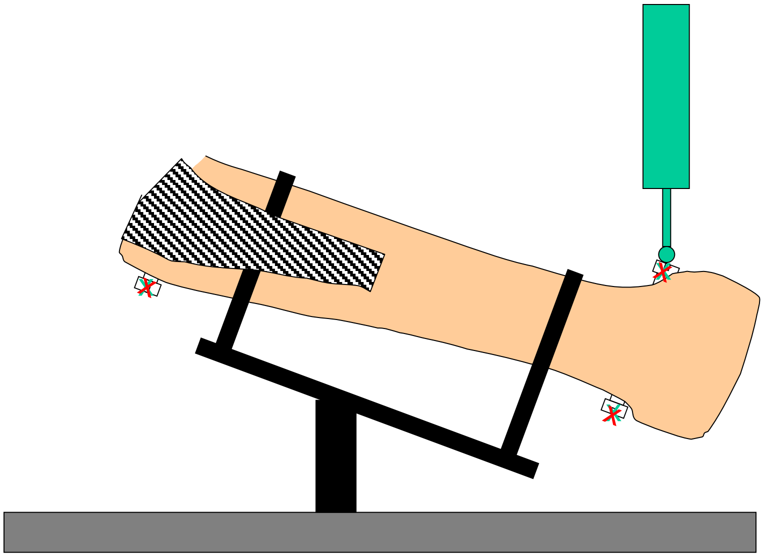
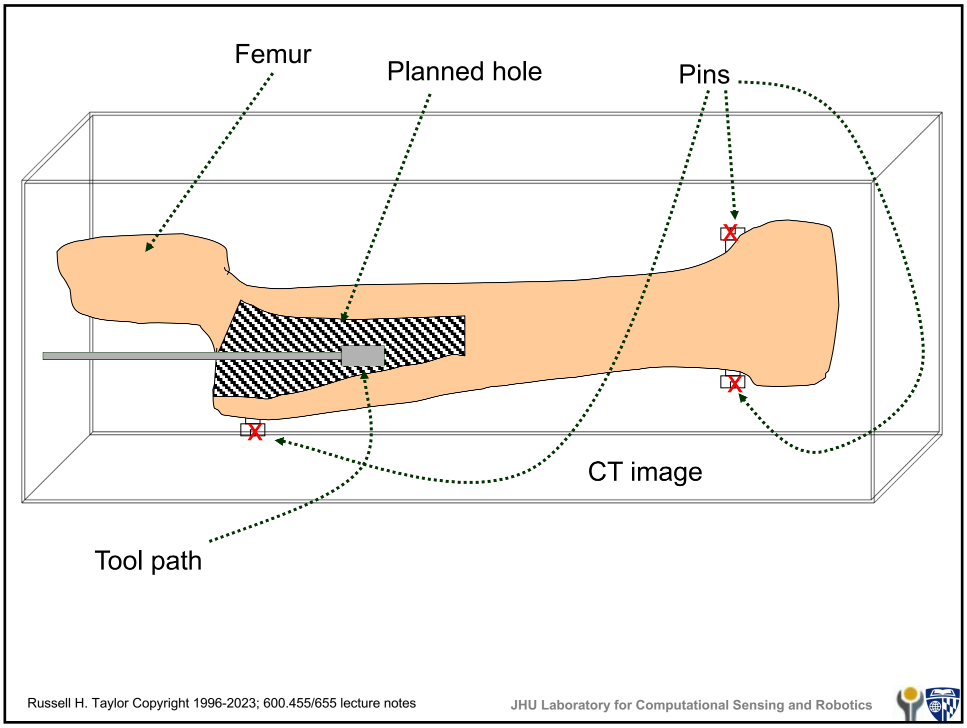
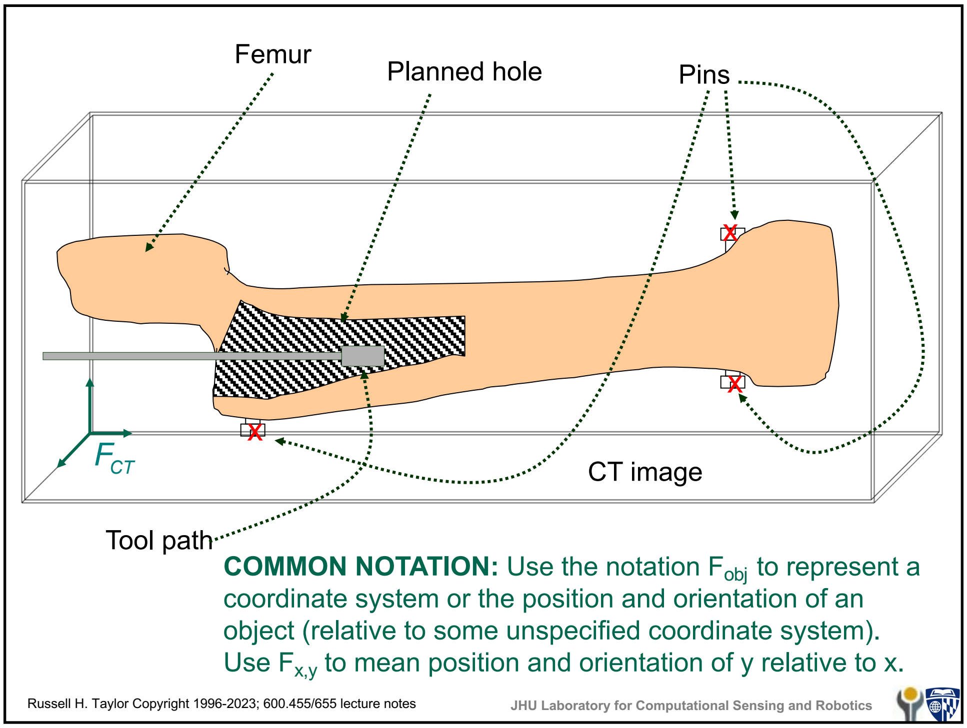
Cartesian Coordinates, Points, and Transformations

CIS - 600.455/655

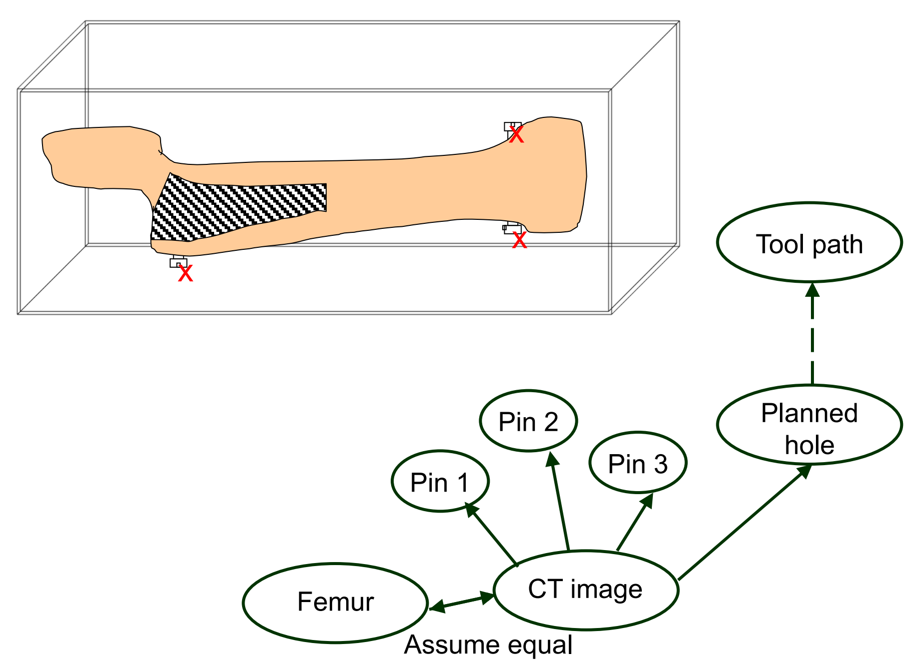
Russell Taylor







COMMON NOTATION: Use the notation F_{obj} to represent a coordinate system or the position and orientation of an object (relative to some unspecified coordinate system). Use $F_{x,y}$ to mean position and orientation of y relative to x.



Tool path

Planned hole

Pin 2

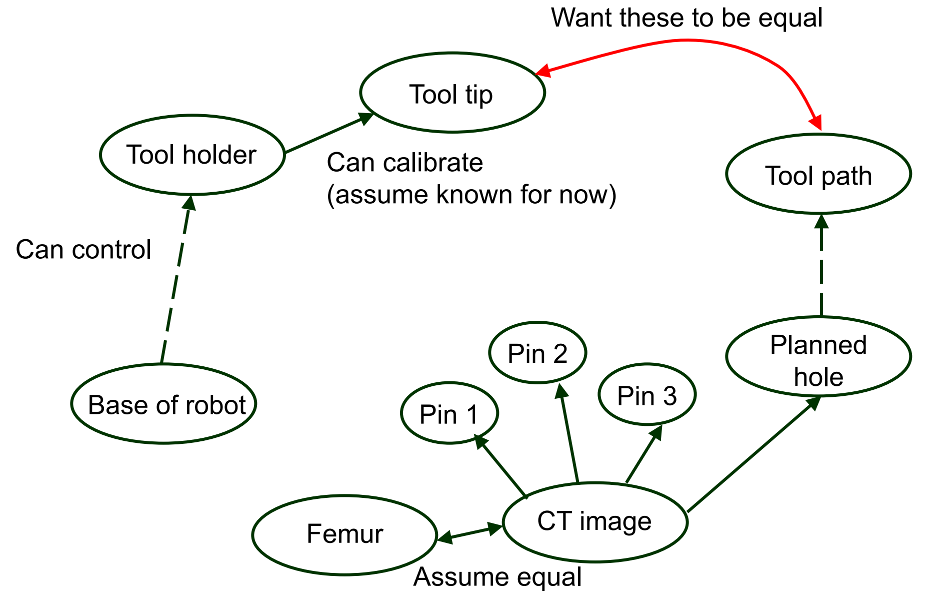
Pin 3

Pin 1

Femur

CT image

Assume equal



Want these to be equal

Tool tip

Tool holder

Can calibrate
(assume known for now)

Tool path

Can control

Base of robot

Pin 2

Pin 3

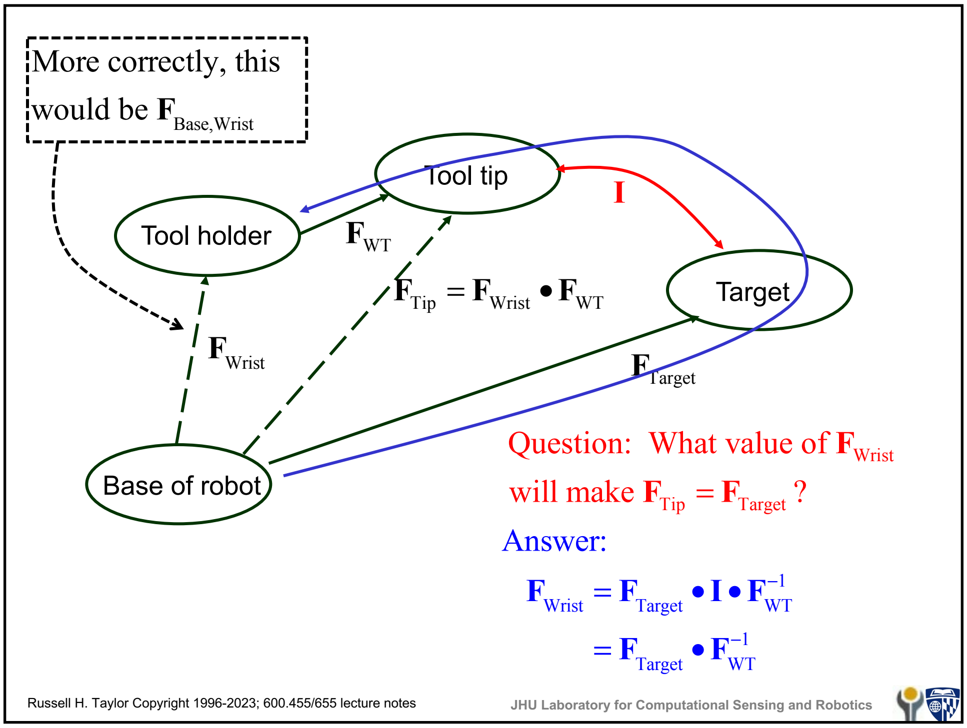
Planned hole

Pin 1

Femur

CT image

Assume equal

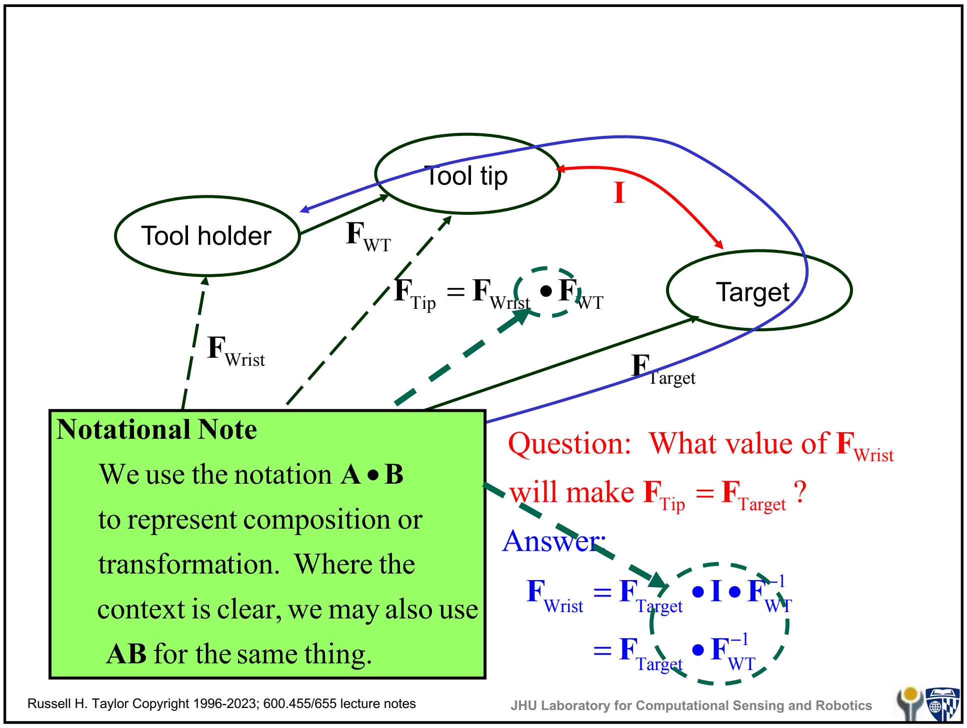


$$\mathbf{F}_{\text{Tip}} = \mathbf{F}_{\text{Wrist}} \bullet \mathbf{F}_{\text{WT}}$$

Question: What value of $\mathbf{F}_{\text{Wrist}}$ will make $\mathbf{F}_{\text{Tip}} = \mathbf{F}_{\text{Target}}$?

