec{\mathbf{a}},\alpha)\bullet\vec{\mathbf{a}}=\vec{\mathbf{a}} \quad \text{and} \quad \left\|Rot(\vec{\mathbf{a}},\alpha)\bullet\vec{\mathbf{b}}\right\|=\left\|\vec{\mathbf{b}}\right\|$$

$$Rot(\vec{\mathbf{a}},\alpha)=Rot(\hat{\mathbf{a}},\alpha) \quad \text{where} \quad \hat{\mathbf{a}}=\frac{\vec{\mathbf{a}}}{\left\|\vec{\mathbf{a}}\right\|}$$

NOTE: Unless otherwise stated, we will usually assume that $\vec{\mathbf{a}}$ in $Rot(\vec{\mathbf{a}},\theta)$ is a unit vector.  I.e., $\left\|\vec{\mathbf{a}}\right\|=1$.

$$Rot(\vec{\mathbf{a}},\alpha)\bullet Rot(\vec{\mathbf{a}},\beta)=Rot(\vec{\mathbf{a}},\alpha+\beta)$$

$$Rot(\vec{\mathbf{a}},\alpha)^{-1}=Rot(\vec{\mathbf{a}},-\alpha)$$

$$Rot(\vec{\mathbf{a}},0)\bullet\vec{\mathbf{b}}=\vec{\mathbf{b}} \quad \text{i.e.,} \quad Rot(\vec{\mathbf{a}},0)=\mathbf{I}_{Rot}= \text{ the identity rotation}$$

$$Rot(\hat{\mathbf{a}},\alpha)\bullet\vec{\mathbf{b}}=\left(\hat{\mathbf{a}}\bullet\vec{\mathbf{b}}\right)\hat{\mathbf{a}}+Rot(\hat{\mathbf{a}},\alpha)\bullet\left(\vec{\mathbf{b}}-\left(\hat{\mathbf{a}}\bullet\vec{\mathbf{b}}\right)\hat{\mathbf{a}}\right)$$

$$Rot(\hat{\mathbf{a}},\alpha)\bullet Rot(\hat{\mathbf{b}},\beta)=Rot(\hat{\mathbf{b}},\beta)\bullet Rot\left(Rot(\hat{\mathbf{b}},-\beta)\bullet\hat{\mathbf{a}},\alpha\right)$$

$$Rot(\hat{\mathbf{a}},\alpha)\bullet\mathbf{R}_{\beta}=\mathbf{R}_{\beta}\bullet Rot(\mathbf{R}_{\beta}^{-1}\bullet\hat{\mathbf{a}},\alpha)$$

$$\mathbf{R}_{\alpha}\bullet Rot(\hat{\mathbf{b}},\beta)=Rot(\mathbf{R}_{\alpha}\bullet\hat{\mathbf{b}},\beta)\bullet\mathbf{R}_{\alpha}$$

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## Rotations: more facts

If $\vec{\mathbf{v}}=[v_x,v_y,v_z]^T$ then a rotation $\mathbf{R}\bullet\vec{\mathbf{v}}$ may be described in terms of the effects of $\mathbf{R}$ on orthogonal unit vectors, $\vec{\mathbf{e}}_x=[1,0,0]^T$, $\vec{\mathbf{e}}_y=[0,1,0]^T$, $\vec{\mathbf{e}}_z=[0,0,1]^T$

$$\mathbf{R}\bullet\vec{\mathbf{v}}=v_x\vec{\mathbf{r}}_x+v_y\vec{\mathbf{r}}_y+v_z\vec{\mathbf{r}}_z$$

where

$$\vec{\mathbf{r}}_x=\mathbf{R}\bullet\vec{\mathbf{e}}_x$$
$$\vec{\mathbf{r}}_y=\mathbf{R}\bullet\vec{\mathbf{e}}_y$$
$$\vec{\mathbf{r}}_z=\mathbf{R}\bullet\vec{\mathbf{e}}_z$$

Note that rotation doesn't affect inner products

$$\left(\mathbf{R}\bullet\vec{\mathbf{b}}\right)\bullet\left(\mathbf{R}\bullet\vec{\mathbf{c}}\right)=\vec{\mathbf{b}}\bullet\vec{\mathbf{c}}$$

or  lengths of vectors

$$\left\|\mathbf{R}\bullet\vec{\mathbf{v}}\right\|=\left\|\vec{\mathbf{v}}\right\|$$

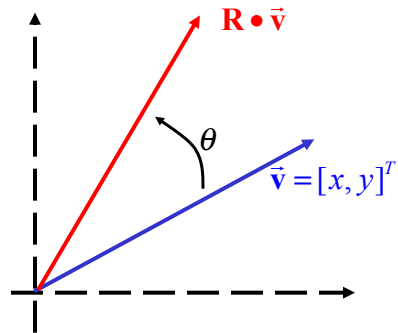JHU Laboratory for Computational Sensing and Robotics

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## Rotations in the plane

$$\mathbf{R} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix}$$



$\mathbf{R} \bullet \vec{\mathbf{v}}$

$\theta$

$\vec{\mathbf{v}} = [x, y]^{T}$

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## Rotations in the plane

$$\mathbf{R} \bullet \begin{bmatrix} \vec{\mathbf{e}}_x & \vec{\mathbf{e}}_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R} \bullet \vec{\mathbf{e}}_x & \mathbf{R} \bullet \vec{\mathbf{e}}_y \end{bmatrix}$$

$$= \begin{bmatrix} \vec{\mathbf{r}}_x & \vec{\mathbf{r}}_y \end{bmatrix}$$
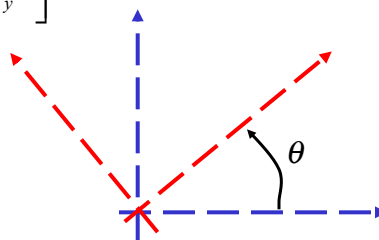


$\theta$

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## 3D Rotation Matrices

$$\mathbf{R} \bullet \begin{bmatrix} \vec{\mathbf{e}}_x & \vec{\mathbf{e}}_y & \vec{\mathbf{e}}_z \end{bmatrix} = \begin{bmatrix} \mathbf{R} \bullet \vec{\mathbf{e}}_x & \mathbf{R} \bullet \vec{\mathbf{e}}_y & \mathbf{R} \bullet \vec{\mathbf{e}}_z \end{bmatrix}$$

$$= \begin{bmatrix} \vec{\mathbf{r}}_x & \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_z \end{bmatrix}$$

$$\mathbf{R}^T \bullet \mathbf{R} = \begin{bmatrix} \hat{\mathbf{r}}_x^{\,T} \\ \hat{\mathbf{r}}_y^{\,T} \\ \hat{\mathbf{r}}_z \end{bmatrix} \bullet \begin{bmatrix} \vec{\mathbf{r}}_x & \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_z \end{bmatrix}$$

$$= \begin{bmatrix} \vec{\mathbf{r}}_x^{\,T} \bullet \vec{\mathbf{r}}_x & \vec{\mathbf{r}}_x^{\,T} \bullet \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_x^{\,T} \bullet \vec{\mathbf{r}}_z \\ \vec{\mathbf{r}}_y^{\,T} \bullet \vec{\mathbf{r}}_x & \vec{\mathbf{r}}_y^{\,T} \bullet \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_y^{\,T} \bullet \vec{\mathbf{r}}_z \\ \vec{\mathbf{r}}_z^{\,T} \bullet \vec{\mathbf{r}}_x & \vec{\mathbf{r}}_z^{\,T} \bullet \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_z^{\,T} \bullet \vec{\mathbf{r}}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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## Properties of Rotation Matrices

Inverse of a Rotation Matrix equals its transpose:

$$\mathbf{R}^{-1} = \mathbf{R}^T$$
$$\mathbf{R}^T \mathbf{R} = \mathbf{R}\,\mathbf{R}^T = \mathbf{I}$$

The Determinant of a Rotation matrix is equal to +1:

$$\det(\mathbf{R}) = +1$$

Any Rotation can be described by consecutive rotations about the three primary axes, x, y, and z:

$$\mathbf{R} = \mathbf{R}_{z,\theta}\,\mathbf{R}_{y,\phi}\,\mathbf{R}_{x,\psi}$$

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# Canonical 3D Rotation Matrices

***Note:** Right-Handed Coordinate System*

$$\mathbf{R}_{\vec{x}}(\theta) = Rot(\vec{x},\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\vec{y}}(\theta) = Rot(\vec{y},\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\vec{z}}(\theta) = Rot(\vec{z},\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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# Axis-angle Representations of Rotations

## Rodrigues' Formula (1840)*

Rotation of a vector $\vec{\mathbf{b}}$ by angle $\alpha$ about axis $\vec{\mathbf{a}}$

(Assumes that $\vec{\mathbf{a}}$ is a unit vector, $\|\vec{\mathbf{a}}\| = 1$)
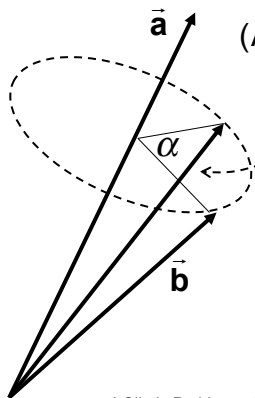
$$\vec{\mathbf{c}} = Rot(\vec{\mathbf{a}},\alpha) \bullet \vec{\mathbf{b}}$$
$$= \vec{\mathbf{b}}\cos\alpha + \vec{\mathbf{a}} \times \vec{\mathbf{b}}\sin\alpha + \vec{\mathbf{a}}(\vec{\mathbf{a}} \bullet \vec{\mathbf{b}})(1-\cos\alpha)$$

In matrix form this is $\vec{\mathbf{c}} = \mathbf{R} \bullet \vec{\mathbf{b}}$ where

$$\mathbf{R} = (\cos\alpha)\mathbf{I} + (\sin\alpha)skew(\vec{\mathbf{a}}) + (1-\cos\alpha)\vec{\mathbf{a}} \bullet \vec{\mathbf{a}}^T$$

* Olinde Rodrigues, "Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendants des causes qui peuvent les produire", *Journal de Mathématiques Pures et Appliquées* 5 (1840), 380–440. (http://sites.mathdoc.fr/JMPA/PDF/JMPA_1840_1_5_A39_0.pdf)

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## Exponential representation

Consider a rotation about axis $\vec{\mathbf{n}}$ by angle $\theta$. Then

$$e^{skew(\vec{\mathbf{n}})\theta} = \mathbf{I} + \theta skew(\vec{\mathbf{n}}) + \frac{\theta^2}{2!} skew(\vec{\mathbf{n}})^2 + \dots$$

By doing some manipulation, you can show

$$Rot(\vec{\mathbf{n}},\theta) = e^{skew(\vec{\mathbf{n}})\theta}$$
$$= \mathbf{I} + skew(\vec{\mathbf{n}})\sin\theta + skew(\vec{\mathbf{n}})^2(1-\cos\theta)$$
$$= \mathbf{I} + skew(\vec{\mathbf{n}})\sin\theta + (\vec{\mathbf{n}}\bullet\vec{\mathbf{n}}^T - \mathbf{I})(1-\cos\theta)$$
$$= \mathbf{I}\cos\theta + skew(\vec{\mathbf{n}})\sin\theta + \vec{\mathbf{n}}\bullet\vec{\mathbf{n}}^T(1-\cos\theta)$$

Note that for small $\theta$, this reduces to

$$Rot(\vec{\mathbf{n}},\theta) \approx \mathbf{I} + skew(\theta\vec{\mathbf{n}})$$

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## Exponential representation

Consider a rotation about axis $\vec{\mathbf{n}}$ by angle $\theta$. Then

$$e^{skew(\vec{\mathbf{n}})\theta} = \mathbf{I} + \theta skew(\vec{\mathbf{n}}) + \frac{\theta^2}{2!} skew(\vec{\mathbf{n}})^2 + \dots$$

By doing some manipulation, you can show

$$Rot(\vec{\mathbf{n}},\theta) = e^{skew(\vec{\mathbf{n}})\theta}$$
$$= \mathbf{I} + skew(\vec{\mathbf{n}})\sin\theta + skew(\vec{\mathbf{n}})^2(1-\cos\theta)$$
$$= \mathbf{I} + skew(\vec{\mathbf{n}})\sin\theta + (\vec{\mathbf{n}}\bullet\vec{\mathbf{n}}^T - \mathbf{I})(1-\cos\theta)$$
$$= \mathbf{I}\cos\theta + skew(\vec{\mathbf{n}})\sin\theta + \vec{\mathbf{n}}\bullet\vec{\mathbf{n}}^T(1-\cos\theta)$$

which is just Rodrigues' formula.

Note that for small $\theta$, this reduces to

$$Rot(\vec{\mathbf{n}},\theta) \approx \mathbf{I} + skew(\theta\vec{\mathbf{n}})$$

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# Cayley Transform Representation

Consider the rotation $\text{Rot}(\vec{\mathbf{n}}, \theta)$ and define

$$\vec{\mathbf{a}} = \left( \tan\frac{\theta}{2} \right) \vec{\mathbf{n}}$$

$$\mathbf{A} = \text{skew}(\vec{\mathbf{a}})$$

Then,

$$\mathbf{R} = (\mathbf{I} - \mathbf{A})^{-1}(\mathbf{I} + \mathbf{A})$$

gives the rotation matrix corresponding to $\text{Rot}(\vec{\mathbf{n}}, \theta)$. Similarly, given $\mathbf{R}$,

$$\mathbf{A} = (\mathbf{R} - \mathbf{I})(\mathbf{I} + \mathbf{R})^{-1}$$

gives the elements of $\text{skew}(\vec{\mathbf{a}})$ and hence for $\vec{\mathbf{a}}$.