Answer:

$$\mathbf{F}_{\text{Wrist}} = \mathbf{F}_{\text{Target}} \bullet \mathbf{I} \bullet \mathbf{F}_{\text{WT}}^{-1}$$
$$= \mathbf{F}_{\text{Target}} \bullet \mathbf{F}_{\text{WT}}^{-1}$$



$$\mathbf{F}_{\text{Tip}} = \mathbf{F}_{\text{Wrist}} \bullet \mathbf{F}_{\text{WT}}$$

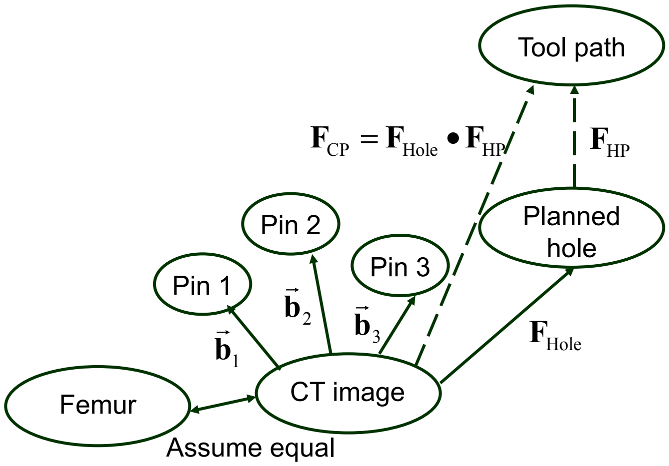
Notational Note

We use the notation $\mathbf{A} \bullet \mathbf{B}$ to represent composition or transformation. Where the context is clear, we may also use $\mathbf{AB}$ for the same thing.

Question: What value of $\mathbf{F}_{\text{Wrist}}$ will make $\mathbf{F}_{\text{Tip}} = \mathbf{F}_{\text{Target}}$?

Answer:

$$\mathbf{F}_{\text{Wrist}} = \mathbf{F}_{\text{Target}} \bullet \mathbf{I} \bullet \mathbf{F}_{\text{WT}}^{-1}$$
$$= \mathbf{F}_{\text{Target}} \bullet \mathbf{F}_{\text{WT}}^{-1}$$



Tool path

$\mathbf{F}_{CP} = \mathbf{F}_{Hole} \bullet \mathbf{F}_{HP}$

$\mathbf{F}_{HP}$

Pin 2

Pin 3

Planned hole

Pin 1

$\vec{\mathbf{b}}_2$

$\vec{\mathbf{b}}_3$

$\vec{\mathbf{b}}_1$

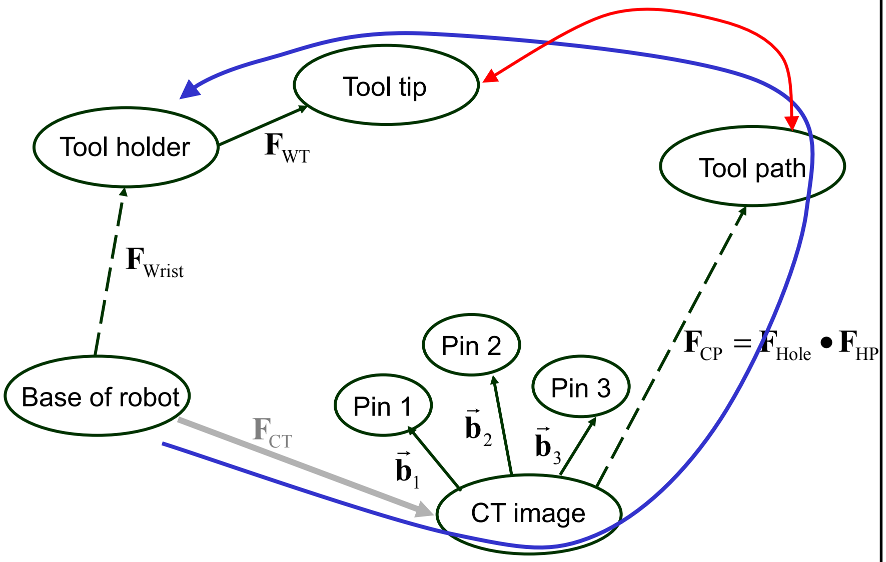
$\mathbf{F}_{Hole}$

Femur

CT image

Assume equal

Question: What value of $\mathbf{F}_{Wrist}$ will make these equal?



Tool tip

Tool holder

$\mathbf{F}_{WT}$

Tool path

$\mathbf{F}_{Wrist}$

$\mathbf{F}_{CP} = \mathbf{F}_{Hole} \bullet \mathbf{F}_{HP}$

Pin 2

Pin 3

Base of robot

Pin 1

$\mathbf{F}_{CT}$

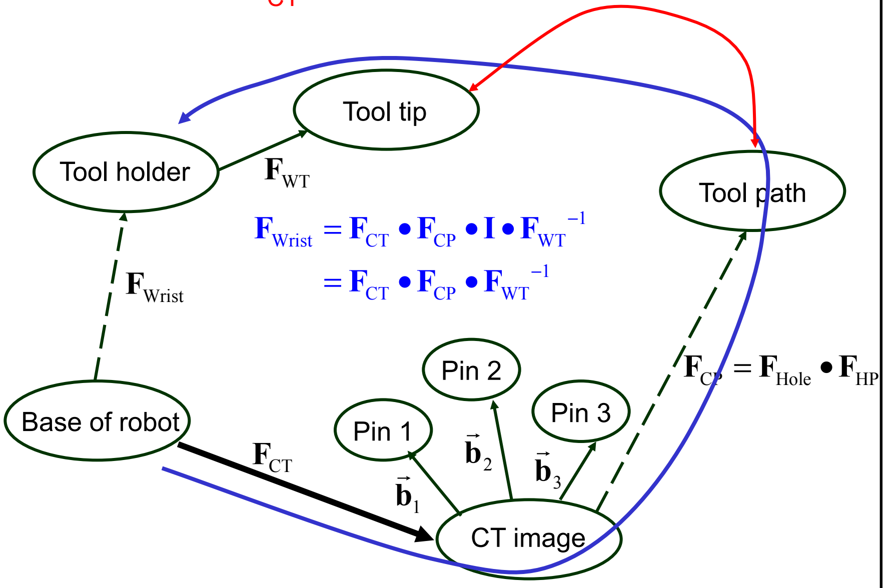
$\vec{\mathbf{b}}_2$

$\vec{\mathbf{b}}_3$

$\vec{\mathbf{b}}_1$

CT image

But: We must find F_{CT} … Let's review some math



Tool tip

Tool holder

Tool path

Base of robot

Pin 1

Pin 2

Pin 3

CT image

$\mathbf{F}_{WT}$

$\mathbf{F}_{Wrist}$

$\mathbf{F}_{CT}$

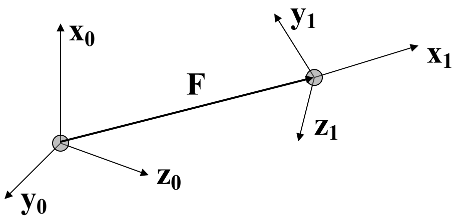
$\vec{\mathbf{b}}_1$ $\vec{\mathbf{b}}_2$ $\vec{\mathbf{b}}_3$

$$\mathbf{F}_{Wrist} = \mathbf{F}_{CT} \bullet \mathbf{F}_{CP} \bullet \mathbf{I} \bullet \mathbf{F}_{WT}^{-1}$$
$$= \mathbf{F}_{CT} \bullet \mathbf{F}_{CP} \bullet \mathbf{F}_{WT}^{-1}$$

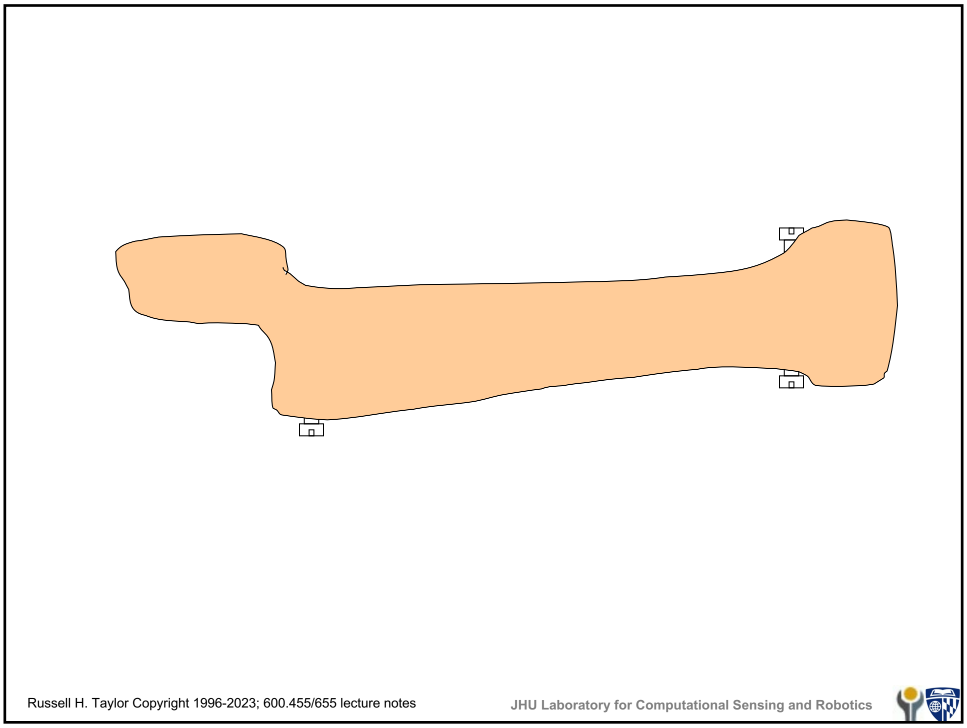
$$\mathbf{F}_{CP} = \mathbf{F}_{Hole} \bullet \mathbf{F}_{HP}$$

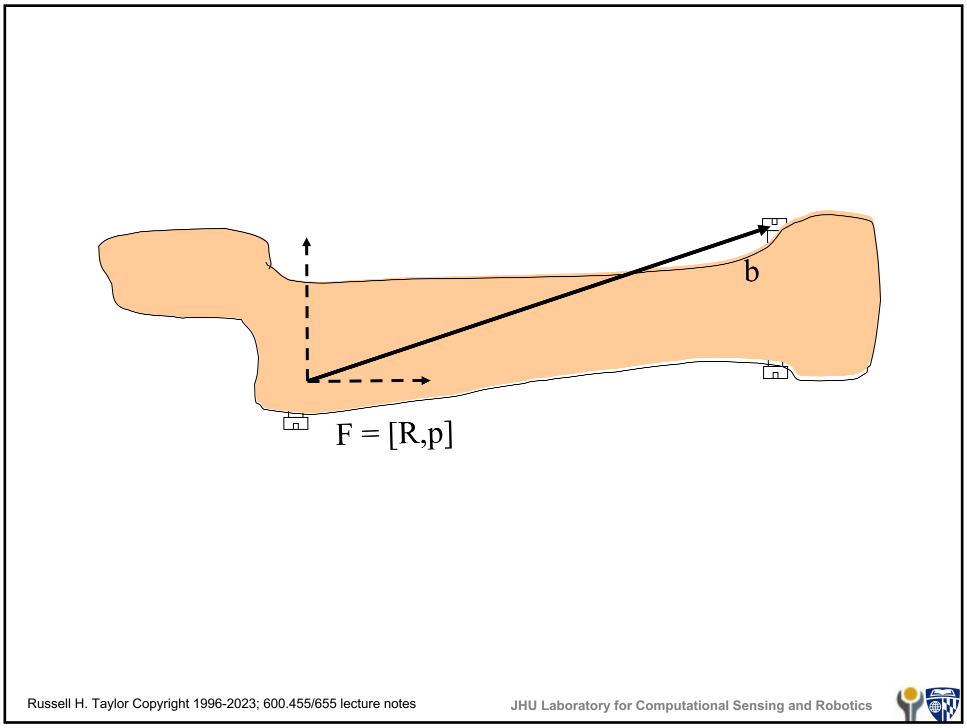
Coordinate Frame Transformation

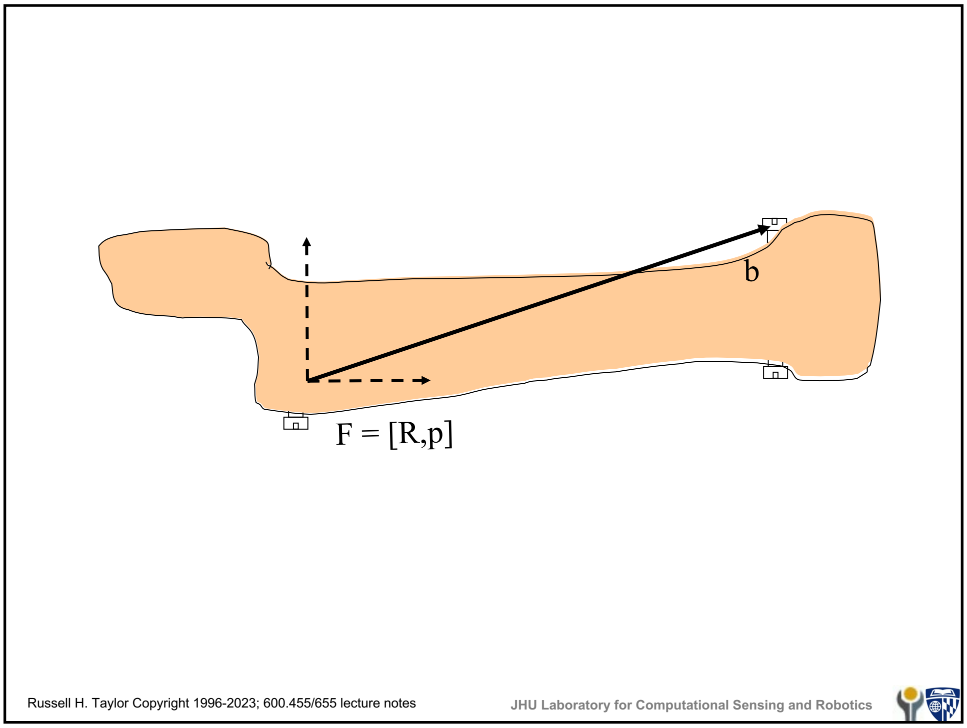
$$\mathbf{F} = [\mathbf{R}, \mathbf{p}]$$