**Note** : The above relations require that $\theta \neq \pm\pi$.

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# Homogeneous Coordinates

- Widely used in graphics, geometric calculations
- Represent 3D vector as 4D quantity
- For our current purposes, we will keep the "scale" *s = 1*

$$\vec{\mathbf{v}} \equiv \begin{bmatrix} xs \\ ys \\ zs \\ s \end{bmatrix} \cong \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

## Representing Frame Transformations as Matrices

$$\mathbf{v} + \mathbf{p} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \mathbf{p}_x \\ 0 & 1 & 0 & \mathbf{p}_y \\ 0 & 0 & 1 & \mathbf{p}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ \mathbf{v}_z \\ 1 \end{bmatrix} = [\mathbf{I}, \vec{\mathbf{p}}] \bullet \mathbf{v}$$
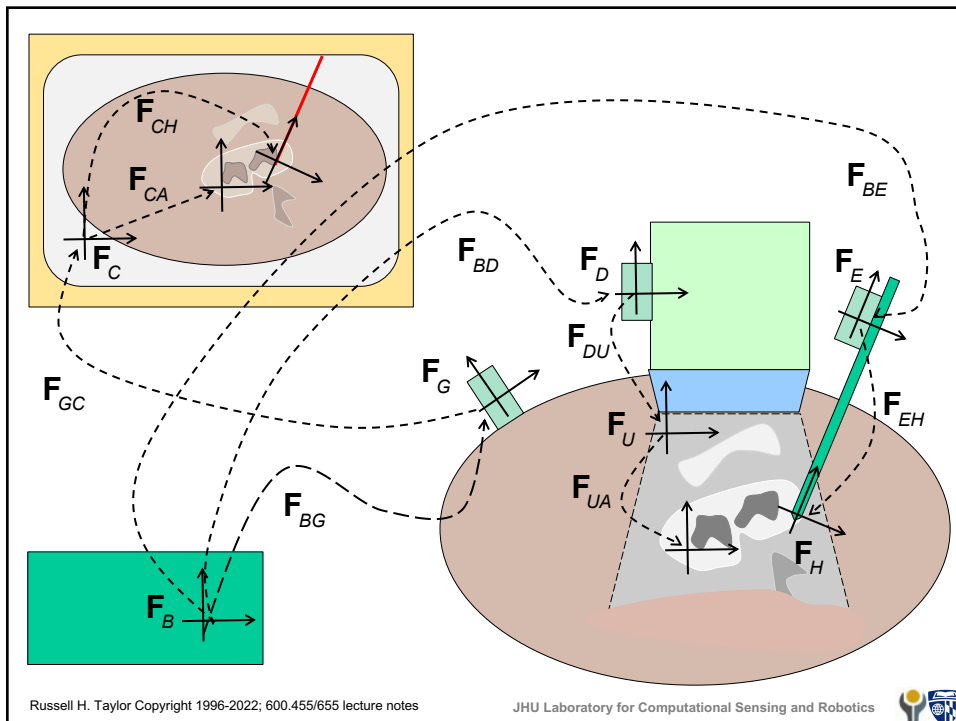
$$\mathbf{R} \bullet \mathbf{v} \rightarrow \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}$$

$$\mathbf{P} \bullet \mathbf{R} \rightarrow \begin{bmatrix} \mathbf{I} & \mathbf{p} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \bullet \begin{bmatrix} \mathbf{R} & 0 \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = [\mathbf{R}, \mathbf{p}] = \mathbf{F}$$

$$\mathbf{F} \bullet \mathbf{v} \rightarrow \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = \begin{bmatrix} (\mathbf{R} \bullet \mathbf{v}) + \mathbf{p} \\ 1 \end{bmatrix}$$

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Give a formula for computing the pose $\mathbf{F}_{GH}$
of the surgical tool coordinate system
relative to the patient rigid body coordinate system $\mathbf{F}_G$
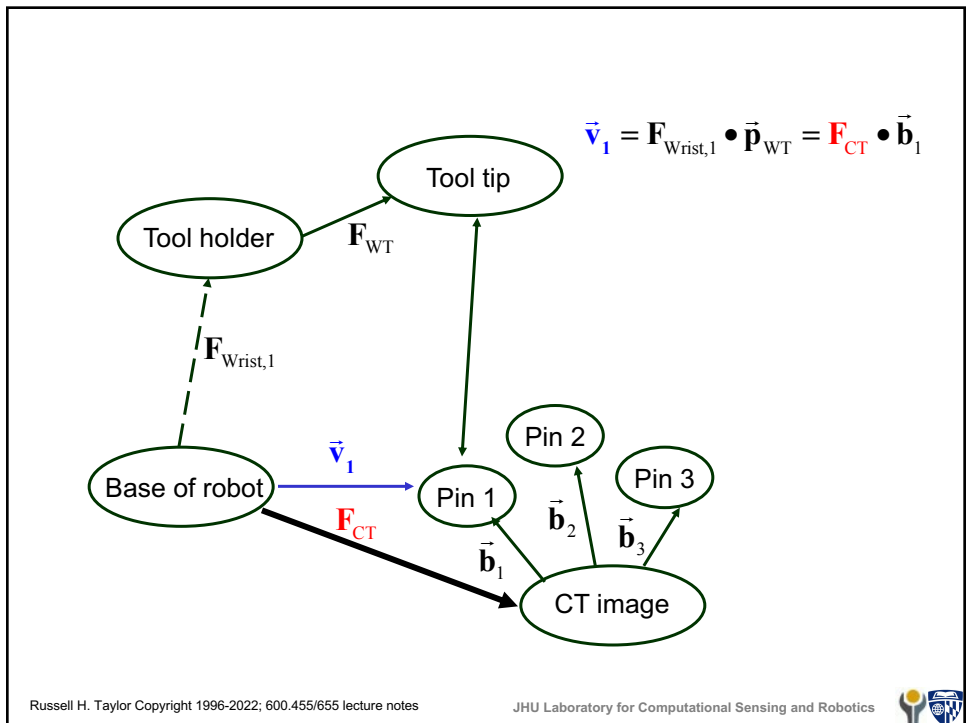
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$$\mathbf{F}_{GH} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\mathbf{F}_{EH}$$
$$= \mathbf{F}_{BG}^{-1}\mathbf{F}_{BH}$$

Give a formula for computing the pose $\mathbf{F}_{GH}$
of the surgical tool coordinate system
relative to the patient rigid body coordinate system $\mathbf{F}_G$ ?

$$\underline{\mathbf{R}}_{GH} = \mathbf{R}_{BG}^{-1}\mathbf{R}_{BE}\mathbf{R}_{EH}$$
$$\vec{\mathbf{p}}_{GH} = \mathbf{F}_{BG}^{-1}\vec{\mathbf{p}}_{BH} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\vec{\mathbf{p}}_{EH}$$
$$\vec{\mathbf{p}}_{GH} = \mathbf{F}_{BG}^{-1}(\mathbf{R}_{BE}\vec{\mathbf{p}}_{EH} + \vec{\mathbf{p}}_{BE})$$
$$\vec{\mathbf{p}}_{GH} = \mathbf{R}_{BG}^{-1}(\mathbf{R}_{BE}\vec{\mathbf{p}}_{EH} + \vec{\mathbf{p}}_{BE} - \vec{\mathbf{p}}_{BG})$$

What are the components $\mathbf{F}_{GH} = [\mathbf{R}_{GH}, \vec{\mathbf{p}}_{GH}]$?

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$$\vec{\mathbf{v}}_1 = \mathbf{F}_{\text{Wrist},1} \bullet \vec{\mathbf{p}}_{\text{WT}} = \mathbf{F}_{\text{CT}} \bullet \vec{\mathbf{b}}_1$$

Tool tip

Tool holder  $\mathbf{F}_{\text{WT}}$

$\mathbf{F}_{\text{Wrist},1}$

Base of robot   $\vec{\mathbf{v}}_1$   Pin 1

Pin 2

Pin 3

$\mathbf{F}_{\text{CT}}$

$\vec{\mathbf{b}}_2$   $\vec{\mathbf{b}}_3$

$\vec{\mathbf{b}}_1$

CT image

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$$\vec{v}_1 = F_{\text{Wrist},1} \bullet \vec{p}_{\text{WT}} = F_{\text{CT}} \bullet \vec{b}_1$$
$$\vec{v}_2 = F_{\text{Wrist},2} \bullet \vec{p}_{\text{WT}} = F_{\text{CT}} \bullet \vec{b}_2$$

Tool tip

Tool holder $F_{\text{WT}}$

$F_{\text{Wrist},2}$

$\vec{v}_2$

Pin 2

Pin 3

Base of robot

$F_{\text{CT}}$

Pin 1

$\vec{b}_2$

$\vec{b}_3$

$\vec{b}_1$

CT image

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$$\vec{v}_1 = F_{\text{Wrist},1} \bullet \vec{p}_{\text{WT}} = F_{\text{CT}} \bullet \vec{b}_1$$
$$\vec{v}_2 = F_{\text{Wrist},2} \bullet \vec{p}_{\text{WT}} = F_{\text{CT}} \bullet \vec{b}_2$$
$$\vec{v}_3 = F_{\text{Wrist},3} \bullet \vec{p}_{\text{WT}} = F_{\text{CT}} \bullet \vec{b}_3$$

Tool tip

Tool holder $F_{\text{WT}}$

$F_{\text{Wrist},3}$

$\vec{v}_3$

Pin 2

Pin 3

Base of robot

$F_{\text{CT}}$

Pin 1

$\vec{b}_2$

$\vec{b}_3$

$\vec{b}_1$

CT image

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# Frame transformation from 3 point pairs

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# Frame transformation from 3 point pairs



$$\mathbf{F}_{rob} \bullet \vec{\mathbf{v}}_k = \mathbf{F}_{CT} \bullet \vec{\mathbf{b}}_k$$

$$\vec{\mathbf{v}}_k = \mathbf{F}_{rob}^{-1} \mathbf{F}_{CT} \bullet \vec{\mathbf{b}}_k$$