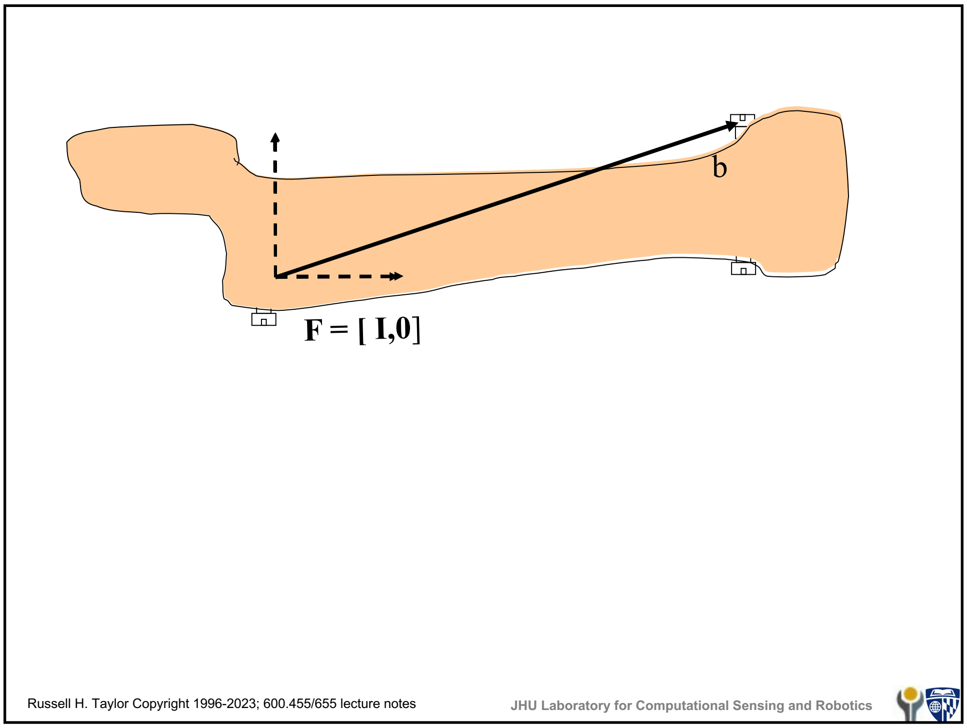
Slide acknowledgment: Sarah Graham and Andy Bzostek



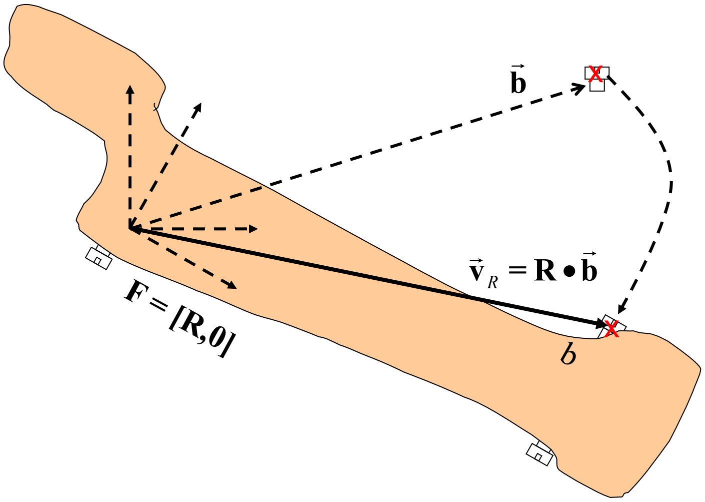

b
F = [R,p]



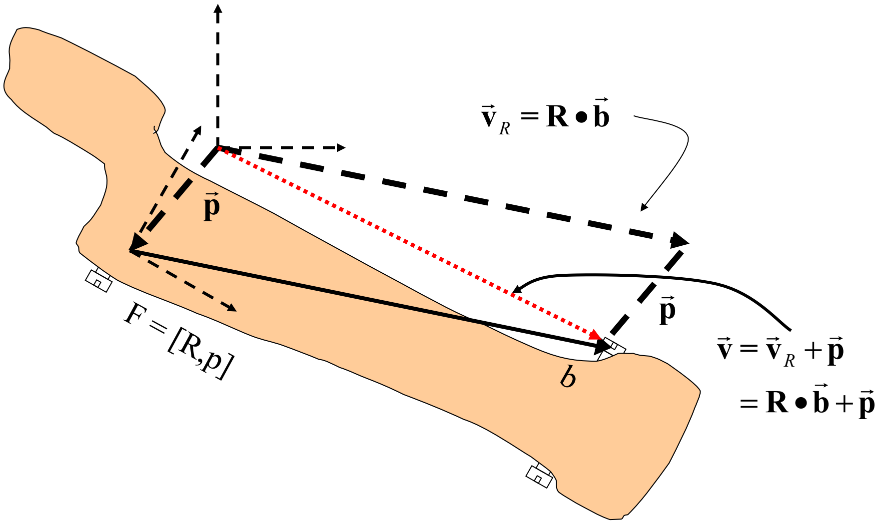
$F = [R,p]$



$\mathbf{F = [\,I,0]}$



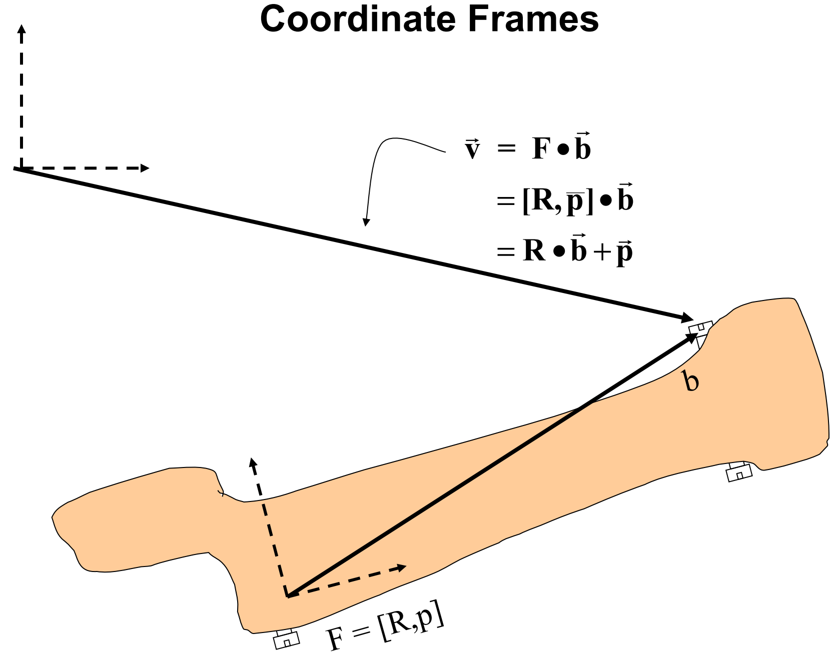
$$\vec{\mathbf{v}}_R = \mathbf{R} \bullet \vec{\mathbf{b}}$$



$$\vec{\mathbf{v}}_R = \mathbf{R} \bullet \vec{\mathbf{b}}$$

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}_R + \vec{\mathbf{p}}$$
$$= \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}$$

Coordinate Frames



$$\begin{aligned}\vec{v} &= F \bullet \vec{b}\\ &= [R,\vec{p}] \bullet \vec{b}\\ &= R \bullet \vec{b} + \vec{p}\end{aligned}$$

Forward and Inverse Frame Transformations

Forward

$$\mathbf{F} = [\mathbf{R},\vec{\mathbf{p}}]$$

$$\begin{aligned}\vec{\mathbf{v}} &= \mathbf{F} \bullet \vec{\mathbf{b}}\\ &= [\mathbf{R},\vec{\mathbf{p}}] \bullet \vec{\mathbf{b}}\\ &= \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}\end{aligned}$$

Inverse

$$\begin{aligned}\mathbf{F}^{-1}\vec{\mathbf{v}} &= \vec{\mathbf{b}}\\ \vec{\mathbf{b}} &= \mathbf{R}^{-1} \bullet (\vec{\mathbf{v}} - \vec{\mathbf{p}})\\ &= \mathbf{R}^{-1} \bullet \vec{\mathbf{v}} - \mathbf{R}^{-1} \bullet \vec{\mathbf{p}}\end{aligned}$$

$$\mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1} \bullet \vec{\mathbf{p}}]$$

Composition

Assume $\mathbf{F}_1 = [\mathbf{R}_1, \vec{\mathbf{p}}_1], \quad \mathbf{F}_2 = [\mathbf{R}_2, \vec{\mathbf{p}}_2]$

Then

$$\begin{aligned}
\mathbf{F}_1 \bullet \mathbf{F}_2 \bullet \vec{\mathbf{b}} &= \mathbf{F}_1 \bullet (\mathbf{F}_2 \bullet \vec{\mathbf{b}}) \\
&= \mathbf{F}_1 \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\
&= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\
&= \mathbf{R}_1 \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) + \vec{\mathbf{p}}_1 \\
&= \mathbf{R}_1 \bullet \mathbf{R}_2 \bullet \vec{\mathbf{b}} + \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1 \\
&= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1] \bullet \vec{\mathbf{b}}
\end{aligned}$$

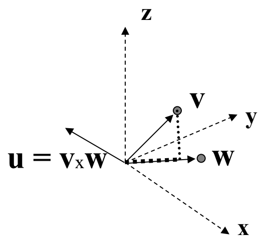
So

$$\begin{aligned}
\mathbf{F}_1 \bullet \mathbf{F}_2 &= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet [\mathbf{R}_2, \vec{\mathbf{p}}_2] \\
&= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1]
\end{aligned}$$

Vectors

$$\vec{\mathbf{v}}_{col} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\vec{\mathbf{v}}_{row} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}$$



length: $\|\vec{\mathbf{v}}\| = \sqrt{v_x^{\,2} + v_y^{\,2} + v_z^{\,2}}$

dot product: $a = \vec{\mathbf{v}} \cdot \vec{\mathbf{w}} \;= \left(v_x w_x + v_y w_y + v_z w_z\right) \;= \|\vec{\mathbf{v}}\| \|\vec{\mathbf{w}}\| \cos\theta$

cross product: $\vec{\mathbf{u}} = \vec{\mathbf{v}} \times \vec{\mathbf{w}} = \begin{bmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{bmatrix}, \|\mathbf{u}\| = \|\mathbf{v}\| \|\mathbf{w}\| \sin\theta$

Slide acknowledgment: Sarah Graham and Andy Bzostek

Matrix representation of cross product operator

Define

$$\hat{\vec{\mathbf{a}}} \overset{\Delta}{=} skew(\vec{\mathbf{a}}) \overset{\Delta}{=} \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Then

$$\vec{\mathbf{a}} \times \vec{\mathbf{v}} = skew(\vec{\mathbf{a}}) \bullet \vec{\mathbf{v}}$$

Rotations: Some Notation

$$Rot(\vec{\mathbf{a}},\alpha) = \text{ Rotation by angle } \alpha \text{ about axis } \vec{\mathbf{a}}$$

$$\mathbf{R}_{\vec{\mathbf{a}}}(\alpha) = \text{ Rotation by angle } \alpha \text{ about axis } \vec{\mathbf{a}}$$

$$\mathbf{R}(\vec{\mathbf{a}}) = Rot(\vec{\mathbf{a}}, \left\|\vec{\mathbf{a}}\right\|)$$

$$\mathbf{R}_{\mathbf{xyz}}(\alpha,\beta,\gamma) = \mathbf{R}(\vec{\mathbf{x}},\alpha) \bullet \mathbf{R}(\vec{\mathbf{y}},\beta) \bullet \mathbf{R}(\vec{\mathbf{z}},\gamma)$$

$$\mathbf{R}_{\mathbf{zyz}}(\alpha,\beta,\gamma) = \mathbf{R}(\vec{\mathbf{z}},\alpha) \bullet \mathbf{R}(\vec{\mathbf{y}},\beta) \bullet \mathbf{R}(\vec{\mathbf{z}},\gamma)$$

Rotations: A few useful facts

$$Rot(s\vec{\mathbf{a}},\alpha)\bullet\vec{\mathbf{a}}=\vec{\mathbf{a}} \quad \text{and} \quad \left\|Rot(\vec{\mathbf{a}},\alpha)\bullet\vec{\mathbf{b}}\right\|=\left\|\vec{\mathbf{b}}\right\|$$

$$Rot(\vec{\mathbf{a}},\alpha)=Rot(\hat{\mathbf{a}},\alpha) \ \text{ where } \hat{\mathbf{a}}=\frac{\vec{\mathbf{a}}}{\left\|\vec{\mathbf{a}}\right\|}$$