$$\vec{\mathbf{v}}_k = \mathbf{F}_{rC} \bullet \vec{\mathbf{b}}_k$$

$$\mathbf{F}_{rC} = \mathbf{F}_{rob}^{-1} \mathbf{F}_{CT}$$
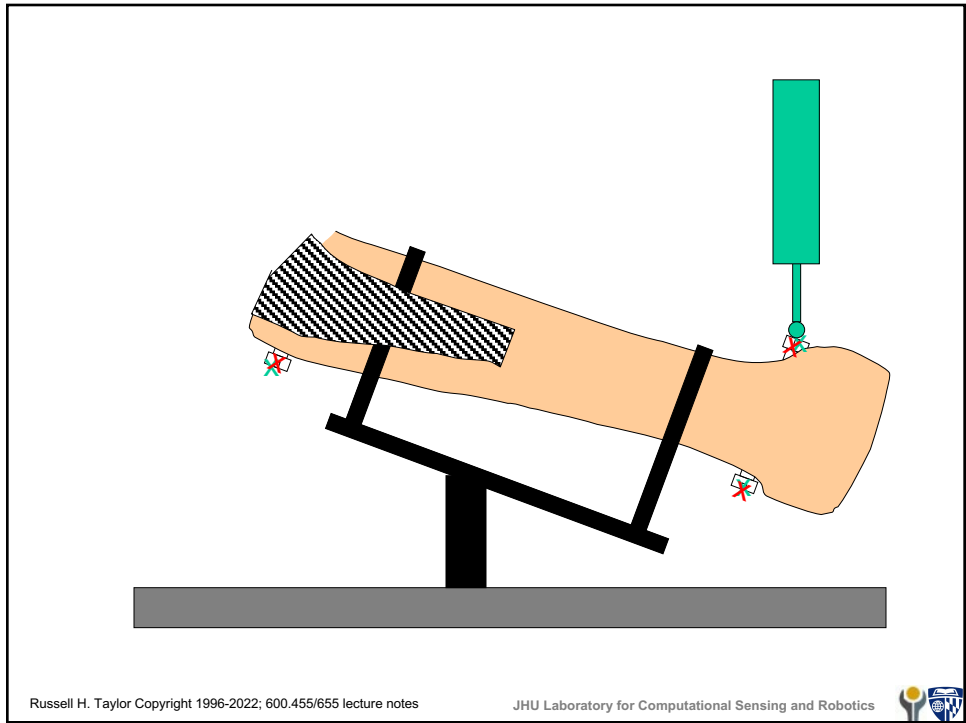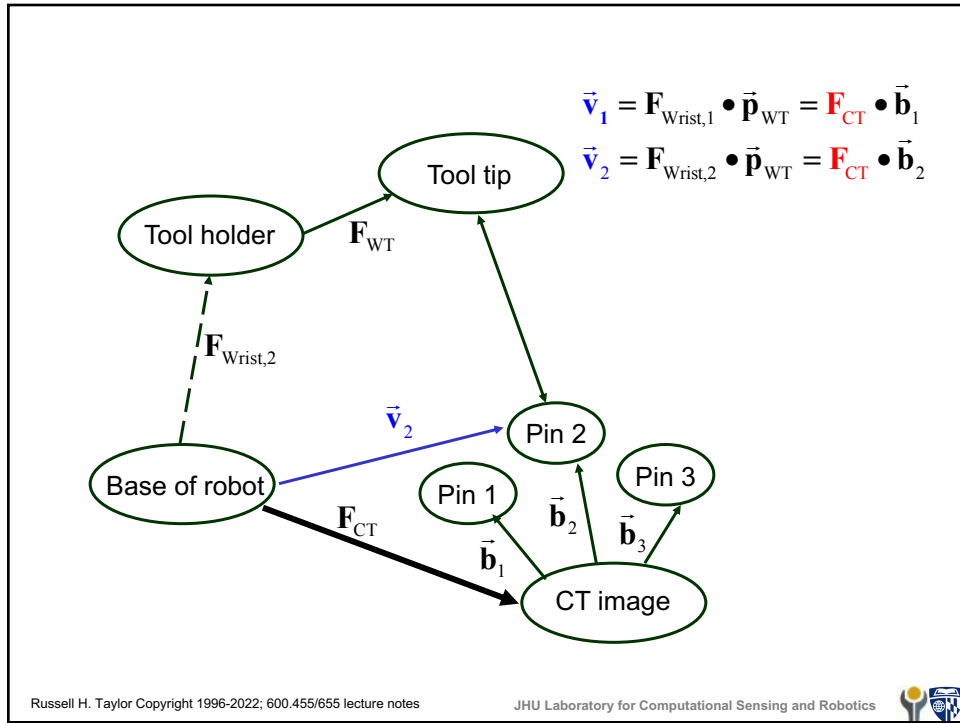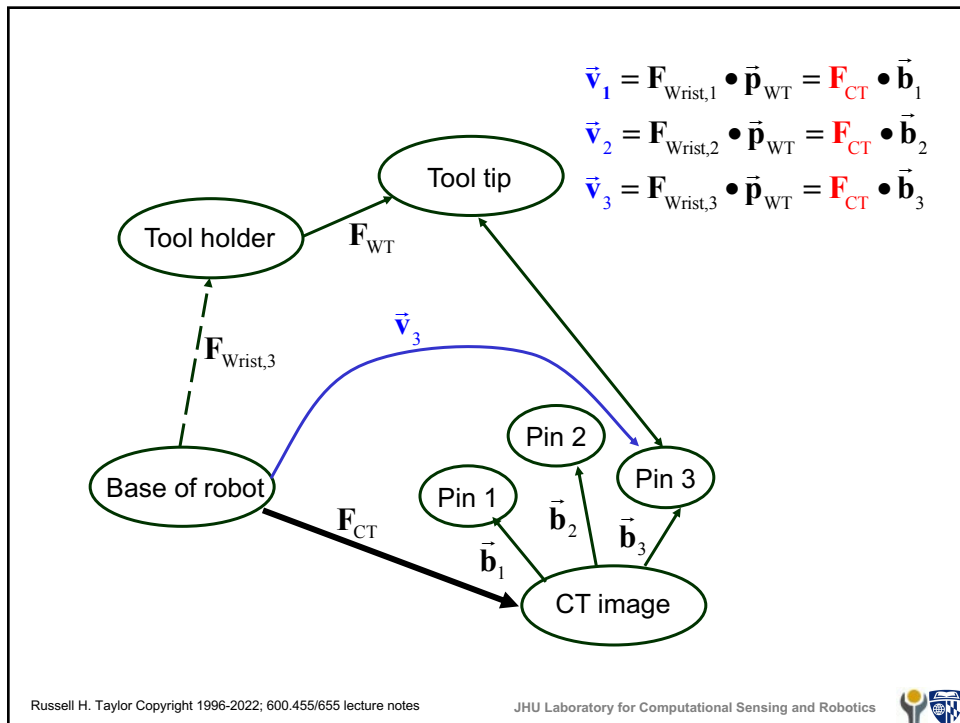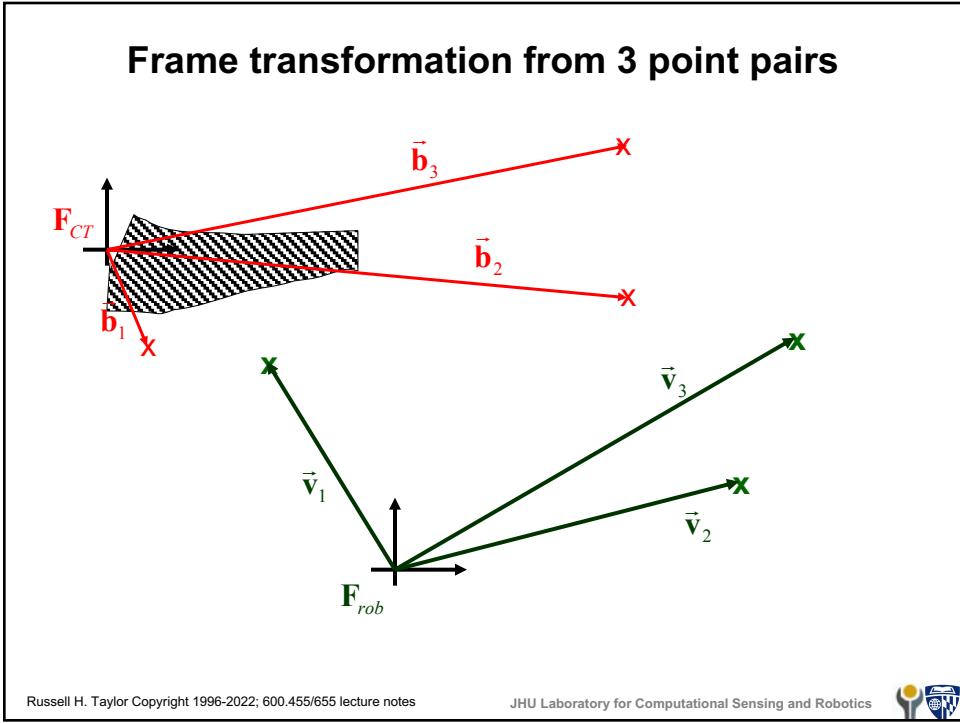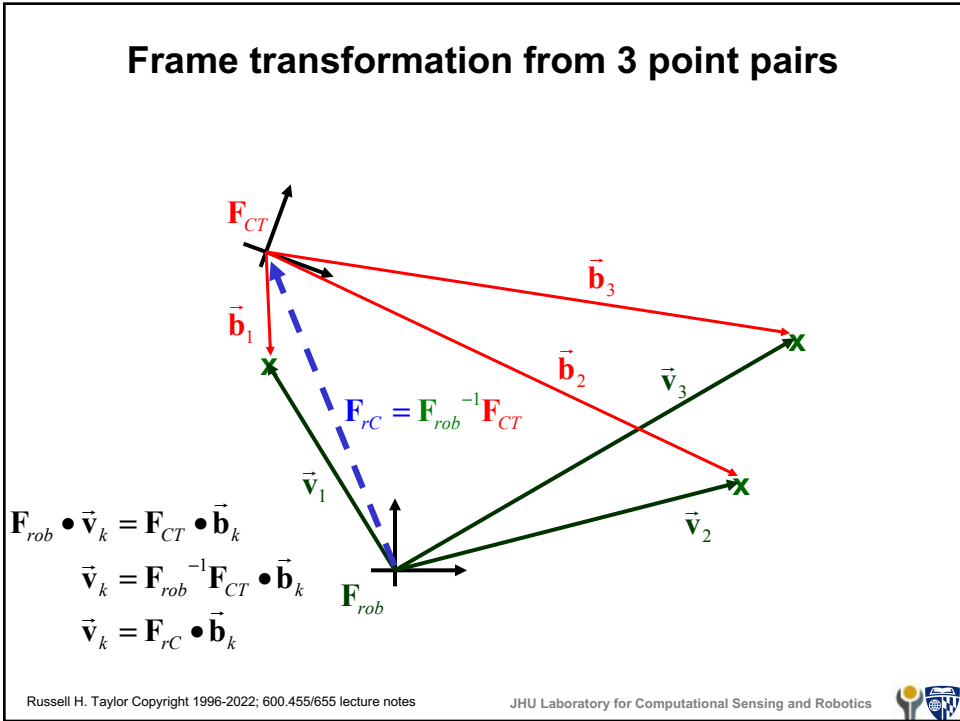
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# Frame transformation from 3 point pairs

$$\vec{\mathbf{v}}_k = \mathbf{F}_{rC}\vec{\mathbf{b}}_k = \mathbf{R}_{rC}\vec{\mathbf{b}}_k + \vec{\mathbf{p}}_{rC}$$

Define

$$\vec{\mathbf{v}}_m = \frac{1}{3}\sum_1^3 \vec{\mathbf{v}}_k \qquad \vec{\mathbf{b}}_m = \frac{1}{3}\sum_1^3 \vec{\mathbf{b}}_k$$

$$\vec{\mathbf{u}}_k = \vec{\mathbf{v}}_k - \vec{\mathbf{v}}_m \qquad \vec{\mathbf{a}}_k = \vec{\mathbf{b}}_k - \vec{\mathbf{b}}_m$$

$$\mathbf{F}_{rC}\vec{\mathbf{a}}_k = \mathbf{R}_{rC}\vec{\mathbf{a}}_k + \vec{\mathbf{p}}_{rC}$$

$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k + \vec{\mathbf{p}}_{rC} = \mathbf{R}_{rC}\left(\vec{\mathbf{b}}_k - \vec{\mathbf{b}}_m\right) + \vec{\mathbf{p}}_{rC}$$

$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k = \mathbf{R}_{rC}\vec{\mathbf{b}}_k + \vec{\mathbf{p}}_{rC} - \mathbf{R}_{rC}\vec{\mathbf{b}}_m - \vec{\mathbf{p}}_{rC}$$

$$\boxed{\begin{aligned}\mathbf{R}_{rC}\vec{\mathbf{a}}_k &= \vec{\mathbf{v}}_k - \vec{\mathbf{v}}_m = \vec{\mathbf{u}}_k \\ \vec{\mathbf{p}}_{rC} &= \vec{\mathbf{v}}_m - \mathbf{R}_{rC}\vec{\mathbf{b}}_m\end{aligned}}$$ **Solve These!!**
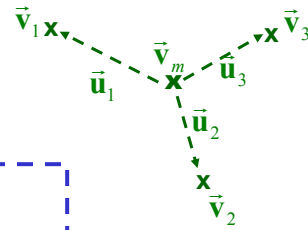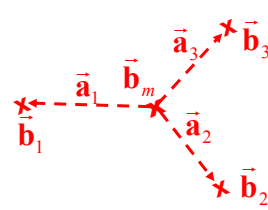
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# Rotation from multiple vector pairs

Given a system $\mathbf{R}\vec{\mathbf{a}}_k = \vec{\mathbf{u}}_k$ for $k = 1, \cdots, n$ the problem is to estimate $\mathbf{R}$. This will require at least three such point pairs. Later in the course we will cover some good ways to solve this system. Here is a not-so-good way that will produce roughly correct answers:

Step 1: Form matrices $\mathbf{U} = \begin{bmatrix}\vec{\mathbf{u}}_1 & \cdots & \vec{\mathbf{u}}_n\end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix}\vec{\mathbf{a}}_1 & \cdots & \vec{\mathbf{a}}_n\end{bmatrix}$

Step 2: Solve the system $\mathbf{R}\mathbf{A} = \mathbf{U}$ for $\mathbf{R}$. E.g., by $\mathbf{R} = \mathbf{U}\mathbf{A}^{-1}$

Step 3: Renormalize $\mathbf{R}$ to guarantee $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

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# Renormalizing Rotation Matrix

Given "rotation" matrix $\mathbf{R} = \begin{bmatrix} \vec{\mathbf{r}}_x & | & \vec{\mathbf{r}}_y & | & \vec{\mathbf{r}}_z \end{bmatrix}$, modify it so $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

Step 1: $\quad \vec{\mathbf{a}} = \vec{\mathbf{r}}_y \times \vec{\mathbf{r}}_z$

Step 2: $\quad \vec{\mathbf{b}} = \vec{\mathbf{r}}_z \times \vec{\mathbf{a}}$

Step 3: $\quad \mathbf{R}_{normalized} = \begin{bmatrix} \dfrac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|} & \bigg| & \dfrac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|} & \bigg| & \dfrac{\vec{\mathbf{r}}_z}{\|\vec{\mathbf{r}}_z\|} \end{bmatrix}$

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# Calibrating a pointer



But what is $\mathbf{b}_{tip}$??
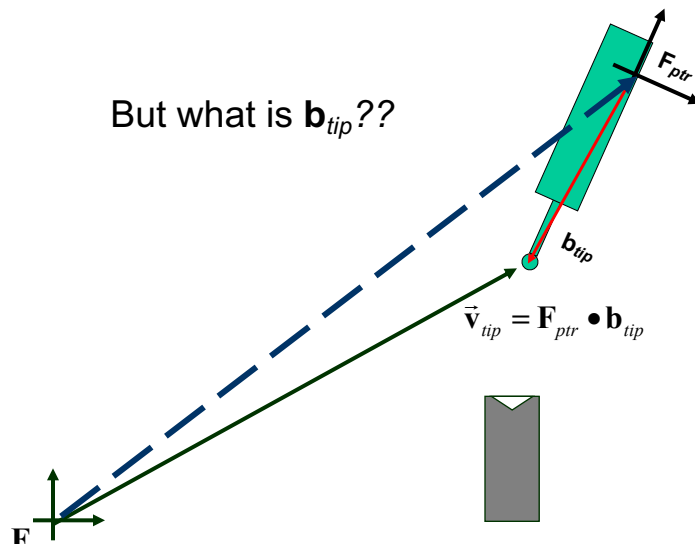
$F_{ptr}$

$\mathbf{b}_{tip}$

$\vec{\mathbf{v}}_{tip} = \mathbf{F}_{ptr} \bullet \mathbf{b}_{tip}$

$\mathbf{F}_{lab}$

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## Calibrating a pointer

$$\vec{\mathbf{b}}_{post} = \mathbf{F}_k \vec{\mathbf{b}}_{tip}$$

$$= \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k$$
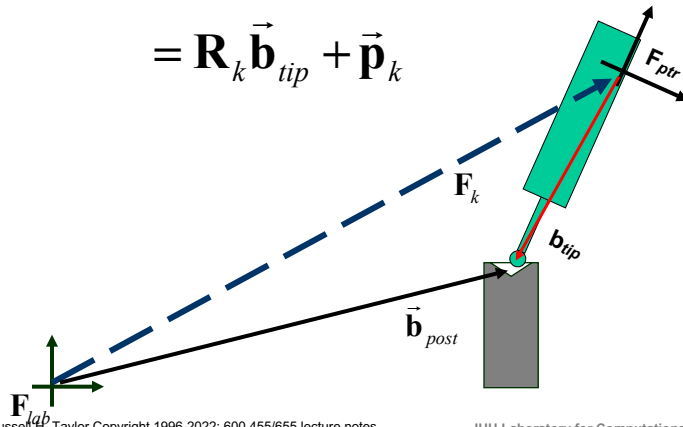
$\mathbf{F}_k$

$\mathbf{b}_{tip}$

$\vec{\mathbf{b}}_{post}$

$\mathbf{F}_{ptr}$

$\mathbf{F}_{lab}$

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## Calibrating a pointer

For each measurement $k$, we have

$$\vec{\mathbf{b}}_{post} = \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k$$

I. e.,

$$\mathbf{R}_k \vec{\mathbf{b}}_{tip} - \vec{\mathbf{b}}_{post} = -\vec{\mathbf{p}}_k$$

Set up a least squares problem

$$\begin{bmatrix} \vdots & | & \vdots \\ \hline \mathbf{R}_k & | & -\mathbf{I} \\ \hline \vdots & | & \vdots \end{bmatrix} \begin{bmatrix} \vec{\mathbf{b}}_{tip} \\ \hline \vec{\mathbf{b}}_{post} \end{bmatrix} \cong \begin{bmatrix} \vdots \\ -\vec{\mathbf{p}}_k \\ \vdots \end{bmatrix}$$
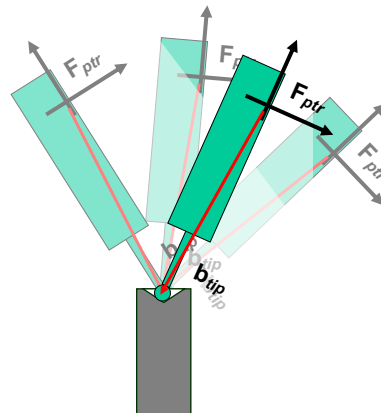
$\mathbf{F}_{ptr}$

$\mathbf{F}_{ptr}$

$\mathbf{F}_{ptr}$

$\mathbf{b}_{tip}$

$\mathbf{b}_{tip}$

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## "Registration Transformations"

Given a coordinate system $\mathbf{F}_C$ and another coordinate system $\mathbf{F}_G$ (e.g., a CT scan and a tracked "rigid body" attached to the patient, and points $\vec{c}_i$ in the coordinate system $\mathbf{F}_C$ and points $\vec{g}_i$ in the coordinate system $\mathbf{F}_G$, then the "registration transformation" $\mathbf{F}_{GC}$ between $\mathbf{F}_G$ and $\mathbf{F}_C$ one in which for $\mathbf{F}_{GC}\vec{c}_i = \vec{g}_i$ if and only if $\vec{c}_i$ and $\vec{g}_i$ refer to the same or corresponding points.

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## Use in surgical navigation

$$\mathbf{F}_{BA} = \mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}$$
$$\mathbf{F}_{BA} = \mathbf{F}_{BG}\mathbf{F}_{GC}\mathbf{F}_{CA}$$

$$\mathbf{F}_{BG}\mathbf{F}_{GC}\mathbf{F}_{CA} = \mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}$$
$$\mathbf{F}_{CA} = \left(\mathbf{F}_{BG}\mathbf{F}_{GC}\right)^{-1}\mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}$$

$$\mathbf{F}_{GC} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}\mathbf{F}_{CA}^{-1}$$
$$\mathbf{F}_{GC} = \mathbf{F}_{GD}\mathbf{F}_{DU}\mathbf{F}_{UA}\mathbf{F}_{CA}^{-1}$$

If an anatomic structure is identified at pose $\mathbf{F}_{UA}$ in ultrasound image coordinates give the formula for computing the corresponding pose $\mathbf{F}_{CA}$ in CT coordinates

where $\mathbf{F}_{GD} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BD}$

NOTE: In cases where there is a reference body (here $\mathbf{F}_G$) affixed to the patient, it is sometimes convenient to assume that the navigation software routinely transforms all tracked markers to the coordinate system of $\mathbf{F}_G$ (i.e., $\mathbf{F}_{Gx} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{Bx}$).

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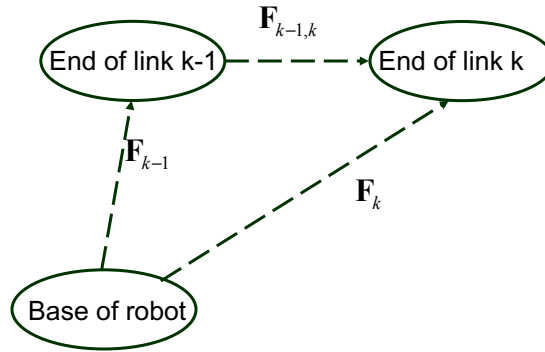
**Kinematic Links**

$$\mathbf{F}_k = \mathbf{F}_{k-1} \bullet \mathbf{F}_{k-1,k}$$

$$\left[ \mathbf{R}_k, \vec{\mathbf{p}}_k \right] = \left[ \mathbf{R}_{k-1}, \mathbf{p}_{k-1} \right] \bullet \left[ \mathbf{R}_{k-1,k}, \mathbf{p}_{k-1,k} \right]$$

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**Kinematic Links**

$$\mathbf{F}_k = \mathbf{F}_{k-1} \bullet \mathbf{F}_{k-1,k}$$

$$\left[ \mathbf{R}_k, \vec{\mathbf{p}}_k \right] = \left[ \mathbf{R}_{k-1}, \mathbf{p}_{k-1} \right] \bullet \left[ \mathbf{R}_{k-1,k}, \mathbf{p}_{k-1,k} \right]$$

$$= \left[ \mathbf{R}_{k-1}, \mathbf{p}_{k-1} \right] \bullet \left[ Rot(\vec{\mathbf{r}}_k, \theta_k), \vec{\mathbf{0}} \right] \bullet \left[ \mathbf{I}, L_k \vec{\mathbf{x}} \right]$$

$$= \left[ \mathbf{R}_{k-1}, \mathbf{p}_{k-1} \right] \bullet \left[ Rot(\vec{\mathbf{r}}_k, \theta_k), L_k Rot(\vec{\mathbf{r}}_k, \theta_k) \bullet \vec{\mathbf{x}} \right]$$

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## Kinematic Chains

$$\mathbf{F}_0 = [\mathbf{I}, \vec{\mathbf{0}}]$$

$$\mathbf{R}_3 = \mathbf{R}_{0,1}\mathbf{R}_{1,2}\mathbf{R}_{2,3} = Rot(\vec{\mathbf{r}}_1,\theta_1)Rot(\vec{\mathbf{r}}_2,\theta_2)Rot(\vec{\mathbf{r}}_3,\theta_3)$$

$$\vec{\mathbf{p}}_3 = \vec{\mathbf{p}}_{0,1} + \mathbf{R}_{0,1}\left(\vec{\mathbf{p}}_{1,2} + \mathbf{R}_{1,2}\vec{\mathbf{p}}_{2,3}\right)$$
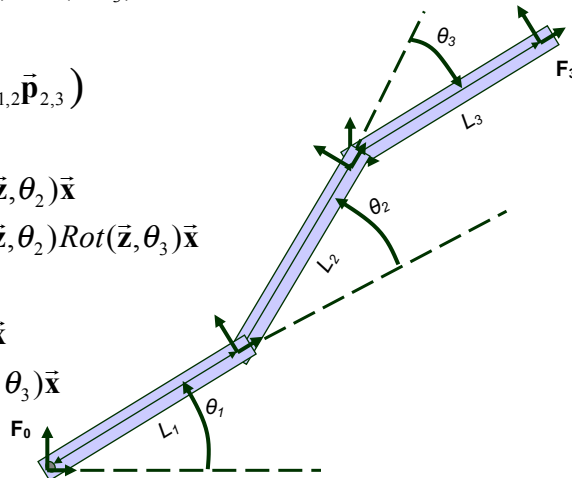
$$= L_1 Rot(\vec{\mathbf{r}}_1,\theta_1)\vec{\mathbf{x}}$$
$$+ L_2 Rot(\vec{\mathbf{r}}_1,\theta_1)Rot(\vec{\mathbf{r}}_2,\theta_2)\vec{\mathbf{x}}$$
$$+ L_3 Rot(\vec{\mathbf{r}}_1,\theta_1)Rot(\vec{\mathbf{r}}_2,\theta_2)Rot(\vec{\mathbf{r}}_3,\theta_3)\vec{\mathbf{x}}$$
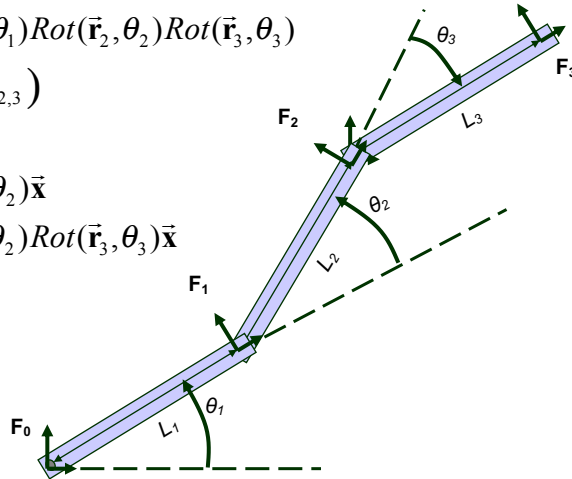
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## Kinematic Chains

If $\vec{\mathbf{r}}_1 = \vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_3 = \vec{\mathbf{z}}$,

$$\mathbf{R}_3 = Rot(\vec{\mathbf{z}},\theta_1)Rot(\vec{\mathbf{z}},\theta_2)Rot(\vec{\mathbf{z}},\theta_3)$$
$$= Rot(\vec{\mathbf{z}},\theta_1+\theta_2+\theta_3)$$

$$\vec{\mathbf{p}}_3 = \vec{\mathbf{p}}_{0,1} + \mathbf{R}_{0,1}\left(\vec{\mathbf{p}}_{1,2} + \mathbf{R}_{1,2}\vec{\mathbf{p}}_{2,3}\right)$$

$$= L_1 Rot(\vec{\mathbf{z}},\theta_1)\vec{\mathbf{x}}$$
$$+ L_2 Rot(\vec{\mathbf{z}},\theta_1)Rot(\vec{\mathbf{z}},\theta_2)\vec{\mathbf{x}}$$
$$+ L_3 Rot(\vec{\mathbf{z}},\theta_1)Rot(\vec{\mathbf{z}},\theta_2)Rot(\vec{\mathbf{z}},\theta_3)\vec{\mathbf{x}}$$

$$= L_1 Rot(\vec{\mathbf{z}},\theta_1)\vec{\mathbf{x}}$$
$$+ L_2 Rot(\vec{\mathbf{z}},\theta_1+\theta_2)\vec{\mathbf{x}}$$
$$+ L_3 Rot(\vec{\mathbf{z}},\theta_1+\theta_2+\theta_3)\vec{\mathbf{x}}$$

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# Kinematic Chains

If $\vec{r}_1 = \vec{r}_2 = \vec{r}_3 = \vec{z}$,

$$\mathbf{R}_3 = \begin{bmatrix} \cos(\theta_1 + \theta_2 + \theta_3) & -\sin(\theta_1 + \theta_2 + \theta_3) & 0 \\ \sin(\theta_1 + \theta_2 + \theta_3) & \cos(\theta_1 + \theta_2 + \theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{\mathbf{p}}_3 = \begin{bmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ 0 \end{bmatrix}$$
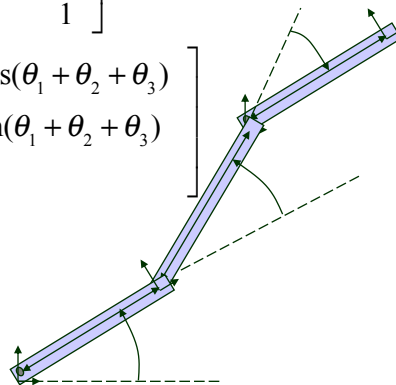
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# "Small" Transformations

- A great deal of CIS is concerned with computing and using geometric information based on imprecise knowledge
- Similarly, one is often concerned with the effects of relatively small rotations and displacements
- Essentially, we will be using fairly straightforward linearizations to model these situations, but a specialized notation is often useful

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## "Small" Frame Transformations

Represent a "small" pose shift consisting of a small rotation $\Delta\mathbf{R}$ followed by a small displacement $\Delta\vec{\mathbf{p}}$ as

$$\Delta\mathbf{F} = [\Delta\mathbf{R}, \Delta\vec{\mathbf{p}}]$$

Then

$$\Delta\mathbf{F} \bullet \vec{\mathbf{v}} = \Delta\mathbf{R} \bullet \vec{\mathbf{v}} + \Delta\vec{\mathbf{p}}$$

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## Small Rotations

$\Delta\mathbf{R} =$ a small rotation

$\mathbf{R}_{\vec{\mathbf{a}}}(\Delta\alpha) =$ a rotation by a small angle $\Delta\alpha$ about axis $\vec{\mathbf{a}}$

$Rot(\vec{\mathbf{a}}, \|\vec{\mathbf{a}}\|) \bullet \vec{\mathbf{b}} \approx \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{b}}$    for $\|\vec{\mathbf{a}}\|$ sufficiently small

$\Delta\mathbf{R}(\vec{\mathbf{a}}) =$ a rotation that is small enough so that any error introduced by this approximation is negligible

$$\Delta\mathbf{R}(\lambda\vec{\mathbf{a}}) \bullet \Delta\mathbf{R}(\mu\vec{\mathbf{b}}) \cong \Delta\mathbf{R}(\lambda\vec{\mathbf{a}} + \mu\vec{\mathbf{b}})$$    (Linearity for small rotations)

Exercise: Work out the linearity proposition by substitution

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## Approximations to "Small" Frames

$$\Delta \mathbf{F}(\vec{\mathbf{a}}, \Delta\vec{\mathbf{p}}) \triangleq [\Delta\mathbf{R}(\vec{\mathbf{a}}), \Delta\vec{\mathbf{p}}]$$

$$\Delta \mathbf{F}(\vec{\mathbf{a}}, \Delta\vec{\mathbf{p}}) \bullet \vec{\mathbf{v}} = \Delta\mathbf{R}(\vec{\mathbf{a}}) \bullet \vec{\mathbf{v}} + \Delta\vec{\mathbf{p}}$$

$$\approx \vec{\mathbf{v}} + \vec{\mathbf{a}} \times \vec{\mathbf{v}} + \Delta\vec{\mathbf{p}}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{v}} = skew(\vec{\mathbf{a}}) \bullet \vec{\mathbf{v}}$$

$$= \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \bullet \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$skew(\vec{\mathbf{a}}) \bullet \vec{\mathbf{a}} = \vec{\mathbf{a}} \times \vec{\mathbf{a}} = \vec{0}$$

$$\Delta\mathbf{R}(\vec{\mathbf{a}}) \approx \mathbf{I} + skew(\vec{\mathbf{a}})$$

$$\Delta\mathbf{R}(\vec{\mathbf{a}})^{-1} \approx \mathbf{I} - skew(\vec{\mathbf{a}}) = \mathbf{I} + skew(-\vec{\mathbf{a}})$$

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## Approximations to "Small" Frames

Notational NOTE:

We often use $\vec{\alpha}$ to represent a vector of small angles

and $\vec{\varepsilon}$ to represent a vector of small displacements

In using these approximations, we typically ignore second order terms.  I.e.,

$\vec{\alpha}_A \vec{\alpha}_B \approx \vec{0}$, $\vec{\alpha}_A \vec{\varepsilon}_B \approx \vec{0}$, $\vec{\varepsilon}_A \vec{\varepsilon}_B \approx \vec{0}$, etc.