NOTE: Unless otherwise stated, we will usually assume that $\vec{\mathbf{a}}$ in $Rot(\vec{\mathbf{a}},\theta)$ is a unit vector. I.e., $\left\|\vec{\mathbf{a}}\right\|=1$.

$$Rot(\vec{\mathbf{a}},\alpha)\bullet Rot(\vec{\mathbf{a}},\beta)=Rot(\vec{\mathbf{a}},\alpha+\beta)$$

$$Rot(\vec{\mathbf{a}},\alpha)^{-1}=Rot(\vec{\mathbf{a}},-\alpha)$$

$$Rot(\vec{\mathbf{a}},0)\bullet\vec{\mathbf{b}}=\vec{\mathbf{b}} \quad \text{i.e., } Rot(\vec{\mathbf{a}},0)=\mathbf{I}_{Rot}= \text{ the identity rotation}$$

$$Rot(\hat{\mathbf{a}},\alpha)\bullet\vec{\mathbf{b}}=\left(\hat{\mathbf{a}}\bullet\vec{\mathbf{b}}\right)\hat{\mathbf{a}}+Rot(\hat{\mathbf{a}},\alpha)\bullet\left(\vec{\mathbf{b}}-\left(\hat{\mathbf{a}}\bullet\vec{\mathbf{b}}\right)\hat{\mathbf{a}}\right)$$

$$Rot(\hat{\mathbf{a}},\alpha)\bullet Rot(\hat{\mathbf{b}},\beta)=Rot(\hat{\mathbf{b}},\beta)\bullet Rot(Rot(\hat{\mathbf{b}},-\beta)\bullet\hat{\mathbf{a}},\alpha)$$

$$Rot(\hat{\mathbf{a}},\alpha)\bullet\mathbf{R}_\beta=\mathbf{R}_\beta\bullet Rot(\mathbf{R}_\beta^{-1}\bullet\hat{\mathbf{a}},\alpha)$$

$$\mathbf{R}_\alpha\bullet Rot(\hat{\mathbf{b}},\beta)=Rot(\mathbf{R}_\alpha\bullet\hat{\mathbf{b}},\beta)\bullet\mathbf{R}_\alpha$$

Rotations: more facts

If $\vec{\mathbf{v}}=[v_x,v_y,v_z]^T$ then a rotation $\mathbf{R}\bullet\vec{\mathbf{v}}$ may be described in terms of the effects of $\mathbf{R}$ on orthogonal unit vectors, $\vec{\mathbf{e}}_x=[1,0,0]^T$, $\vec{\mathbf{e}}_y=[0,1,0]^T$, $\vec{\mathbf{e}}_z=[0,0,1]^T$

$$\mathbf{R}\bullet\vec{\mathbf{v}}=v_x\vec{\mathbf{r}}_x+v_y\vec{\mathbf{r}}_y+v_z\vec{\mathbf{r}}_z$$

where

$$\vec{\mathbf{r}}_x=\mathbf{R}\bullet\vec{\mathbf{e}}_x$$
$$\vec{\mathbf{r}}_y=\mathbf{R}\bullet\vec{\mathbf{e}}_y$$
$$\vec{\mathbf{r}}_z=\mathbf{R}\bullet\vec{\mathbf{e}}_z$$

Note that rotation doesn't affect inner products

$$\left(\mathbf{R}\bullet\vec{\mathbf{b}}\right)\bullet\left(\mathbf{R}\bullet\vec{\mathbf{c}}\right)=\vec{\mathbf{b}}\bullet\vec{\mathbf{c}}$$

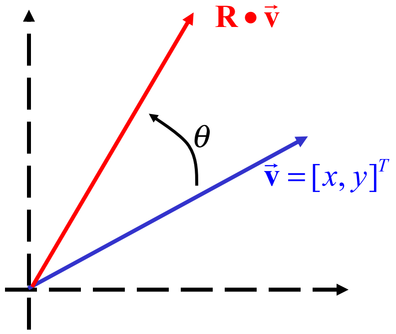
or lengths of vectors

$$\left\|\mathbf{R}\cdot\vec{\mathbf{v}}\right\|=\left\|\vec{\mathbf{v}}\right\|$$

Rotations in the plane

$$\mathbf{R}\bullet\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\cos\theta-y\sin\theta\\x\sin\theta+y\cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}\bullet\begin{bmatrix}x\\y\end{bmatrix}$$

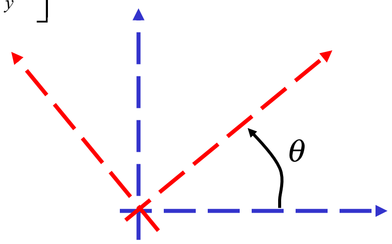


Rotations in the plane

$$\mathbf{R}\bullet\begin{bmatrix}\vec{\mathbf{e}}_x & \vec{\mathbf{e}}_y\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}\bullet\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\mathbf{R}\bullet\vec{\mathbf{e}}_x & \mathbf{R}\bullet\vec{\mathbf{e}}_y\end{bmatrix}$$

$$=\begin{bmatrix}\vec{\mathbf{r}}_x & \vec{\mathbf{r}}_y\end{bmatrix}$$



3D Rotation Matrices

$$\mathbf{R}\bullet\begin{bmatrix}\vec{\mathbf{e}}_x & \vec{\mathbf{e}}_y & \vec{\mathbf{e}}_z\end{bmatrix}=\begin{bmatrix}\mathbf{R}\bullet\vec{\mathbf{e}}_x & \mathbf{R}\bullet\vec{\mathbf{e}}_y & \mathbf{R}\bullet\vec{\mathbf{e}}_z\end{bmatrix}$$

$$=\begin{bmatrix}\vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_z\end{bmatrix}$$

$$\mathbf{R}^T\bullet\mathbf{R}=\begin{bmatrix}\hat{\mathbf{r}}_{\mathbf{x}}{}^T\\ \hat{\mathbf{r}}_y{}^T\\ \hat{\mathbf{r}}_z\end{bmatrix}\bullet\begin{bmatrix}\vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_z\end{bmatrix}$$

$$=\begin{bmatrix}\vec{\mathbf{r}}_{\mathbf{x}}{}^T\bullet\vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_{\mathbf{x}}{}^T\bullet\vec{\mathbf{r}}_y & \vec{\mathbf{r}}_{\mathbf{x}}{}^T\bullet\vec{\mathbf{r}}_z\\ \vec{\mathbf{r}}_y{}^T\bullet\vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_y{}^T\bullet\vec{\mathbf{r}}_y & \vec{\mathbf{r}}_y{}^T\bullet\vec{\mathbf{r}}_z\\ \vec{\mathbf{r}}_z{}^T\bullet\vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_z{}^T\bullet\vec{\mathbf{r}}_y & \vec{\mathbf{r}}_z{}^T\bullet\vec{\mathbf{r}}_z\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Properties of Rotation Matrices

Inverse of a Rotation Matrix equals its transpose:

$\mathbf{R}^{-1} = \mathbf{R}^{T}$

$\mathbf{R}^{T}\,\mathbf{R}=\mathbf{R}\,\mathbf{R}^{T} = \mathbf{I}$

The Determinant of a Rotation matrix is equal to +1:

$\mathbf{det(R)= +1}$

Any Rotation can be described by consecutive rotations about the three primary axes, x, y, and z:

$\mathbf{R} = \mathbf{R}_{z,\theta}\,\mathbf{R}_{y,\phi}\,\mathbf{R}_{x,\psi}$

Canonical 3D Rotation Matrices

Note: **Right-Handed Coordinate System**

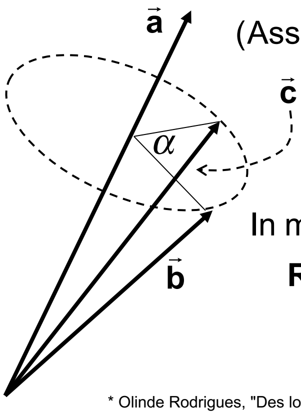
$$\mathbf{R}_{\vec{\mathbf{x}}}(\theta) = Rot(\vec{\mathbf{x}},\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\vec{\mathbf{y}}}(\theta) = Rot(\vec{\mathbf{y}},\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\vec{\mathbf{z}}}(\theta) = Rot(\vec{\mathbf{z}},\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Axis-angle Representations of Rotations

Rodrigues' Formula (1840)*



Rotation of a vector $\vec{\mathbf{b}}$ by angle α about axis $\vec{\mathbf{a}}$
(Assumes that $\vec{\mathbf{a}}$ is a unit vector, $\|\vec{\mathbf{a}}\| = 1$)

$$\begin{aligned} \vec{\mathbf{c}} &= Rot(\vec{\mathbf{a}},\alpha) \bullet \vec{\mathbf{b}} \\ &= \vec{\mathbf{b}}\cos\alpha + \vec{\mathbf{a}} \times \vec{\mathbf{b}}\sin\alpha + \vec{\mathbf{a}}(\vec{\mathbf{a}} \bullet \vec{\mathbf{b}})\left(1-\cos\alpha\right) \end{aligned}$$

In matrix form this is $\vec{\mathbf{c}} = \mathbf{R} \bullet \vec{\mathbf{b}}$ where

$$\mathbf{R} = \left(\cos\alpha\right)\mathbf{I} + \left(\sin\alpha\right) skew(\vec{\mathbf{a}}) + \left(1-\cos\alpha\right)\vec{\mathbf{a}} \bullet \vec{\mathbf{a}}^T$$

* Olinde Rodrigues, "Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendants des causes qui peuvent les produire", *Journal de Mathématiques Pures et Appliquées* 5 (1840), 380–440. (http://sites.mathdoc.fr/JMPA/PDF/JMPA_1840_1_5_A39_0.pdf)

Exponential representation

Consider a rotation about axis $\vec{\mathbf{n}}$ by angle θ. Then

$$e^{skew(\vec{\mathbf{n}})\theta} = \mathbf{I} + \theta skew(\vec{\mathbf{n}}) + \frac{\theta^2}{2!} skew(\vec{\mathbf{n}})^2 + \ldots$$

By doing some manipulation, you can show

$$\begin{aligned} Rot(\vec{\mathbf{n}},\theta) &= e^{skew(\vec{\mathbf{n}})\theta} \\ &= \mathbf{I} + skew(\vec{\mathbf{n}})\sin\theta + skew(\vec{\mathbf{n}})^2(1-\cos\theta) \\ &= \mathbf{I} + skew(\vec{\mathbf{n}})\sin\theta + (\vec{\mathbf{n}} \bullet \vec{\mathbf{n}}^T - \mathbf{I})(1-\cos\theta) \\ &= \mathbf{I}\cos\theta + skew(\vec{\mathbf{n}})\sin\theta + \vec{\mathbf{n}} \bullet \vec{\mathbf{n}}^T(1-\cos\theta) \end{aligned}$$

which is just Rodrigues' formula.

Note that for small θ, this reduces to

$$Rot(\vec{\mathbf{n}},\theta) \approx \mathbf{I} + skew(\theta\vec{\mathbf{n}})$$

Cayley Transform Representation

Consider the rotation Rot($\vec{\mathbf{n}},\theta$) and define

$$\vec{\mathbf{a}} = \left(\tan\frac{\theta}{2}\right)\vec{\mathbf{n}}$$

$$\mathbf{A} = \text{skew}(\vec{\mathbf{a}})$$

Then,

$$\mathbf{R} = (\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A}) \quad = (\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}$$

gives the rotation matrix corresponding to Rot($\vec{\mathbf{n}},\theta$). Similarly, given $\mathbf{R}$,

$$\mathbf{A} = (\mathbf{R}-\mathbf{I})(\mathbf{I}+\mathbf{R})^{-1}$$

gives the elements of skew($\vec{\mathbf{a}}$) and hence for $\vec{\mathbf{a}}$.

Note : The above relations require that $\theta \neq \pm\pi$.