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# Errors & sensitivity

Often, we do not have an accurate value for a transformation,
so we need to model the error. We model this as a composition
of a "nominal" frame and a small displacement

$$\mathbf{F}_{actual} = \mathbf{F}_{nominal} \bullet \Delta\mathbf{F}$$

Often, we will use the notation $\mathbf{F}^*$ for $\mathbf{F}_{actual}$ and will just use
$\mathbf{F}$ for $\mathbf{F}_{nominal}$. Thus we may write something like

$$\mathbf{F}^* = \mathbf{F} \bullet \Delta\mathbf{F}$$

or (less often) $\mathbf{F}^* = \Delta\mathbf{F} \bullet \mathbf{F}$. We also use $\vec{\mathbf{v}}^* = \vec{\mathbf{v}} + \Delta\vec{\mathbf{v}}, etc.$
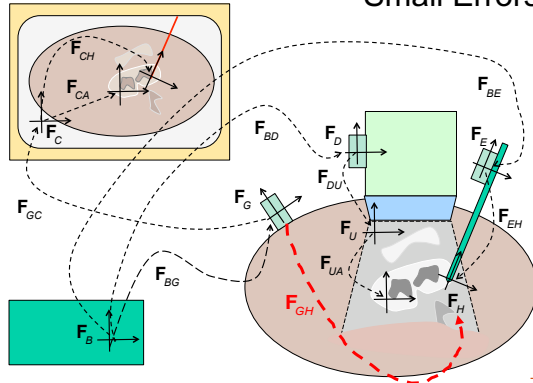Thus, if we use the former form (error on the right), and
have nominal relationship $\vec{\mathbf{v}} = \mathbf{F} \bullet \vec{\mathbf{b}}$, we get

$$\vec{\mathbf{v}}^* = \mathbf{F}^* \bullet \vec{\mathbf{b}}^*$$
$$= \mathbf{F} \bullet \Delta\mathbf{F} \bullet (\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}}) = \mathbf{F} \bullet (\Delta\mathbf{R} \bullet \vec{\mathbf{b}} + \Delta\mathbf{R} \bullet \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}})$$
$$\approx \mathbf{R} \bullet ((\mathbf{I} + sk(\vec{\alpha})) \bullet (\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}}) + \Delta\vec{\mathbf{p}}) + \vec{\mathbf{p}} = \mathbf{R} \bullet (\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}) + \vec{\mathbf{p}}$$
$$\approx \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}) + \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}} = \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}) + \vec{\mathbf{v}}$$
$$\Delta\vec{\mathbf{v}} \approx \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}})$$

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# "Small Errors"



Suppose that there is a small systematic
error in the tracking system so that

$$\mathbf{F}_{Bx}^* = \Delta\mathbf{F}_B\mathbf{F}_{Bx}$$

for $\mathbf{F}_{BG}, \mathbf{F}_{BD}, \mathbf{F}_{BE}$. How does this affect
the calculation of $\mathbf{F}_{GH}$?

$$\mathbf{F}_{GH}^* = (\mathbf{F}_{BG}^*)^{-1}\mathbf{F}_{BE}^* \mathbf{F}_{EH}$$
$$\mathbf{F}_{GH}\Delta\mathbf{F}_{GH} = (\Delta\mathbf{F}_B\mathbf{F}_{BG})^{-1}\Delta\mathbf{F}_B\mathbf{F}_{BE}\mathbf{F}_{EH}$$
$$\Delta\mathbf{F}_{GH} = \mathbf{F}_{GH}^{-1}(\Delta\mathbf{F}_B\mathbf{F}_{BG})^{-1}\Delta\mathbf{F}_B\mathbf{F}_{BE}\mathbf{F}_{EH}$$
$$= \mathbf{F}_{GH}^{-1}\mathbf{F}_{BG}^{-1}\Delta\mathbf{F}_B^{-1}\Delta\mathbf{F}_B\mathbf{F}_{BE}\mathbf{F}_{EH}$$
$$= \mathbf{F}_{GH}^{-1}\mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\mathbf{F}_{EH} = \mathbf{F}_{GH}^{-1}\mathbf{F}_{GH} = \mathbf{I}$$

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## "Small Errors"



Suppose that there are additional errors in the tracking of each tracker body so that

$$\mathbf{F}_{Bx}^{*} = \Delta\mathbf{F}_{B}\mathbf{F}_{Bx}\Delta\mathbf{F}_{Bx}$$

for $\mathbf{F}_{BG}, \mathbf{F}_{BD}, \mathbf{F}_{BE}$. How does this affect the calculation of $\mathbf{F}_{GH}$?

$$\mathbf{F}_{GH}^{*} = \mathbf{F}_{GH}\Delta\mathbf{F}_{GH} = (\mathbf{F}_{BG}^{*})^{-1}\mathbf{F}_{BE}^{*}\ \mathbf{F}_{EH}$$

$$\Delta\mathbf{F}_{GH} = \mathbf{F}_{GH}^{-1}\left(\Delta\mathbf{F}_{B}\mathbf{F}_{BG}\Delta\mathbf{F}_{BG}\right)^{-1}\left(\Delta\mathbf{F}_{B}\mathbf{F}_{BE}\Delta\mathbf{F}_{BE}\right)\mathbf{F}_{EH}$$

$$\Delta\mathbf{F}_{GH} = \left(\mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\mathbf{F}_{EH}\right)^{-1}\Delta\mathbf{F}_{BG}^{-1}\mathbf{F}_{BG}^{-1}\Delta\mathbf{F}_{B}^{-1}\Delta\mathbf{F}_{B}\mathbf{F}_{BE}\Delta\mathbf{F}_{BG}\mathbf{F}_{EH}$$

$$= \mathbf{F}_{EH}^{-1}\mathbf{F}_{BE}^{-1}\mathbf{F}_{BG}\Delta\mathbf{F}_{BG}^{-1}\mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\Delta\mathbf{F}_{BG}\mathbf{F}_{EH}$$
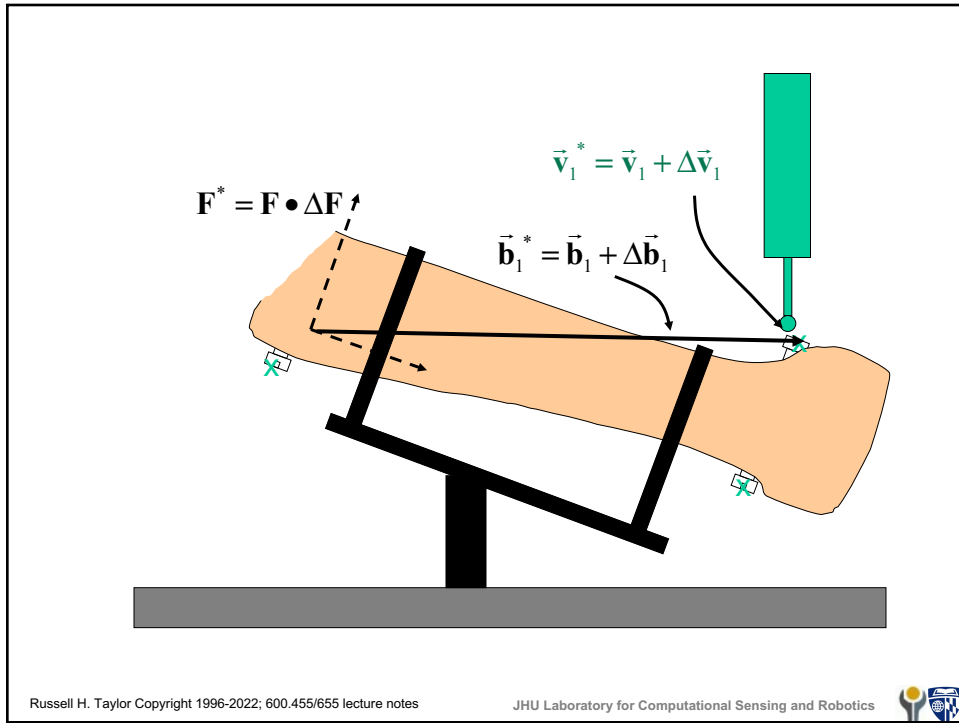
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$F = [R, p]$      $\vec{\mathbf{b}}_1$    $\vec{\mathbf{v}}_1$

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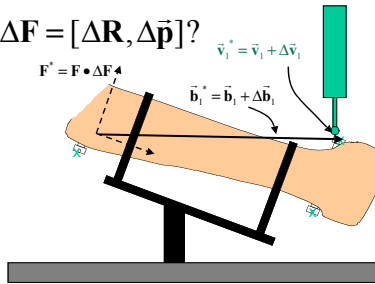
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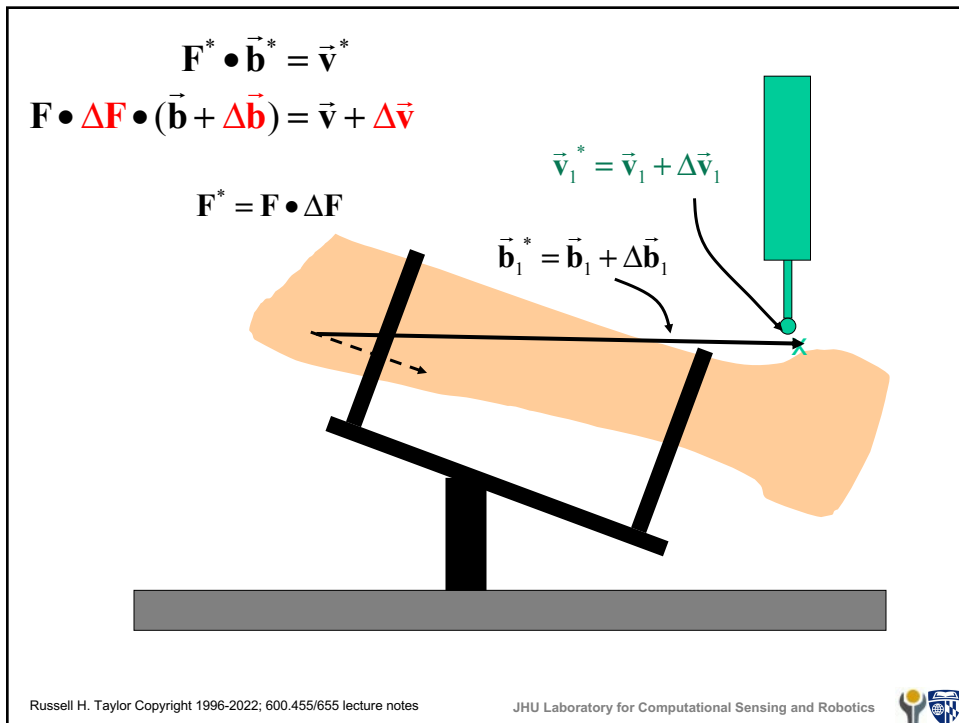
## Errors & Sensitivity

Suppose that we know nominal values for $\mathbf{F}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{v}}$ and that

$$\left[-\varepsilon,-\varepsilon,-\varepsilon\right]^T \le \Delta\vec{\mathbf{v}}_1 \le \left[\varepsilon,\varepsilon,\varepsilon\right]^T \quad (i.e., \left\|\Delta\vec{\mathbf{v}}_1\right\|_\infty \le \varepsilon)$$

What does this tell us about $\Delta\mathbf{F} = [\Delta\mathbf{R},\Delta\vec{\mathbf{p}}]$?