Note on the difference between two rotations

One often wants to consider the "difference" between two rotations

$$\mathbf{R}_{12} = \mathbf{R}_1^{-1} \bullet \mathbf{R}_2$$

In these cases, it is useful to consider an axis-angle representation

$$\mathbf{R}_{12} = Rot(\vec{\mathbf{n}}_{12}, \theta_{12})$$

Given $\mathbf{R}_{12}$, there are several ways to extract $\vec{\mathbf{n}}_{12}$ and $\vec{\theta}_{12}$. For example, you can use the Cayley formula to compute $\vec{\mathbf{a}}_{12} = \tan(\theta_{12}/2)\vec{\mathbf{n}}_{12}$ from

$$sk(\vec{\mathbf{a}}_{12}) = (\mathbf{R}_{12} - \mathbf{I})(\mathbf{I} + \mathbf{R}_{12})^{-1}$$

and then

$$\theta_{12} = 2\arctan\left(\left\|\vec{\mathbf{a}}_{12}\right\|\right) \text{ and } \vec{\mathbf{n}}_{12} = \vec{\mathbf{a}}_{12} / \left\|\vec{\mathbf{a}}_{12}\right\|$$

Also, if the rotations are very close to each other so that $\vec{\theta}_{12}$ is small, then

$$\mathbf{R}_{12} \approx \mathbf{I} + sk(\vec{\alpha}_{12})$$

So

$$\theta_{12} \approx \left\|\vec{\alpha}_{12}\right\| \text{ and } \vec{\mathbf{n}}_{12} \approx \vec{\alpha}_{12} / \theta_{12}$$

This last relationship is very useful in reporting things like registration error since the elements $\vec{\alpha}_{12}$ are small rotations about the x,y,z axes

Homogeneous Coordinates

- Widely used in graphics, geometric calculations
- Represent 3D vector as 4D quantity
- For our current purposes, we will keep the “scale” *s = 1*

$$\vec{\mathbf{v}} \equiv \begin{bmatrix} xs \\ ys \\ zs \\ s \end{bmatrix} \cong \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

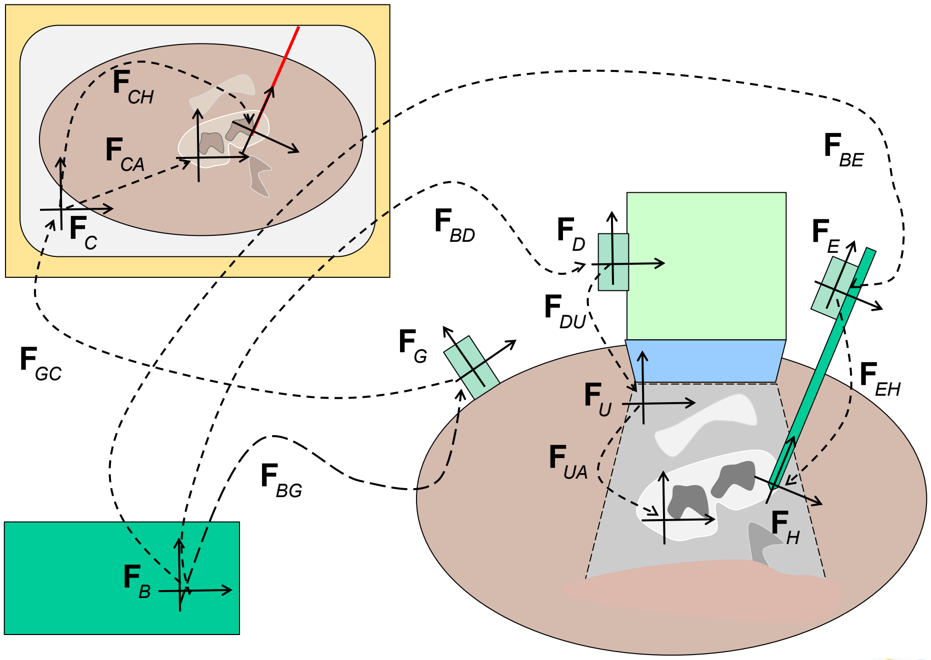
Representing Frame Transformations as Matrices

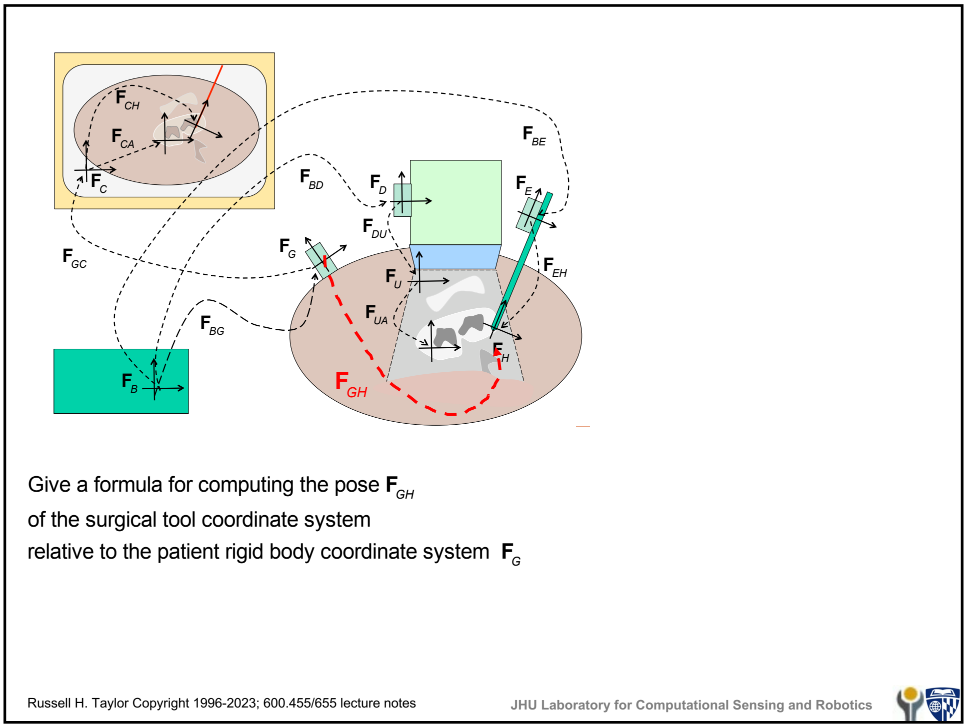
$$\mathbf{v}+\mathbf{p}\rightarrow\begin{bmatrix}1&0&0&\mathbf{p}_x\\0&1&0&\mathbf{p}_y\\0&0&1&\mathbf{p}_z\\0&0&0&1\end{bmatrix}\begin{bmatrix}\mathbf{v}_x\\\mathbf{v}_y\\\mathbf{v}_z\\1\end{bmatrix}=[\mathbf{I},\vec{\mathbf{p}}]\bullet\mathbf{v}$$

$$\mathbf{R}\bullet\mathbf{v}\rightarrow\begin{bmatrix}\mathbf{R}&\mathbf{0}\\\mathbf{0}&1\end{bmatrix}\begin{bmatrix}\mathbf{v}\\1\end{bmatrix}$$

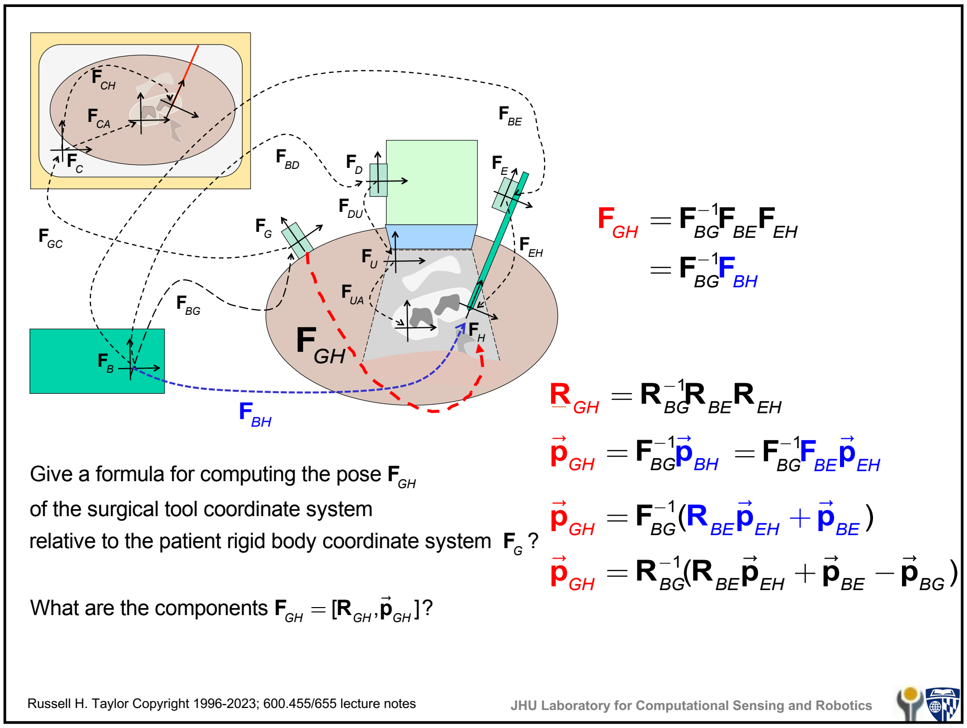
$$\mathbf{P}\bullet\mathbf{R}\rightarrow\begin{bmatrix}\mathbf{I}&\mathbf{p}\\\mathbf{0}&\mathbf{1}\end{bmatrix}\bullet\begin{bmatrix}\mathbf{R}&0\\\mathbf{0}&\mathbf{1}\end{bmatrix}=\begin{bmatrix}\mathbf{R}&\mathbf{p}\\\mathbf{0}&\mathbf{1}\end{bmatrix}=[\mathbf{R},\mathbf{p}]=\mathbf{F}$$

$$\mathbf{F}\bullet\mathbf{v}\rightarrow\begin{bmatrix}\mathbf{R}&\mathbf{p}\\\mathbf{0}&1\end{bmatrix}\begin{bmatrix}\mathbf{v}\\1\end{bmatrix}=\begin{bmatrix}(\mathbf{R}\bullet\mathbf{v})+\mathbf{p}\\1\end{bmatrix}$$





Give a formula for computing the pose $\mathbf{F}_{GH}$
of the surgical tool coordinate system
relative to the patient rigid body coordinate system $\mathbf{F}_G$

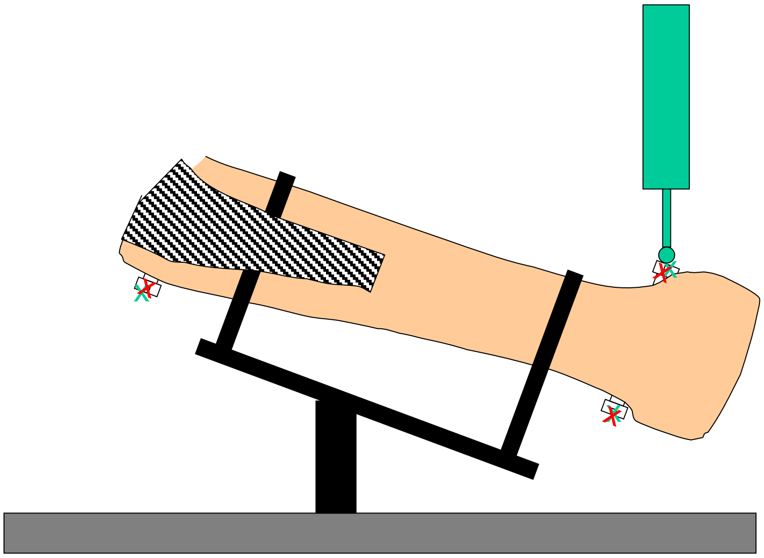


$$\mathbf{F}_{GH} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\mathbf{F}_{EH}$$
$$= \mathbf{F}_{BG}^{-1}\mathbf{F}_{BH}$$

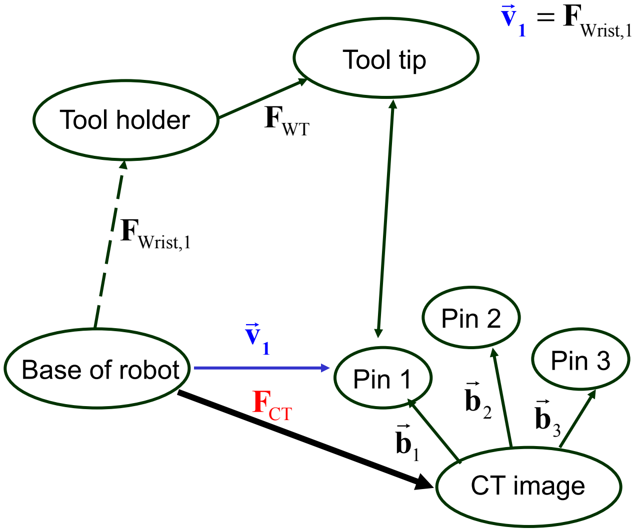
$$\mathbf{R}_{GH} = \mathbf{R}_{BG}^{-1}\mathbf{R}_{BE}\mathbf{R}_{EH}$$
$$\vec{\mathbf{p}}_{GH} = \mathbf{F}_{BG}^{-1}\vec{\mathbf{p}}_{BH} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\vec{\mathbf{p}}_{EH}$$
$$\vec{\mathbf{p}}_{GH} = \mathbf{F}_{BG}^{-1}(\mathbf{R}_{BE}\vec{\mathbf{p}}_{EH} + \vec{\mathbf{p}}_{BE})$$
$$\vec{\mathbf{p}}_{GH} = \mathbf{R}_{BG}^{-1}(\mathbf{R}_{BE}\vec{\mathbf{p}}_{EH} + \vec{\mathbf{p}}_{BE} - \vec{\mathbf{p}}_{BG})$$

Give a formula for computing the pose $\mathbf{F}_{GH}$
of the surgical tool coordinate system
relative to the patient rigid body coordinate system $\mathbf{F}_G$?

What are the components $\mathbf{F}_{GH} = [\mathbf{R}_{GH}, \vec{\mathbf{p}}_{GH}]$?

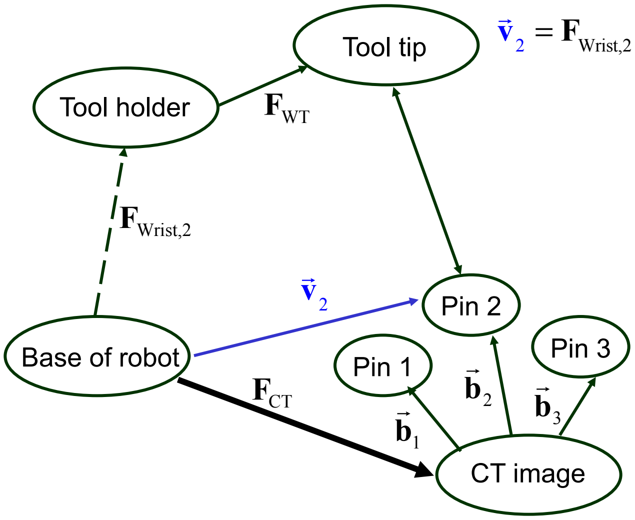


$$\vec{\mathbf{v}}_1 = \mathbf{F}_{\mathrm{Wrist},1} \bullet \vec{\mathbf{p}}_{\mathrm{WT}} = \mathbf{F}_{\mathrm{CT}} \bullet \vec{\mathbf{b}}_1$$



$$\vec{\mathbf{v}}_1 = \mathbf{F}_{\mathrm{Wrist},1} \bullet \vec{\mathbf{p}}_{\mathrm{WT}} = \mathbf{F}_{\mathrm{CT}} \bullet \vec{\mathbf{b}}_1$$

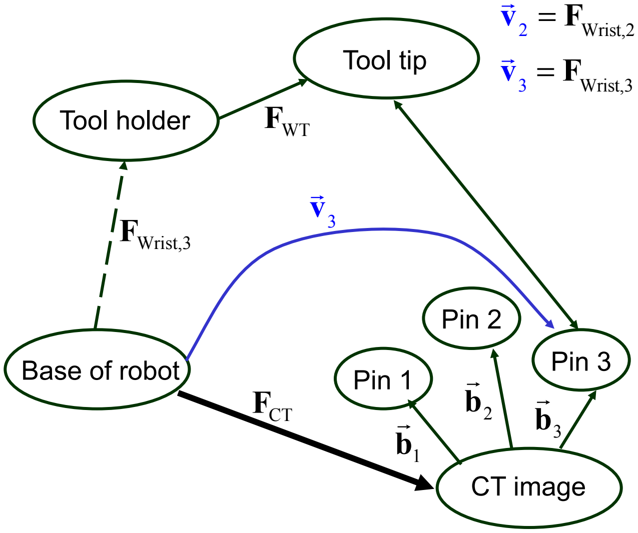
$$\vec{\mathbf{v}}_2 = \mathbf{F}_{\mathrm{Wrist},2} \bullet \vec{\mathbf{p}}_{\mathrm{WT}} = \mathbf{F}_{\mathrm{CT}} \bullet \vec{\mathbf{b}}_2$$



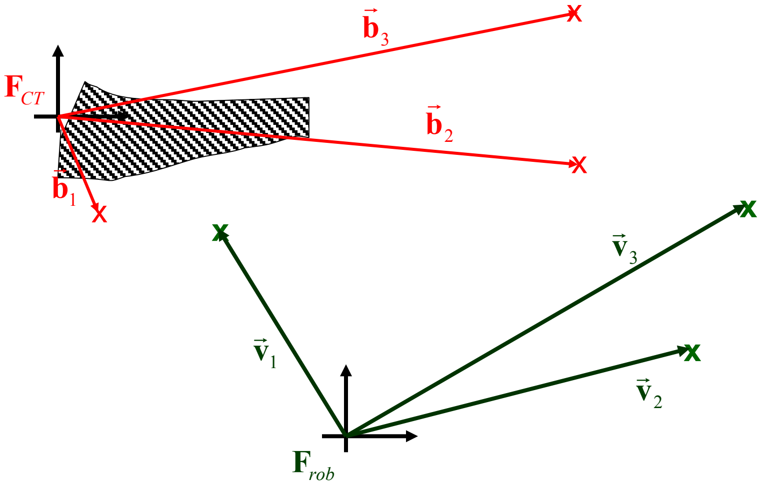
$$\vec{\mathbf{v}}_1 = \mathbf{F}_{\mathrm{Wrist},1} \bullet \vec{\mathbf{p}}_{\mathrm{WT}} = \mathbf{F}_{\mathrm{CT}} \bullet \vec{\mathbf{b}}_1$$

$$\vec{\mathbf{v}}_2 = \mathbf{F}_{\mathrm{Wrist},2} \bullet \vec{\mathbf{p}}_{\mathrm{WT}} = \mathbf{F}_{\mathrm{CT}} \bullet \vec{\mathbf{b}}_2$$

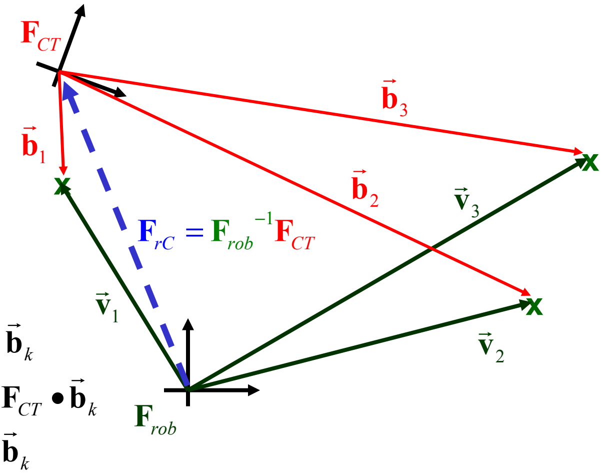
$$\vec{\mathbf{v}}_3 = \mathbf{F}_{\mathrm{Wrist},3} \bullet \vec{\mathbf{p}}_{\mathrm{WT}} = \mathbf{F}_{\mathrm{CT}} \bullet \vec{\mathbf{b}}_3$$



Frame transformation from 3 point pairs



Frame transformation from 3 point pairs



$$\mathbf{F}_{rC} = \mathbf{F}_{rob}^{-1}\mathbf{F}_{CT}$$

$$\mathbf{F}_{rob} \bullet \vec{\mathbf{v}}_k = \mathbf{F}_{CT} \bullet \vec{\mathbf{b}}_k$$

$$\vec{\mathbf{v}}_k = \mathbf{F}_{rob}^{-1}\mathbf{F}_{CT} \bullet \vec{\mathbf{b}}_k$$

$$\vec{\mathbf{v}}_k = \mathbf{F}_{rC} \bullet \vec{\mathbf{b}}_k$$

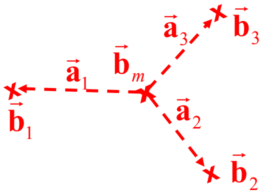
Frame transformation from 3 point pairs

$$\vec{\mathbf{v}}_k = \mathbf{F}_{rC}\vec{\mathbf{b}}_k = \mathbf{R}_{rC}\vec{\mathbf{b}}_k + \vec{\mathbf{p}}_{rC}$$

Define

$$\vec{\mathbf{v}}_m = \frac{1}{3}\sum_1^3 \vec{\mathbf{v}}_k \qquad \vec{\mathbf{b}}_m = \frac{1}{3}\sum_1^3 \vec{\mathbf{b}}_k$$

$$\vec{\mathbf{u}}_k = \vec{\mathbf{v}}_k - \vec{\mathbf{v}}_m \qquad \vec{\mathbf{a}}_k = \vec{\mathbf{b}}_k - \vec{\mathbf{b}}_m$$



$$\mathbf{F}_{rC}\vec{\mathbf{a}}_k = \mathbf{R}_{rC}\vec{\mathbf{a}}_k + \vec{\mathbf{p}}_{rC}$$

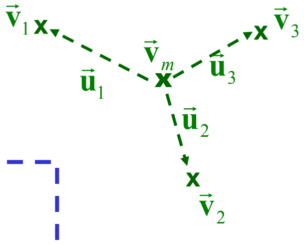
$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k + \vec{\mathbf{p}}_{rC} = \mathbf{R}_{rC}\left(\vec{\mathbf{b}}_k - \vec{\mathbf{b}}_m\right) + \vec{\mathbf{p}}_{rC}$$

$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k = \mathbf{R}_{rC}\vec{\mathbf{b}}_k + \vec{\mathbf{p}}_{rC} - \mathbf{R}_{rC}\vec{\mathbf{b}}_m - \vec{\mathbf{p}}_{rC}$$

$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k = \vec{\mathbf{v}}_k - \vec{\mathbf{v}}_m = \vec{\mathbf{u}}_k$$

$$\vec{\mathbf{p}}_{rC} = \vec{\mathbf{v}}_m - \mathbf{R}_{rC}\vec{\mathbf{b}}_m$$

Solve These!!



Rotation from multiple vector pairs

Given a system $\mathbf{R}\vec{\mathbf{a}}_k = \vec{\mathbf{u}}_k$ for $k = 1, \cdots, n$ the problem is to estimate $\mathbf{R}$. This will require at least three such point pairs. Later in the course we will cover some good ways to solve this system. Here is a not-so-good way that will produce roughly correct answers:

Step 1: Form matrices $\mathbf{U} = [\vec{\mathbf{u}}_1 \ \cdots \ \vec{\mathbf{u}}_n]$ and $\mathbf{A} = [\vec{\mathbf{a}}_1 \ \cdots \ \vec{\mathbf{a}}_n]$

Step 2: Solve the system $\mathbf{RA} = \mathbf{U}$ for $\mathbf{R}$. E.g., by $\mathbf{R} = \mathbf{UA}^{-1}$

Step 3: Renormalize $\mathbf{R}$ to guarantee $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

Renormalizing Rotation Matrix

Given "rotation" matrix $\mathbf{R} = \left[\vec{\mathbf{r}}_x \mid \vec{\mathbf{r}}_y \mid \vec{\mathbf{r}}_z \right]$, modify it so $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

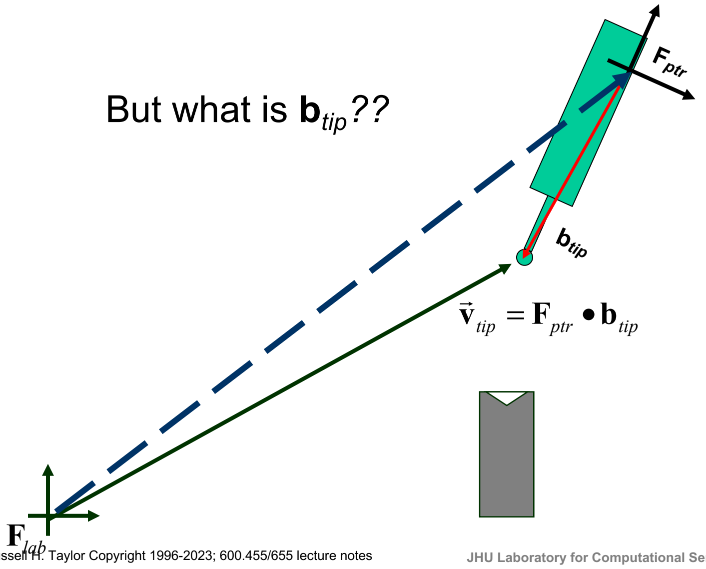
Step 1: $\vec{\mathbf{a}} = \vec{\mathbf{r}}_y \times \vec{\mathbf{r}}_z$

Step 2: $\vec{\mathbf{b}} = \vec{\mathbf{r}}_z \times \vec{\mathbf{a}}$

Step 3: $\mathbf{R}_{normalized} = \left[\frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|} \;\middle|\; \frac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|} \;\middle|\; \frac{\vec{\mathbf{r}}_z}{\|\vec{\mathbf{r}}_z\|} \right]$

Calibrating a pointer

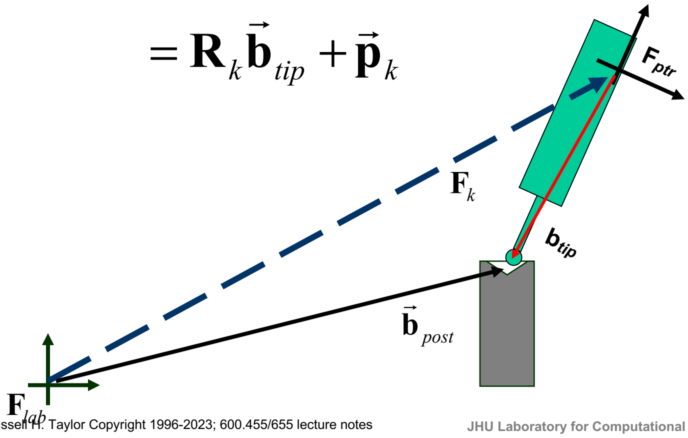
But what is $\mathbf{b}_{tip}$??



$$\vec{\mathbf{v}}_{tip} = \mathbf{F}_{ptr} \bullet \mathbf{b}_{tip}$$

Calibrating a pointer

$$\begin{aligned}\vec{\mathbf{b}}_{post} &= \mathbf{F}_k \vec{\mathbf{b}}_{tip} \\ &= \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k\end{aligned}$$



Calibrating a pointer

For each measurement k, we have

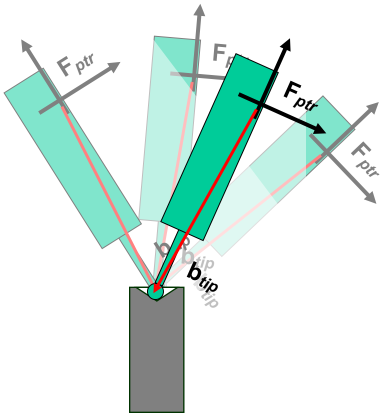
$$\vec{\mathbf{b}}_{post} = \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k$$

I. e.,

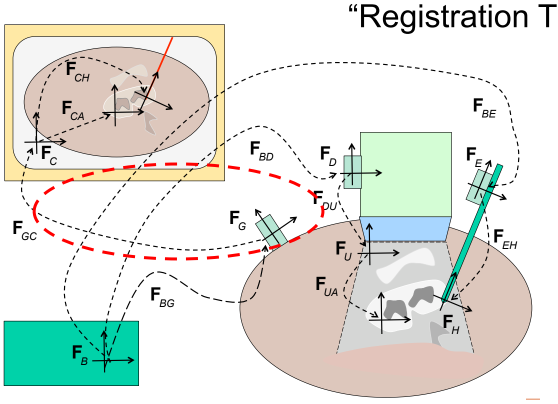
$$\mathbf{R}_k \vec{\mathbf{b}}_{tip} - \vec{\mathbf{b}}_{post} = -\vec{\mathbf{p}}_k$$

Set up a least squares problem

$$\left[\begin{array}{c:c} \vdots & \vdots \\ \hdashline \mathbf{R}_k & -\mathbf{I} \\ \hdashline \vdots & \vdots \end{array}\right] \left[\begin{array}{c} \vec{\mathbf{b}}_{tip} \\ \hdashline \vec{\mathbf{b}}_{post} \end{array}\right] \cong \left[\begin{array}{c} \vdots \\ -\vec{\mathbf{p}}_k \\ \vdots \end{array}\right]$$

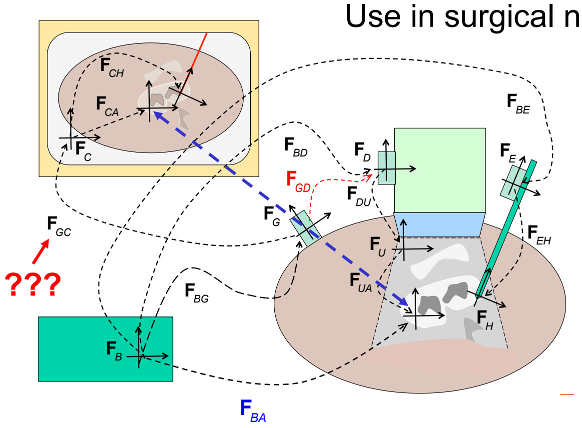


“Registration Transformations”



Given a coordinate system $\mathbf{F}_C$ and another coordinate system $\mathbf{F}_G$ (e.g., a CT scan and a tracked "rigid body" attached to the patient, and points $\vec{\mathbf{c}}_i$ in the coordinate system $\mathbf{F}_C$ and points $\vec{\mathbf{g}}_i$ in the coordinate system $\mathbf{F}_G$, then the "registration transformation" $\mathbf{F}_{GC}$ between $\mathbf{F}_G$ and $\mathbf{F}_C$ one in which for $\mathbf{F}_{GC}\vec{\mathbf{c}}_i = \vec{\mathbf{g}}_i$ if and only if $\vec{\mathbf{c}}_i$ and $\vec{\mathbf{g}}_i$ refer to the same or corresponding points.