$\vec{\mathbf{v}}_1^{\,*} = \vec{\mathbf{v}}_1 + \Delta\vec{\mathbf{v}}_1$

$\mathbf{F}^* = \mathbf{F}\bullet\Delta\mathbf{F}$

$\vec{\mathbf{b}}_1^{\,*} = \vec{\mathbf{b}}_1 + \Delta\vec{\mathbf{b}}_1$

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## Errors & Sensitivity

$$\vec{\mathbf{v}}^{\,*} = \mathbf{F}^* \bullet \vec{\mathbf{b}}^{\,*}$$

$$= \mathbf{F} \bullet \Delta\mathbf{F} \bullet (\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}})$$

$$= \mathbf{R} \bullet \left(\Delta\mathbf{R}(\vec{\alpha}) \bullet \left(\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}}\right) + \Delta\vec{\mathbf{p}}\right) + \vec{\mathbf{p}}$$

$$\approx \mathbf{R} \bullet \left(\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \vec{\alpha}\times\vec{\mathbf{b}} + \vec{\alpha}\times\Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}\right) + \vec{\mathbf{p}}$$

$$= \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}} + \mathbf{R} \bullet \left(\Delta\vec{\mathbf{b}} + \vec{\alpha}\times\vec{\mathbf{b}} + \vec{\alpha}\times\Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}\right)$$

$$\approx \vec{\mathbf{v}} + \mathbf{R} \bullet \left(\Delta\vec{\mathbf{b}} + \vec{\alpha}\times\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}\right)$$

if $\left\|\vec{\alpha}\times\Delta\vec{\mathbf{b}}\right\| \le \left\|\vec{\alpha}\right\|\left\|\Delta\vec{\mathbf{b}}\right\|$ is negligible (it usually is)

so

$$\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}^{\,*} - \vec{\mathbf{v}} \approx \mathbf{R} \bullet \left(\Delta\vec{\mathbf{b}} + \vec{\alpha}\times\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}\right) = \mathbf{R} \bullet \Delta\vec{\mathbf{b}} + \mathbf{R} \bullet \vec{\alpha}\times\vec{\mathbf{b}} + \mathbf{R} \bullet \Delta\vec{\mathbf{p}}$$

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# Digression: "rotation triple product"

Expressions like $\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}}$ are linear in $\vec{\mathbf{a}}$, but are not always convenient to work with. Often we would prefer something like $\mathbf{M}(\mathbf{R}, \vec{\mathbf{b}}) \bullet \vec{\mathbf{a}}$.

$$\begin{aligned}
\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}} &= -\mathbf{R} \bullet \vec{\mathbf{b}} \times \vec{\mathbf{a}} \\
&= \mathbf{R} \bullet skew(-\vec{\mathbf{b}}) \bullet \vec{\mathbf{a}} \\
&= \left[ \mathbf{R} \bullet skew(\vec{\mathbf{b}})^T \right] \bullet \vec{\mathbf{a}}
\end{aligned}$$

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# Digression: "rotation triple product"

Here are a few more useful facts:

$$\mathbf{R} \bullet \left( \vec{\mathbf{a}} \times \vec{\mathbf{b}} \right) = \left( \mathbf{R} \bullet \vec{\mathbf{a}} \right) \times \left( \mathbf{R} \bullet \vec{\mathbf{b}} \right)$$

$$\vec{\mathbf{a}} \times \left( \mathbf{R} \bullet \vec{\mathbf{b}} \right) = \mathbf{R} \bullet \left( \left( \mathbf{R}^{-1} \bullet \vec{\mathbf{a}} \right) \times \vec{\mathbf{b}} \right)$$

Consequently

$$skew(\vec{\mathbf{a}}) \bullet \mathbf{R} = \mathbf{R} \bullet skew(\mathbf{R}^{-1} \bullet \vec{\mathbf{a}})$$

$$\mathbf{R}^{-1} skew(\vec{\mathbf{a}}) \bullet \mathbf{R} = skew(\mathbf{R}^{-1} \bullet \vec{\mathbf{a}})$$

## A "standard form" for linearized error expressions

It is often convenient to use identities to rearrange expressions involving small error variables into sums of terms with the general form $\mathbf{M}_k \vec{\eta}_k$, where $\mathbf{M}_k$ involve things known to the computer, and the $\vec{\eta}_k$ are error variables.

For example,
$$\vec{\gamma}=\mathbf{R}sk(\vec{\alpha})\vec{\mathbf{a}}+sk(\vec{\beta})\vec{\mathbf{b}}$$
would be rewritten as
$$\vec{\gamma}=\text{-}\mathbf{R}sk(\vec{\mathbf{a}})\vec{\alpha}-sk(\vec{\mathbf{b}})\vec{\beta}$$
or
$$\vec{\gamma}=\mathbf{R}sk(-\vec{\mathbf{a}})\vec{\alpha}+sk(-\vec{\mathbf{b}})\vec{\beta}$$

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# Errors & Sensitivity



Computed pose

Measured position

Error window

F

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# Errors & Sensitivity

Computed pose

Measured position

Error window

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# Errors & Sensitivity

Computed pose

Measured position

Error window

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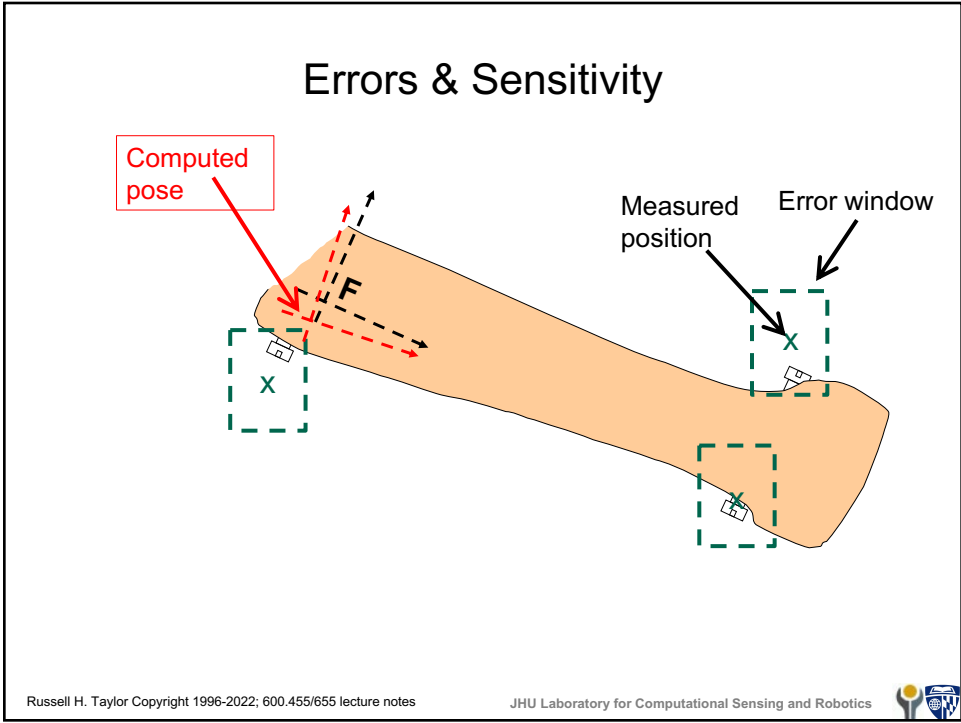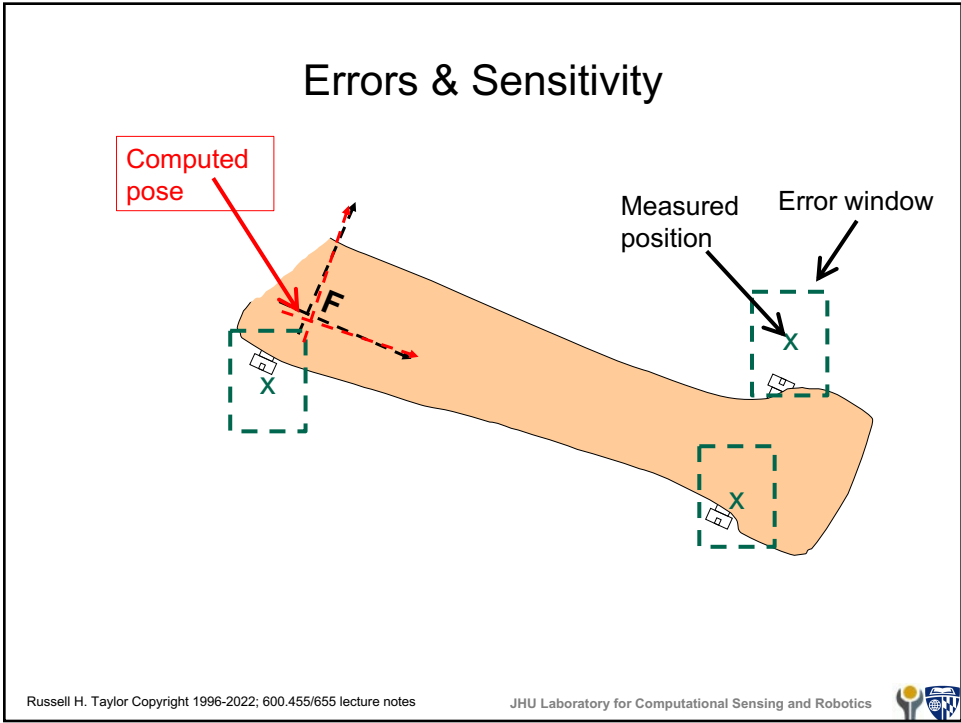
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Errors & Sensitivity

Computed pose

Measured position

Error window

**F**

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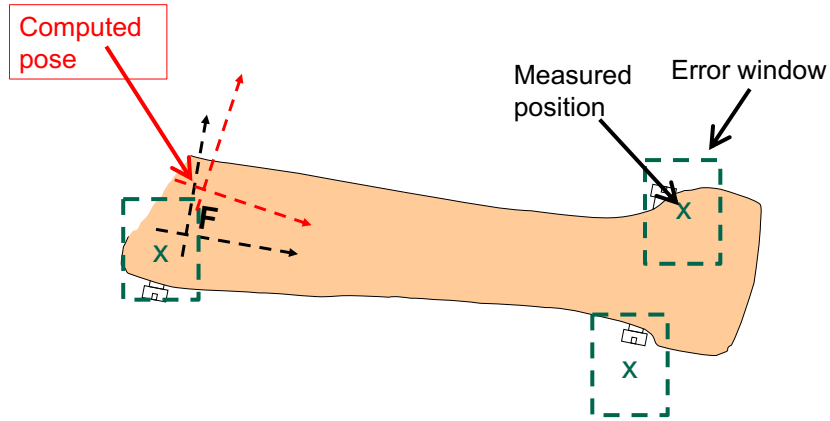
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Errors & Sensitivity

Computed pose

Measured position

Error window

**F**

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# Errors & Sensitivity

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# Errors & Sensitivity

Previous expression was

$$\Delta \vec{\mathbf{v}}_1 \approx \mathbf{R} \bullet \left( \Delta \vec{\mathbf{b}}_1 + \vec{\alpha} \times \vec{\mathbf{b}} + \Delta \vec{\mathbf{p}}_1 \right)$$

Substituting triple product and rearranging gives

$$\Delta \vec{\mathbf{v}}_1 \approx \left[ \ \mathbf{R} \ \middle| \ \mathbf{R} \ \middle| \ \mathbf{R} \bullet skew(-\vec{\mathbf{b}}) \ \right] \bullet \begin{bmatrix} \Delta \vec{\mathbf{b}}_1 \\ \Delta \vec{\mathbf{p}} \\ \vec{\alpha} \end{bmatrix}$$

So

$$\begin{bmatrix} -\varepsilon \\ -\varepsilon \\ -\varepsilon \end{bmatrix} \leq \left[ \ \mathbf{R} \ \middle| \ \mathbf{R} \ \middle| \ \mathbf{R} \bullet skew(-\vec{\mathbf{b}}) \ \right] \begin{bmatrix} \Delta \vec{\mathbf{b}}_1 \\ \Delta \vec{\mathbf{p}} \\ \vec{\alpha} \end{bmatrix} \leq \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix}$$

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## Errors & Sensitivity

Now, suppose we know that $\left|\Delta\vec{\mathbf{b}}_1\right| \le \beta$, this will give us

a system of linear constraints

$$\begin{bmatrix} -\varepsilon \\ -\varepsilon \\ -\varepsilon \\ \hline -\beta \\ -\beta \\ -\beta \end{bmatrix} \le \left[ \begin{array}{c|c|c} \mathbf{R} & \mathbf{R} & \mathbf{R} \bullet skew(-\vec{\mathbf{b}}) \\ \hline \mathbf{I} & \mathbf{0} & \mathbf{0} \end{array} \right] \begin{bmatrix} \Delta\vec{\mathbf{b}}_1 \\ \Delta\vec{\mathbf{p}}_1 \\ \vec{\alpha} \end{bmatrix} \le \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \\ \hline \beta \\ \beta \\ \beta \end{bmatrix}$$

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## Error from frame composition

Consider $\mathbf{R}_1^* \mathbf{R}_2^* = \mathbf{R}_3^*$ where $\mathbf{R}_1^* = \mathbf{R}_1 \Delta\mathbf{R}_1, \mathbf{R}_2^* = \mathbf{R}_2 \Delta\mathbf{R}_2, \ \mathbf{R}_3^* = \mathbf{R}_3 \Delta\mathbf{R}_3$
and $\Delta\mathbf{R}_1 \approx \mathbf{I} + sk\left(\vec{\alpha}_1\right), \ \Delta\mathbf{R}_2 \approx \mathbf{I} + sk\left(\vec{\alpha}_2\right)$, estimate $\Delta\mathbf{R}_3 \approx \mathbf{I} + sk\left(\vec{\alpha}_3\right)$

$$\mathbf{R}_1 \Delta\mathbf{R}_1 \mathbf{R}_2 \Delta\mathbf{R}_2 = \mathbf{R}_1 \mathbf{R}_2 \Delta\mathbf{R}_3$$

$$\mathbf{R}_1(\mathbf{I} + sk(\vec{\alpha}_1))\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_2)) \approx \mathbf{R}_1 \mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_3))$$

$$\left(\mathbf{R}_1 \mathbf{R}_2\right)^{-1} \mathbf{R}_1(\mathbf{I} + sk(\vec{\alpha}_1))\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_2)) \approx \mathbf{I} + sk(\vec{\alpha}_3)$$

$$\mathbf{R}_2^{-1} \mathbf{R}_1^{-1} \mathbf{R}_1(\mathbf{I} + sk(\vec{\alpha}_1))\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_2)) \approx \mathbf{I} + sk(\vec{\alpha}_3)$$

$$\mathbf{I} + \mathbf{R}_2^{-1} sk(\vec{\alpha}_1)\mathbf{R}_2 + sk(\vec{\alpha}_2) + \mathbf{R}_2^{-1} sk(\vec{\alpha}_1)\mathbf{R}_2 sk(\vec{\alpha}_2) \approx \mathbf{I} + sk(\vec{\alpha}_3)$$

$$\mathbf{R}_2^{-1} sk(\vec{\alpha}_1)\mathbf{R}_2 + sk(\vec{\alpha}_2) \approx sk(\vec{\alpha}_3)$$

Since $\mathbf{R}^{-1} \bullet (\vec{\mathbf{a}} \times \mathbf{R}\vec{\mathbf{b}}) = (\mathbf{R}^{-1}\vec{\mathbf{a}}) \times \vec{\mathbf{b}}$ for all $\mathbf{R}, \vec{\mathbf{a}}, \ \vec{\mathbf{b}}$ we get $\mathbf{R}_2^{-1} sk(\vec{\alpha}_1)\mathbf{R}_2 = sk(\mathbf{R}_2^{-1}\vec{\alpha}_1)$

$$sk(\vec{\alpha}_3) \approx sk(\mathbf{R}_2^{-1}\vec{\alpha}_1) + sk(\vec{\alpha}_2) = sk(\mathbf{R}_2^{-1}\vec{\alpha}_1 + \vec{\alpha}_2)$$

$$\vec{\alpha}_3 \approx \mathbf{R}_2^{-1}\vec{\alpha}_1 + \vec{\alpha}_2$$

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## Error from frame composition