Use in surgical navigation



$$\mathbf{F}_{BA} = \mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}$$
$$\mathbf{F}_{BA} = \mathbf{F}_{BG}\mathbf{F}_{GC}\mathbf{F}_{CA}$$

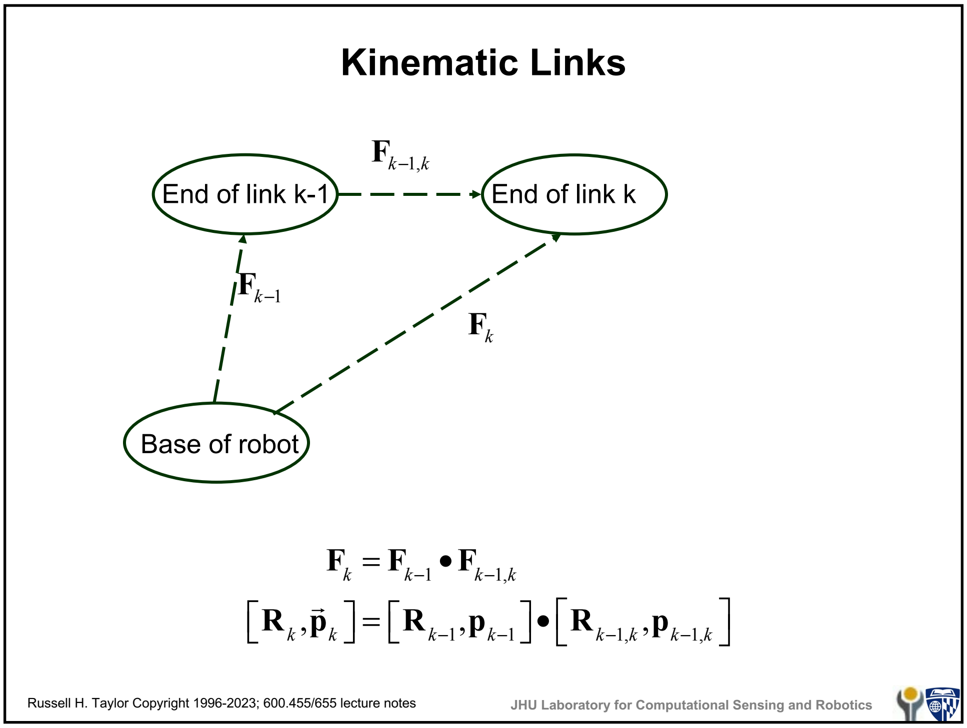
$$\mathbf{F}_{BG}\mathbf{F}_{GC}\mathbf{F}_{CA} = \mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}$$
$$\mathbf{F}_{CA} = \left(\mathbf{F}_{BG}\mathbf{F}_{GC}\right)^{-1}\mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}$$

$$\mathbf{F}_{GC} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}\mathbf{F}_{CA}^{-1}$$
$$\mathbf{F}_{GC} = \mathbf{F}_{GD}\mathbf{F}_{DU}\mathbf{F}_{UA}\mathbf{F}_{CA}^{-1}$$

where $\mathbf{F}_{GD} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BD}$

If an anatomic structure is identified at pose $\mathbf{F}_{UA}$ in ultrasound image coordinates give the formula for computing the corresponding pose $\mathbf{F}_{CA}$ in CT coordinates

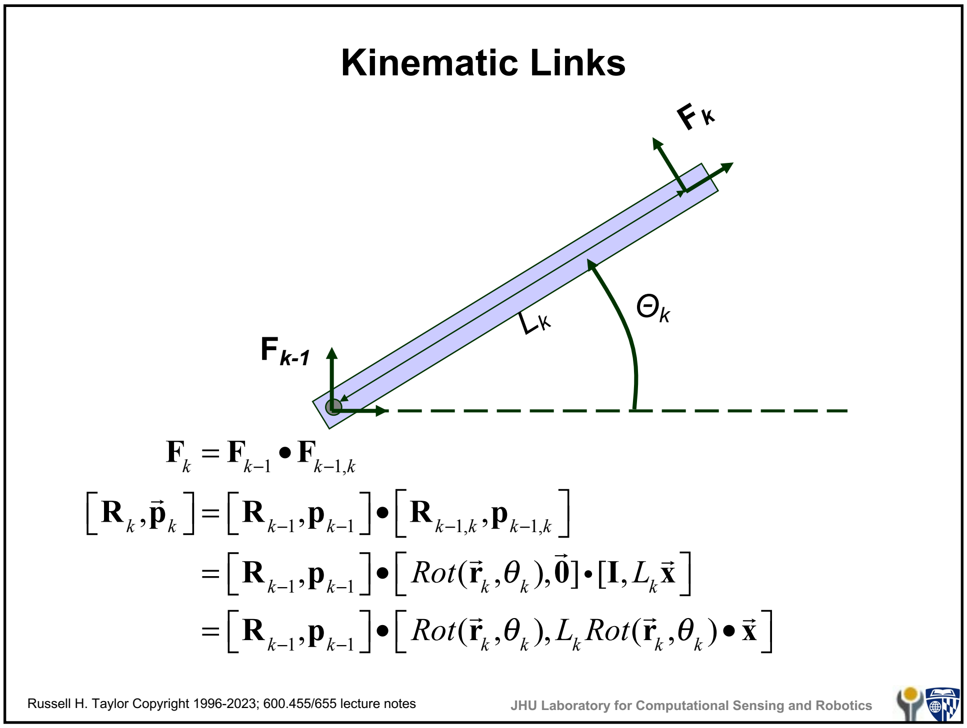
NOTE: In cases where there is a reference body (here $\mathbf{F}_G$) affixed to the patient, it is sometimes convenient to assume that the navigation software routinely transforms all tracked markers to the coordinate system of $\mathbf{F}_G$ (i.e., $\mathbf{F}_{Gx} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{Bx}$).



Kinematic Links

$$\mathbf{F}_k = \mathbf{F}_{k-1} \bullet \mathbf{F}_{k-1,k}$$

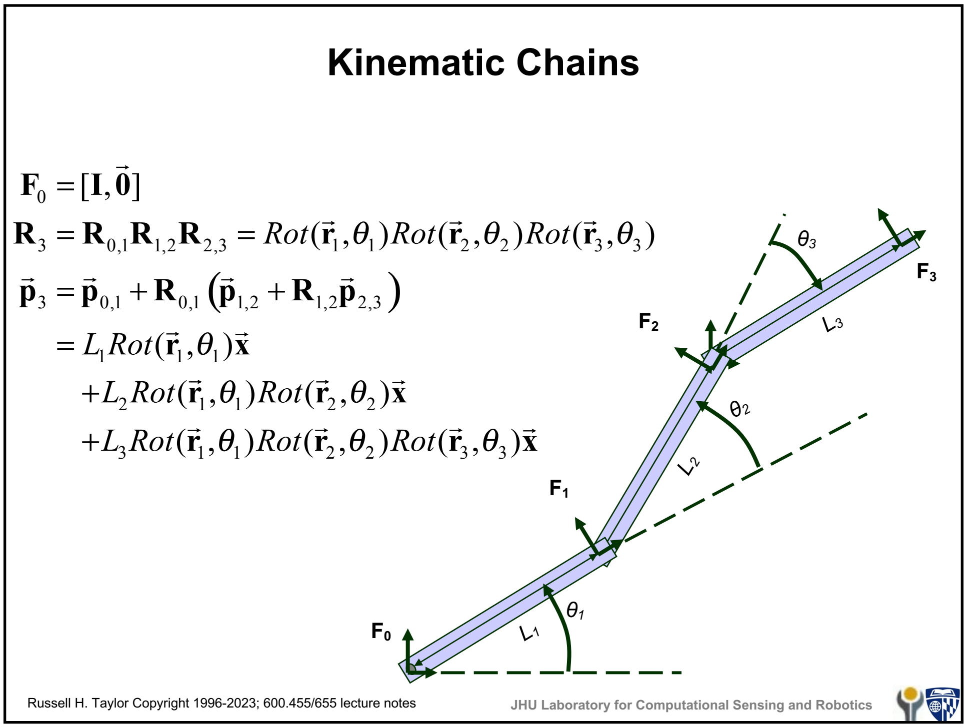
$$\left[\mathbf{R}_k, \vec{\mathbf{p}}_k\right] = \left[\mathbf{R}_{k-1}, \mathbf{p}_{k-1}\right] \bullet \left[\mathbf{R}_{k-1,k}, \mathbf{p}_{k-1,k}\right]$$



Kinematic Links

$$\mathbf{F}_k = \mathbf{F}_{k-1} \bullet \mathbf{F}_{k-1,k}$$

$$\begin{aligned}
\left[\mathbf{R}_k, \vec{\mathbf{p}}_k\right] &= \left[\mathbf{R}_{k-1}, \mathbf{p}_{k-1}\right] \bullet \left[\mathbf{R}_{k-1,k}, \mathbf{p}_{k-1,k}\right] \\
&= \left[\mathbf{R}_{k-1}, \mathbf{p}_{k-1}\right] \bullet \left[Rot(\vec{\mathbf{r}}_k, \theta_k), \vec{\mathbf{0}}\right] \bullet [\mathbf{I}, L_k \vec{\mathbf{x}}] \\
&= \left[\mathbf{R}_{k-1}, \mathbf{p}_{k-1}\right] \bullet \left[Rot(\vec{\mathbf{r}}_k, \theta_k), L_k Rot(\vec{\mathbf{r}}_k, \theta_k) \bullet \vec{\mathbf{x}}\right]
\end{aligned}$$

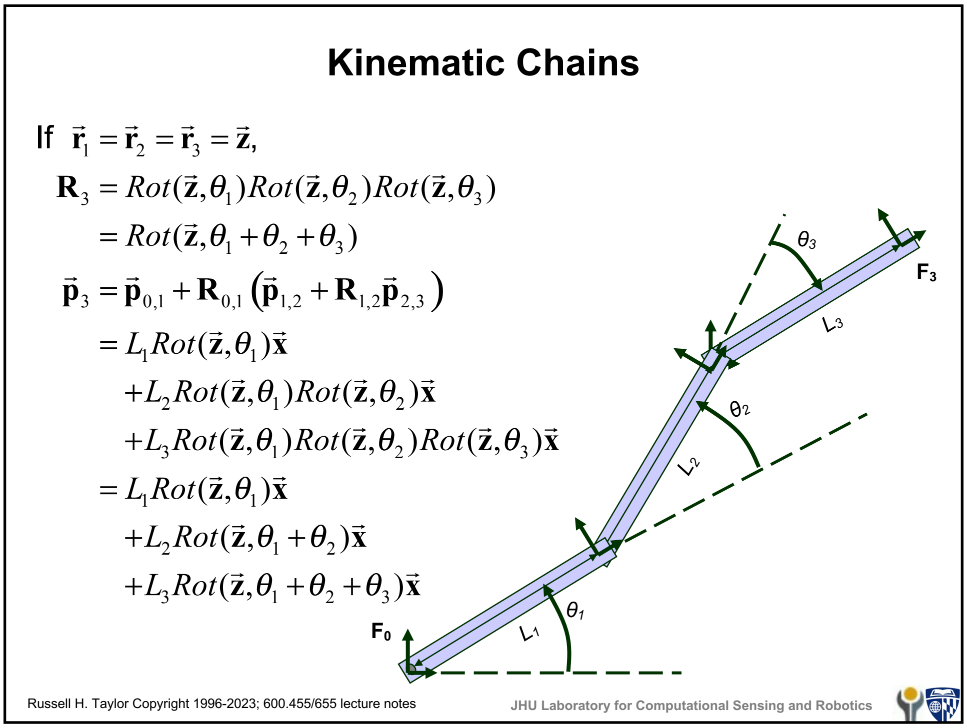


Kinematic Chains

$$\mathbf{F}_0 = [\mathbf{I}, \vec{\mathbf{0}}]$$

$$\mathbf{R}_3 = \mathbf{R}_{0,1}\mathbf{R}_{1,2}\mathbf{R}_{2,3} = Rot(\vec{\mathbf{r}}_1,\theta_1)Rot(\vec{\mathbf{r}}_2,\theta_2)Rot(\vec{\mathbf{r}}_3,\theta_3)$$

$$\begin{aligned}\vec{\mathbf{p}}_3 &= \vec{\mathbf{p}}_{0,1} + \mathbf{R}_{0,1}\left(\vec{\mathbf{p}}_{1,2} + \mathbf{R}_{1,2}\vec{\mathbf{p}}_{2,3}\right) \\ &= L_1 Rot(\vec{\mathbf{r}}_1,\theta_1)\vec{\mathbf{x}} \\ &\quad + L_2 Rot(\vec{\mathbf{r}}_1,\theta_1)Rot(\vec{\mathbf{r}}_2,\theta_2)\vec{\mathbf{x}} \\ &\quad + L_3 Rot(\vec{\mathbf{r}}_1,\theta_1)Rot(\vec{\mathbf{r}}_2,\theta_2)Rot(\vec{\mathbf{r}}_3,\theta_3)\vec{\mathbf{x}}\end{aligned}$$