Consider $\mathbf{F}_1^*\mathbf{F}_2^* = \mathbf{F}_3^*$ where $\mathbf{F}_1^* = \mathbf{F}_1\Delta\mathbf{F}_1, \mathbf{F}_2^* = \mathbf{F}_2\Delta\mathbf{F}_2, \mathbf{F}_3^* = \mathbf{F}_3\Delta\mathbf{F}_3$

and $\Delta\mathbf{F}_1 \approx \left[\mathbf{I} + sk\left(\vec{\alpha}_1\right), \vec{\varepsilon}_1\right]$, $\Delta\mathbf{F}_2 \approx \left[\mathbf{I} + sk\left(\vec{\alpha}_2\right), \vec{\varepsilon}_2\right]$,

estimate $\Delta\mathbf{F}_3 \approx \left[\mathbf{I} + sk\left(\vec{\alpha}_3\right), \vec{\varepsilon}_3\right]$

From before, we have $\vec{\alpha}_3 \approx \mathbf{R}_2^{-1}\vec{\alpha}_1 + \vec{\alpha}_2$. So now we just need $\vec{\varepsilon}_3$.

$$\mathbf{F}_3\Delta\mathbf{F}_3 = [\mathbf{R}_3\Delta\mathbf{R}_3, \mathbf{R}_3\Delta\vec{\mathbf{p}}_3 + \vec{\mathbf{p}}_3]$$

$$\vec{\mathbf{p}}_3 + \mathbf{R}_3\Delta\vec{\mathbf{p}}_3 = \mathbf{R}_1(\Delta\mathbf{R}_1(\vec{\mathbf{p}}_2 + \mathbf{R}_2\vec{\varepsilon}_2) + \varepsilon_1) + \vec{\mathbf{p}}_1$$

$$\vec{\mathbf{p}}_3 + \mathbf{R}_3\vec{\varepsilon}_3 \approx \mathbf{R}_1\left(\mathbf{I} + sk(\vec{\alpha}_1)\right)(\vec{\mathbf{p}}_2 + \mathbf{R}_2\vec{\varepsilon}_2) + \mathbf{R}_1\vec{\varepsilon}_1 + \vec{\mathbf{p}}_1$$

$$= \mathbf{R}_1\vec{\mathbf{p}}_2 + \mathbf{R}_1\mathbf{R}_2\vec{\varepsilon}_2 + \mathbf{R}_1\cdot\left(\vec{\alpha}_1\times\vec{\mathbf{p}}_2 + \vec{\alpha}_1\times\vec{\varepsilon}_2\right) + \vec{\mathbf{p}}_1 + \mathbf{R}_1\vec{\varepsilon}_1$$

$$= \vec{\mathbf{p}}_3 + \mathbf{R}_1\mathbf{R}_2\vec{\varepsilon}_2 + \mathbf{R}_1\cdot\left(\vec{\alpha}_1\times\vec{\mathbf{p}}_2 + \vec{\alpha}_1\times\vec{\varepsilon}_2\right) + \mathbf{R}_1\vec{\varepsilon}_1$$

$$\mathbf{R}_3\vec{\varepsilon}_3 \approx \mathbf{R}_1\mathbf{R}_2\vec{\varepsilon}_2 + \mathbf{R}_1\cdot\vec{\alpha}_1\times\vec{\mathbf{p}}_2 + \mathbf{R}_1\vec{\varepsilon}_1$$

$$\vec{\varepsilon}_3 \approx \mathbf{R}_2^{-1}\mathbf{R}_1^{-1}\mathbf{R}_1\mathbf{R}_2\vec{\varepsilon}_2 + \mathbf{R}_2^{-1}\mathbf{R}_1^{-1}\mathbf{R}_1\cdot\vec{\alpha}_1\times\vec{\mathbf{p}}_2 + \mathbf{R}_2^{-1}\mathbf{R}_1^{-1}\mathbf{R}_1\vec{\varepsilon}_1$$

$$= \vec{\varepsilon}_2 + \mathbf{R}_2^{-1}\cdot\vec{\alpha}_1\times\vec{\mathbf{p}}_2 + \mathbf{R}_2^{-1}\vec{\varepsilon}_1$$

$$= \vec{\varepsilon}_2 - \mathbf{R}_2^{-1}sk(\vec{\mathbf{p}}_2)\vec{\alpha}_1 + \mathbf{R}_2^{-1}\vec{\varepsilon}_1$$

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## Inverse of frame transformation with errors

$$\mathbf{F}_i = \mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1}\vec{\mathbf{p}}]$$

$$\left(\mathbf{F}_i^*\right)^{-1} = \left(\mathbf{F}\Delta\mathbf{F}\right)^{-1}$$

$$\left(\mathbf{F}_i\Delta\mathbf{F}_i\right)^{-1} = \left[\mathbf{R}\Delta\mathbf{R}, \mathbf{R}\Delta\vec{\mathbf{p}} + \vec{\mathbf{p}}\right]^{-1}$$

$$= [(\mathbf{R}\Delta\mathbf{R})^{-1}, -(\mathbf{R}\Delta\mathbf{R})^{-1}(\mathbf{R}\Delta\mathbf{p} + \vec{\mathbf{p}})]$$

$$= [\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}(\mathbf{R}\Delta\mathbf{p} + \vec{\mathbf{p}})]$$

$$= [\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}}]$$

$$\left(\Delta\mathbf{F}_i\right)^{-1} = \left(\mathbf{F}^{-1}\right)^{-1}[\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}}]$$

$$= [\mathbf{R}, \vec{\mathbf{p}}]\cdot[\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}}]$$

$$= [\mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\mathbf{R}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}} + \vec{\mathbf{p}}]$$

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## Inverse of frame transformation with errors

Suppose we know that $\Delta \mathbf{R}$ is "small", i.e., $\Delta \mathbf{R} \approx \mathbf{I} + sk(\vec{\alpha})$, and for notational convenience we write $\Delta \vec{\mathbf{p}} = \vec{\varepsilon}$, we get

$$\Delta \mathbf{R}_i = \mathbf{R}\Delta \mathbf{R}^{-1}\mathbf{R}^{-1} \approx \mathbf{R}\left(\mathbf{I} + sk(\vec{\alpha})\right)^{-1}\mathbf{R}^{-1}$$

$$\approx \mathbf{R}\left(\mathbf{I} - sk(\vec{\alpha})\right)\mathbf{R}^{-1}$$

$$= \mathbf{R}\mathbf{R}^{-1} - \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1}$$

$$= \mathbf{I} - \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1}$$

$$= \mathbf{I} - sk\left(\left(\mathbf{R}^{-1}\right)^{-1}\vec{\alpha}\right) = \mathbf{I} - sk(\mathbf{R}\vec{\alpha})$$

$$\Delta \vec{\mathbf{p}}_i = -\mathbf{R}\Delta \mathbf{R}^{-1}\Delta \vec{\mathbf{p}} - \mathbf{R}\Delta \mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}} + \vec{\mathbf{p}}$$

$$\approx -\mathbf{R}\left(\mathbf{I} - sk(\vec{\alpha})\right)\vec{\varepsilon} - \left(\mathbf{I} - sk(\mathbf{R}\vec{\alpha})\right)\vec{\mathbf{p}} + \vec{\mathbf{p}}$$

$$= -\mathbf{R}\vec{\varepsilon} + \mathbf{R}\left(\vec{\alpha} \times \vec{\varepsilon}\right) - \vec{\mathbf{p}} + (\mathbf{R}\vec{\alpha}) \times \vec{\mathbf{p}} + \vec{\mathbf{p}}$$

$$\approx -\mathbf{R}\vec{\varepsilon} + (\mathbf{R}\vec{\alpha}) \times \vec{\mathbf{p}} = -\mathbf{R}\vec{\varepsilon} - \vec{\mathbf{p}} \times (\mathbf{R}\vec{\alpha}) = -\mathbf{R}\vec{\varepsilon} - sk(\vec{\mathbf{p}})\mathbf{R}\vec{\alpha}$$

$$\text{(also)} = -\mathbf{R}\vec{\varepsilon} - \left((\mathbf{R}\mathbf{R}^{-1}\vec{\mathbf{p}}) \times (\mathbf{R}\vec{\alpha})\right) = -\mathbf{R}\vec{\varepsilon} - \mathbf{R}\left((\mathbf{R}^{-1}\vec{\mathbf{p}}) \times \vec{\alpha}\right)$$

$$= -\mathbf{R} \bullet \left(\vec{\varepsilon} + (\mathbf{R}^{-1}\vec{\mathbf{p}}) \times \vec{\alpha}\right) = -\mathbf{R}\vec{\varepsilon} - \mathbf{R}sk(\mathbf{R}^{-1}\vec{\mathbf{p}})\vec{\alpha}$$
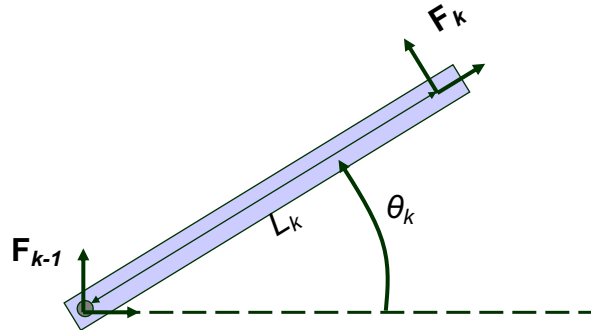
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## Error Propagation in Chains



$$\mathbf{F}^{*}_{k} = \mathbf{F}^{*}_{k-1} \bullet \mathbf{F}^{*}_{k-1,k}$$

$$\mathbf{F}_{k}\Delta \mathbf{F}_{k} = \mathbf{F}_{k-1}\Delta \mathbf{F}_{k-1}\mathbf{F}_{k-1,k}\Delta \mathbf{F}_{k-1,k}$$

$$\Delta \mathbf{F}_{k} = \left(\mathbf{F}_{k}^{-1}\mathbf{F}_{k-1}\right)\Delta \mathbf{F}_{k-1}\mathbf{F}_{k-1,k}\Delta \mathbf{F}_{k-1,k}$$

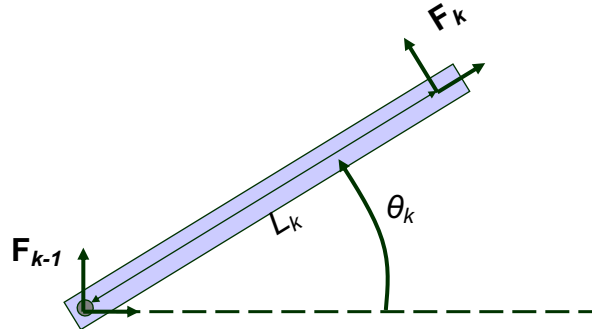$$= \left(\mathbf{F}_{k-1,k}^{-1}\Delta \mathbf{F}_{k-1}\mathbf{F}_{k-1,k}\right)\Delta \mathbf{F}_{k-1,k}$$

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## Error Propagation in Chains



$$\Delta \mathbf{F}_k = \left( \mathbf{F}_k^{-1} \mathbf{F}_{k-1} \right) \Delta \mathbf{F}_{k-1} \mathbf{F}_{k-1,k} \Delta \mathbf{F}_{k-1,k}$$
$$= \left( \mathbf{F}_{k-1,k}^{-1} \Delta \mathbf{F}_{k-1} \mathbf{F}_{k-1,k} \right) \Delta \mathbf{F}_{k-1,k}$$
$$\Delta \mathbf{R}_k = \left( \mathbf{R}_{k-1,k}^{-1} \Delta \mathbf{R}_{k-1} \mathbf{R}_{k-1,k} \right) \Delta \mathbf{R}_{k-1,k}$$
$$\approx \left( \mathbf{R}_{k-1,k}^{-1} \left( \mathbf{I} + skew(\vec{\alpha}_{k-1}) \right) \mathbf{R}_{k-1,k} \right) \left( \mathbf{I} + skew(\vec{\alpha}_{k-1,k}) \right)$$
$$\approx \mathbf{I} + \left( \mathbf{R}_{k-1,k}^{-1} skew(\vec{\alpha}_{k-1}) \mathbf{R}_{k-1,k} \right) + skew(\vec{\alpha}_{k-1,k}) = \mathbf{I} + skew(\mathbf{R}_{k-1,k}^{-1} \vec{\alpha}_{k-1} + \vec{\alpha}_{k-1,k})$$

Note: This is same as what we could have obtained by substituting in formulas from the "error from frame composition" slides given earlier.

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## Exercise

Suppose that you have

$$\Delta \mathbf{R}_{k-1,k} = \Delta \mathbf{R}(\vec{\mathbf{a}}_k) \cong \mathbf{I} + skew(\vec{\mathbf{a}}_k)$$
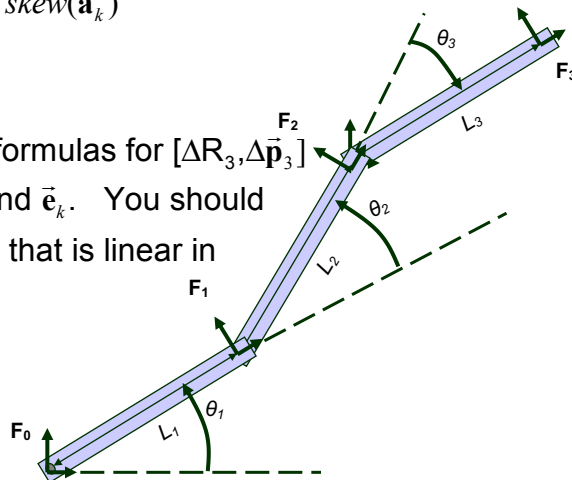$$\Delta \vec{\mathbf{p}}_{k-1,k} = \vec{\mathbf{e}}_k$$



Work out approximate formulas for $[\Delta \mathbf{R}_3, \Delta \vec{\mathbf{p}}_3]$ in terms of $L_k, \vec{\mathbf{r}}_k, \theta_k, \vec{\mathbf{a}}_k$ and $\vec{\mathbf{e}}_k$. You should come up with a formula that is linear in $L_k, \vec{\mathbf{a}}_k$, and $\vec{\mathbf{e}}_k$.

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## Exercise

Suppose we want to know error in $\mathbf{F}_{0,3} = \mathbf{F}_0^{-1}\mathbf{F}_3$

$$\mathbf{F}_{0,3} = \mathbf{F}_0^{-1}\mathbf{F}_0\mathbf{F}_{0,1}\mathbf{F}_{1,2}\mathbf{F}_{2,3}$$
$$\mathbf{F}_{0,3}{}^* = \mathbf{F}_{0,1}{}^*\mathbf{F}_{1,2}{}^*\mathbf{F}_{2,3}{}^*$$
$$\mathbf{F}_{0,3}\Delta\mathbf{F}_{0,3} = \mathbf{F}_{0,1}{}^*\mathbf{F}_{1,2}{}^*\mathbf{F}_{2,3}{}^*$$
$$\Delta\mathbf{F}_3 = \mathbf{F}_{0,3}{}^{-1}\mathbf{F}_{0,1}\Delta\mathbf{F}_{0,1}\mathbf{F}_{1,2}\Delta\mathbf{F}_{1,2}\mathbf{F}_{2,3}\Delta\mathbf{F}_{2,3}$$
$$= \mathbf{F}_{2,3}{}^{-1}\mathbf{F}_{1,2}{}^{-1}\mathbf{F}_{0,1}{}^{-1}\mathbf{F}_{0,1}\Delta\mathbf{F}_{0,1}\mathbf{F}_{1,2}\Delta\mathbf{F}_{1,2}\mathbf{F}_{2,3}\Delta\mathbf{F}_{2,3}$$
$$= \mathbf{F}_{2,3}{}^{-1}\mathbf{F}_{1,2}{}^{-1}\Delta\mathbf{F}_{0,1}\mathbf{F}_{1,2}\Delta\mathbf{F}_{1,2}\mathbf{F}_{2,3}\Delta\mathbf{F}_{2,3}$$

Now substitute and simplify

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## Another Example

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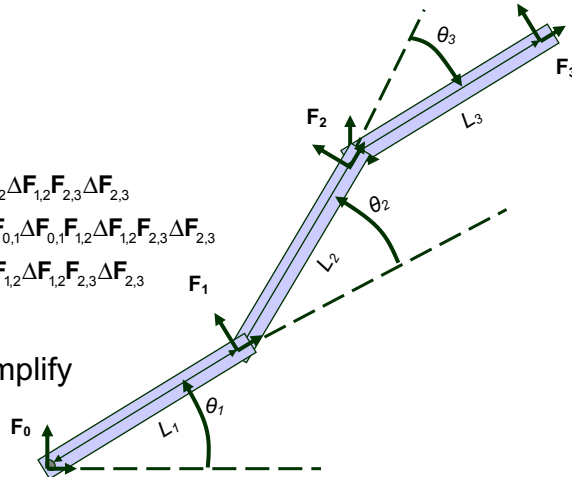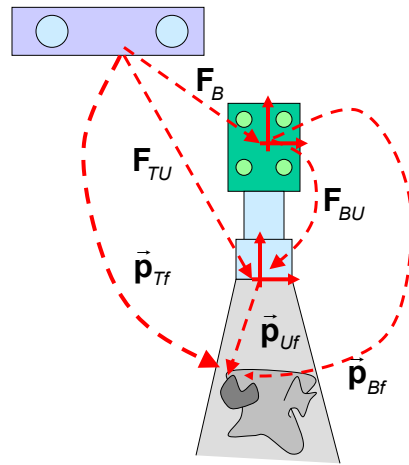
## Another Example

$$\vec{\mathbf{p}}_{Tf} = \mathbf{F}_{TU} \bullet \vec{\mathbf{p}}_{Uf}$$

$$\mathbf{F}_{TU} = \mathbf{F}_B \bullet \mathbf{F}_{BU}$$

$$= [\mathbf{R}_B \bullet \mathbf{R}_{BU}, \mathbf{R}_B \bullet \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_B]$$

$$\vec{\mathbf{p}}_{Tf} = \mathbf{R}_B \bullet \mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} + \mathbf{R}_B \bullet \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_B$$

### Also

$$\vec{\mathbf{p}}_{Tf} = \mathbf{F}_B \bullet \vec{\mathbf{p}}_{Bf}$$

$$\vec{\mathbf{p}}_{Bf} = \mathbf{F}_{BU} \bullet \vec{\mathbf{p}}_{Uf}$$

$$= \mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} + \vec{\mathbf{p}}_{BU}$$

$$\vec{\mathbf{p}}_{Tf} = \mathbf{R}_B \bullet \mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} + \mathbf{R}_B \bullet \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_B$$

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## Another Example

Suppose that the track body to US calibration is not perfect

$$\mathbf{F}_{BU}{}^* = \mathbf{F}_{BU}\Delta\mathbf{F}_{BU}$$

$$= [\mathbf{R}_{BU}\Delta\mathbf{R}_{BU}, \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_{BU}]$$

$$\vec{\mathbf{p}}_{Bf}{}^* = \mathbf{F}_{BU}{}^* \bullet \vec{\mathbf{p}}_{Uf} \quad \text{I.e.,} \quad \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{BF} = \mathbf{F}_{BU}\Delta\mathbf{F}_{BU}\vec{\mathbf{p}}_{Uf}$$

$$\Delta\vec{\mathbf{p}}_{Bf} = \mathbf{F}_{BU}\Delta\mathbf{F}_{BU}\vec{\mathbf{p}}_{Uf} - \vec{\mathbf{p}}_{Bf}$$

$$= \mathbf{F}_{BU}(\Delta\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} + \Delta\vec{\mathbf{p}}_{BU}) - (\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} + \vec{\mathbf{p}}_{BU})$$

$$= \mathbf{R}_{BU}\Delta\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} - \vec{\mathbf{p}}_{BU}$$

$$= \mathbf{R}_{BU}\Delta\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}$$

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## Another Example

Continuing …

$$\Delta \vec{\mathbf{p}}_{Bf} = \mathbf{R}_{BU} \Delta \mathbf{R}_{BU} \vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU} \Delta \vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU} \vec{\mathbf{p}}_{Uf}$$

$$\approx \mathbf{R}_{BU} \left( \mathbf{I} + \mathbf{skew}(\vec{\alpha}_{BU}) \right) \vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU} \Delta \vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU} \vec{\mathbf{p}}_{Uf}$$

$$= \mathbf{R}_{BU} \vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU} \bullet \vec{\alpha}_{BU} \times \vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU} \Delta \vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU} \vec{\mathbf{p}}_{Uf}$$

$$= \mathbf{R}_{BU} \bullet \vec{\alpha}_{BU} \times \vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU} \Delta \vec{\mathbf{p}}_{BU}$$

$$= -\mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} \times \vec{\alpha}_{BU} + \mathbf{R}_{BU} \Delta \vec{\mathbf{p}}_{BU}$$

$$= \mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{Uf}) \vec{\alpha}_{BU} + \mathbf{R}_{BU} \Delta \vec{\mathbf{p}}_{BU}$$

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## Another Example

$$\vec{\mathbf{p}}_{Tf} + \Delta \vec{\mathbf{p}}_{Tf} = \mathbf{F}_B \Delta \mathbf{F}_B \left( \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_{Bf} \right)$$

$$\Delta \vec{\mathbf{p}}_{Tf} = \mathbf{F}_B \Delta \mathbf{F}_B \left( \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_{Bf} \right) - \mathbf{F}_B \vec{\mathbf{p}}_{Bf}$$

$$\Delta \mathbf{F}_B \left( \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_{Bf} \right) = \Delta \mathbf{R}_B \left( \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_{Bf} \right) + \Delta \vec{\mathbf{p}}_B$$

$$\approx \left( \mathbf{I} + skew(\vec{\alpha}_B) \right) \left( \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_{Bf} \right) + \Delta \vec{\mathbf{p}}_B$$

$$= \left( \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_{Bf} \right) + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \Delta \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_B$$

$$\approx \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_B$$

$$\Delta \vec{\mathbf{p}}_{Tf} \approx \mathbf{F}_B \left( \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_B \right) - \mathbf{F}_B \vec{\mathbf{p}}_{Bf}$$

$$= \mathbf{R}_B \left( \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_B \right) + \vec{\mathbf{p}}_B - \left( \mathbf{R}_B \vec{\mathbf{p}}_{Bf} + \vec{\mathbf{p}}_B \right)$$

$$= \mathbf{R}_B \left( \Delta \vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta \vec{\mathbf{p}}_B \right)$$

$$\Delta \vec{\mathbf{p}}_{Bf} \approx \mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU}) \vec{\alpha}_{BU} + \mathbf{R}_{BU} \Delta \vec{\mathbf{p}}_{BU}$$

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## Another Example

$$\Delta\vec{\mathbf{p}}_{Tf} \approx \mathbf{R}_B\left(\Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right)$$

$$\Delta\vec{\mathbf{p}}_{Bf} \approx \mathbf{R}_{BU}skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}$$

$$\Delta\vec{\mathbf{p}}_{Tf} \approx \mathbf{R}_B\left(\mathbf{R}_{BU}skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right)$$

$$= \left(\begin{array}{c} \mathbf{R}_B\mathbf{R}_{BU}skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_B\mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} \\ +\mathbf{R}_B skew(-\vec{\mathbf{p}}_{Bf})\vec{\alpha}_B + \mathbf{R}_B\Delta\vec{\mathbf{p}}_B \end{array}\right)$$

$$= \left[\mathbf{R}_B\mathbf{R}_{BU}skew(-\vec{\mathbf{p}}_{BU}) \;\;|\;\; \mathbf{R}_B\mathbf{R}_{BU} \;\;|\;\; \mathbf{R}_B skew(-\vec{\mathbf{p}}_{Bf}) \;\;|\;\; \mathbf{R}_B \right]\begin{bmatrix} \vec{\alpha}_{BU} \\ \Delta\vec{\mathbf{p}}_{BU} \\ \vec{\alpha}_B \\ \Delta\vec{\mathbf{p}}_B \end{bmatrix}$$

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## Parametric Sensitivity

Suppose you have an explicit formula like

$$\vec{\mathbf{p}}_3 = \begin{bmatrix} L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) \\ L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) + L_3\sin(\theta_1+\theta_2+\theta_3) \\ 0 \end{bmatrix}$$

and know that the only variation is in parameters like $L_k$ and $\theta_k$. Then you can estimate the variation in $\vec{\mathbf{p}}_3$ as a function of variation in $L_k$ and $\theta_k$ by remembering your calculus.

$$\Delta\vec{\mathbf{p}}_3 \cong \begin{bmatrix} \dfrac{\partial\vec{\mathbf{p}}_3}{\partial\vec{L}} & \dfrac{\partial\vec{\mathbf{p}}_3}{\partial\vec{\theta}} \end{bmatrix}\begin{bmatrix} \Delta\vec{L} \\ \Delta\vec{\theta} \end{bmatrix}$$

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# Parametric Sensitivity

Grinding this out gives:

$$\Delta\vec{\mathbf{p}}_3 \cong \begin{bmatrix} \dfrac{\partial\vec{\mathbf{p}}_3}{\partial\vec{L}} & \dfrac{\partial\vec{\mathbf{p}}_3}{\partial\vec{\theta}} \end{bmatrix} \begin{bmatrix} \Delta\vec{L} \\ \Delta\vec{\theta} \end{bmatrix}$$

where

$$\vec{L} = [L_1, L_2, L_3]^T$$

$$\vec{\theta} = [\theta_1, \theta_2, \theta_3]^T$$

$$\frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{L}} = \begin{bmatrix} \cos(\theta_1) & \cos(\theta_1+\theta_2) & \cos(\theta_1+\theta_2+\theta_3) \\ \sin(\theta_1) & \sin(\theta_1+\theta_2) & \sin(\theta_1+\theta_2+\theta_3) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{\theta}} = \begin{bmatrix} -L_1\sin(\theta_1) - L_2\sin(\theta_1+\theta_2) - L_3\sin(\theta_1+\theta_2+\theta_3) & -L_2\sin(\theta_1+\theta_2) - L_3\sin(\theta_1+\theta_2+\theta_3) & -L_3\sin(\theta_1+\theta_2+\theta_3) \\ L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) & L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) & L_3\cos(\theta_1+\theta_2+\theta_3) \\ 0 & 0 & 0 \end{bmatrix}$$

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# More generally …

Suppose that we have a vector function

$$\vec{\mathbf{v}} = \vec{\mathbf{g}}(\vec{\mathbf{q}}) = [g_1(\vec{\mathbf{q}}),...,g_m(\vec{\mathbf{q}})]^T$$

of parameters $\vec{\mathbf{q}} = [q_1,...,q_n]$. Then we can estimate the value of

$$\vec{\mathbf{v}} + \Delta\vec{\mathbf{v}} = \vec{\mathbf{g}}(\vec{\mathbf{q}} + \Delta\vec{\mathbf{q}})$$

by

$$\vec{\mathbf{v}} + \Delta\vec{\mathbf{v}} \approx \vec{\mathbf{g}}(\vec{\mathbf{q}}) + \mathbf{J}_G(\vec{\mathbf{q}}) \bullet \Delta\vec{\mathbf{q}}$$

where

$$\mathbf{J}_G(\vec{\mathbf{q}}) = \begin{bmatrix} \dfrac{\partial g_1}{\partial q_1} & & \dfrac{\partial g_1}{\partial q_j} & & \dfrac{\partial g_1}{\partial q_n} \\ \vdots & & \vdots & & \vdots \\ \dfrac{\partial g_i}{\partial q_1} & \cdots & \dfrac{\partial g_i}{\partial q_j} & \cdots & \dfrac{\partial g_i}{\partial q_n} \\ \vdots & & \vdots & & \vdots \\ \dfrac{\partial g_m}{\partial q_1} & & \dfrac{\partial g_m}{\partial q_j} & & \dfrac{\partial g_m}{\partial q_n} \end{bmatrix}$$

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