Kinematic Chains

If $\vec{\mathbf{r}}_1 = \vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_3 = \vec{\mathbf{z}},$

$$\begin{aligned}\mathbf{R}_3 &= Rot(\vec{\mathbf{z}},\theta_1)Rot(\vec{\mathbf{z}},\theta_2)Rot(\vec{\mathbf{z}},\theta_3) \\ &= Rot(\vec{\mathbf{z}},\theta_1+\theta_2+\theta_3)\end{aligned}$$

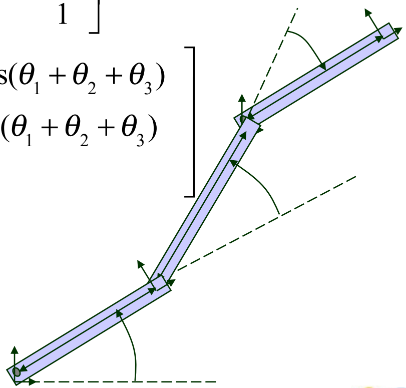
$$\begin{aligned}\vec{\mathbf{p}}_3 &= \vec{\mathbf{p}}_{0,1} + \mathbf{R}_{0,1}\left(\vec{\mathbf{p}}_{1,2} + \mathbf{R}_{1,2}\vec{\mathbf{p}}_{2,3}\right) \\ &= L_1 Rot(\vec{\mathbf{z}},\theta_1)\vec{\mathbf{x}} \\ &\quad + L_2 Rot(\vec{\mathbf{z}},\theta_1)Rot(\vec{\mathbf{z}},\theta_2)\vec{\mathbf{x}} \\ &\quad + L_3 Rot(\vec{\mathbf{z}},\theta_1)Rot(\vec{\mathbf{z}},\theta_2)Rot(\vec{\mathbf{z}},\theta_3)\vec{\mathbf{x}} \\ &= L_1 Rot(\vec{\mathbf{z}},\theta_1)\vec{\mathbf{x}} \\ &\quad + L_2 Rot(\vec{\mathbf{z}},\theta_1+\theta_2)\vec{\mathbf{x}} \\ &\quad + L_3 Rot(\vec{\mathbf{z}},\theta_1+\theta_2+\theta_3)\vec{\mathbf{x}}\end{aligned}$$

Kinematic Chains

If $\vec{\mathbf{r}}_1 = \vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_3 = \vec{\mathbf{z}},$

$$\mathbf{R}_3 = \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{\mathbf{p}}_3 = \begin{bmatrix} L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) \\ L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) + L_3\sin(\theta_1+\theta_2+\theta_3) \\ 0 \end{bmatrix}$$



“Small” Transformations

- A great deal of CIS is concerned with computing and using geometric information based on imprecise knowledge
- Similarly, one is often concerned with the effects of relatively small rotations and displacements
- Essentially, we will be using fairly straightforward linearizations to model these situations, but a specialized notation is often useful

“Small” Frame Transformations

Represent a "small" pose shift consisting of a small rotation $\Delta\mathbf{R}$ followed by a small displacement $\Delta\vec{\mathbf{p}}$ as

$$\Delta\mathbf{F} = [\Delta\mathbf{R}, \Delta\vec{\mathbf{p}}]$$

Then

$$\Delta\mathbf{F} \bullet \vec{\mathbf{v}} = \Delta\mathbf{R} \bullet \vec{\mathbf{v}} + \Delta\vec{\mathbf{p}}$$

Small Rotations

$$\begin{aligned}
\Delta\mathbf{R} &= \text{ a small rotation} \\
\mathbf{R}_{\vec{\mathbf{a}}}(\Delta\alpha) &= \text{ a rotation by a small angle } \Delta\alpha \text{ about axis } \vec{\mathbf{a}} \\
Rot(\vec{\mathbf{a}}, \|\vec{\mathbf{a}}\|) \bullet \vec{\mathbf{b}} &\approx \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{b}} \quad \text{ for } \|\vec{\mathbf{a}}\| \text{ sufficiently small} \\
\Delta\mathbf{R}(\vec{\mathbf{a}}) &= \text{ a rotation that is small enough so that any error} \\
&\quad\ \text{ introduced by this approximation is negligible}
\end{aligned}$$

$$\Delta\mathbf{R}(\lambda\,\vec{\mathbf{a}}) \bullet \Delta\mathbf{R}(\mu\,\vec{\mathbf{b}}) \cong \Delta\mathbf{R}(\lambda\,\vec{\mathbf{a}} + \mu\,\vec{\mathbf{b}}) \quad \text{(Linearity for small rotations)}$$

Exercise: Work out the linearity proposition by substitution

Approximations to “Small” Frames

$$\Delta\mathbf{F}(\vec{\mathbf{a}},\Delta\vec{\mathbf{p}}) \triangleq [\Delta\mathbf{R}(\vec{\mathbf{a}}),\Delta\vec{\mathbf{p}}]$$

$$\begin{aligned}\Delta\mathbf{F}(\vec{\mathbf{a}},\Delta\vec{\mathbf{p}})\bullet\vec{\mathbf{v}} &= \Delta\mathbf{R}(\vec{\mathbf{a}})\bullet\vec{\mathbf{v}}+\Delta\vec{\mathbf{p}}\\ &\approx \vec{\mathbf{v}}+\vec{\mathbf{a}}\times\vec{\mathbf{v}}+\Delta\vec{\mathbf{p}}\end{aligned}$$

$$\begin{aligned}\vec{\mathbf{a}}\times\vec{\mathbf{v}} &= skew(\vec{\mathbf{a}})\bullet\vec{\mathbf{v}}\\ &= \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}\bullet\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}\end{aligned}$$

$$skew(\vec{\mathbf{a}})\bullet\vec{\mathbf{a}} = \vec{\mathbf{a}}\times\vec{\mathbf{a}} = \vec{0}$$

$$\Delta\mathbf{R}(\vec{\mathbf{a}}) \approx \mathbf{I}+skew(\vec{\mathbf{a}})$$

$$\Delta\mathbf{R}(\vec{\mathbf{a}})^{-1} \approx \mathbf{I}-skew(\vec{\mathbf{a}}) = \mathbf{I}+skew(-\vec{\mathbf{a}})$$

Approximations to “Small” Frames

Notational NOTE:

We often use $\vec{\alpha}$ to represent a vector of small angles
and $\vec{\varepsilon}$ to represent a vector of small displacements

In using these approximations, we typically ignore second order terms. I.e.,

$\vec{\alpha}_A\vec{\alpha}_B \approx \vec{0}$, $\vec{\alpha}_A\vec{\varepsilon}_B \approx \vec{0}$, $\vec{\varepsilon}_A\vec{\varepsilon}_B \approx \vec{0}$, etc.

Errors & sensitivity

Often, we do not have an accurate value for a transformation, so we need to model the error. We model this as a composition of a "nominal" frame and a small displacement

$$\mathbf{F}_{\text{actual}} = \mathbf{F}_{\text{nominal}} \bullet \Delta\mathbf{F}$$

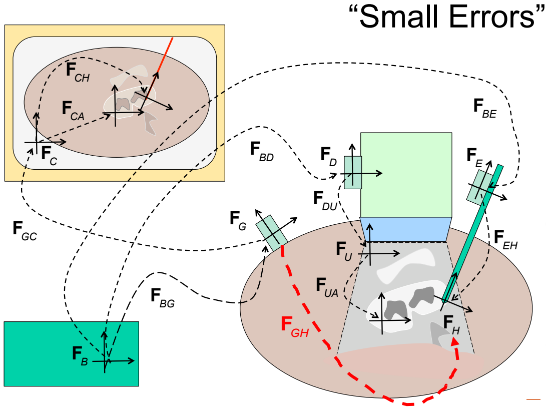
Often, we will use the notation $\mathbf{F}^*$ for $\mathbf{F}_{\text{actual}}$ and will just use $\mathbf{F}$ for $\mathbf{F}_{\text{nominal}}$. Thus we may write something like

$$\mathbf{F}^* = \mathbf{F} \bullet \Delta\mathbf{F}$$

or (less often) $\mathbf{F}^* = \Delta\mathbf{F} \bullet \mathbf{F}$. We also use $\vec{\mathbf{v}}^* = \vec{\mathbf{v}} + \Delta\vec{\mathbf{v}}, \mathit{etc}$.
Thus, if we use the former form (error on the right), and have nominal relationship $\vec{\mathbf{v}} = \mathbf{F} \bullet \vec{\mathbf{b}}$, we get

$$\begin{aligned}
\vec{\mathbf{v}}^* &= \mathbf{F}^* \bullet \vec{\mathbf{b}}^* \\
&= \mathbf{F} \bullet \Delta\mathbf{F} \bullet (\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}}) = \mathbf{F} \bullet (\Delta\mathbf{R} \bullet \vec{\mathbf{b}} + \Delta\mathbf{R} \bullet \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}) \\
&\approx \mathbf{R} \bullet ((\mathbf{I} + sk(\vec{\alpha})) \bullet \left(\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}}\right) + \Delta\vec{\mathbf{p}}) + \vec{\mathbf{p}} = \mathbf{R} \bullet (\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}) + \vec{\mathbf{p}} \\
&\approx \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}) + \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}} = \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}) + \vec{\mathbf{v}} \\
\Delta\vec{\mathbf{v}} &\approx \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}})
\end{aligned}$$

"Small Errors"



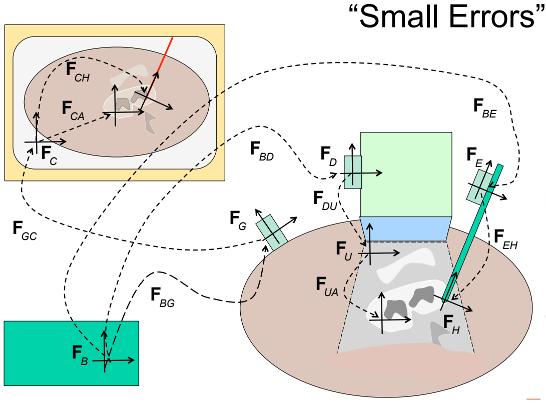
Suppose that there is a small systematic error in the tracking system so that

$$\mathbf{F}^*_{Bx} = \Delta\mathbf{F}_B\mathbf{F}_{Bx}$$

for $\mathbf{F}_{BG}, \mathbf{F}_{BD}, \mathbf{F}_{BE}$. How does this affect the calculation of $\mathbf{F}_{GH}$?

$$\begin{aligned}
\mathbf{F}_{GH}{}^* &= (\mathbf{F}_{BG}{}^*)^{-1}\mathbf{F}_{BE}{}^*\,\mathbf{F}_{EH} \\
\mathbf{F}_{GH}\Delta\mathbf{F}_{GH} &= \left(\Delta\mathbf{F}_B\mathbf{F}_{BG}\right)^{-1}\Delta\mathbf{F}_B\mathbf{F}_{BE}\mathbf{F}_{EH} \\
\Delta\mathbf{F}_{GH} &= \mathbf{F}_{GH}{}^{-1}\left(\Delta\mathbf{F}_B\mathbf{F}_{BG}\right)^{-1}\Delta\mathbf{F}_B\mathbf{F}_{BE}\mathbf{F}_{EH} \\
&= \mathbf{F}_{GH}{}^{-1}\mathbf{F}_{BG}{}^{\cancel{-1}\cancel{\Delta\mathbf{F}_B{}^{-1}\Delta\mathbf{F}_B}}\mathbf{F}_{BE}\mathbf{F}_{EH} \\
&= \mathbf{F}_{GH}{}^{-1}\mathbf{F}_{BG}{}^{-1}\mathbf{F}_{BE}\mathbf{F}_{EH} = \mathbf{F}_{GH}{}^{-1}\mathbf{F}_{GH} = \mathbf{I}
\end{aligned}$$

“Small Errors”



Suppose that there are additional errors in the tracking of each tracker body so that

$$\mathbf{F}^*_{Bx} = \Delta\mathbf{F}_B\mathbf{F}_{Bx}\Delta\mathbf{F}_{Bx}$$

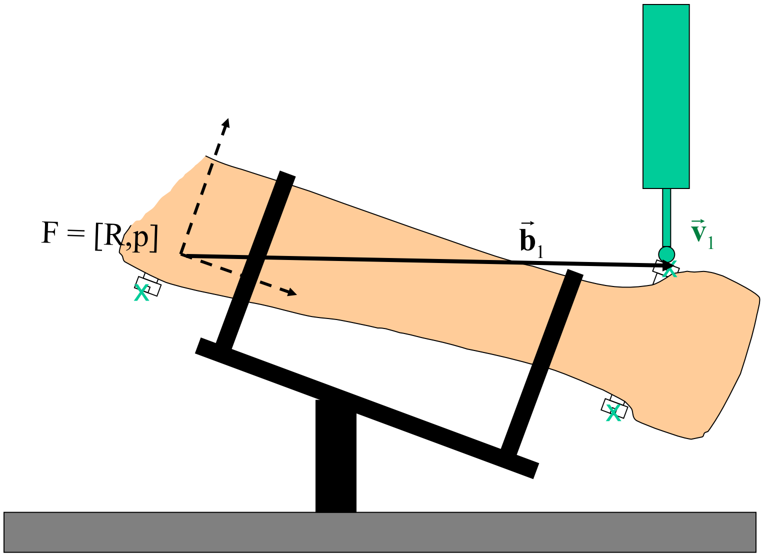
for $\mathbf{F}_{BG}$, $\mathbf{F}_{BD}$, $\mathbf{F}_{BE}$. How does this affect the calculation of $\mathbf{F}_{GH}$?

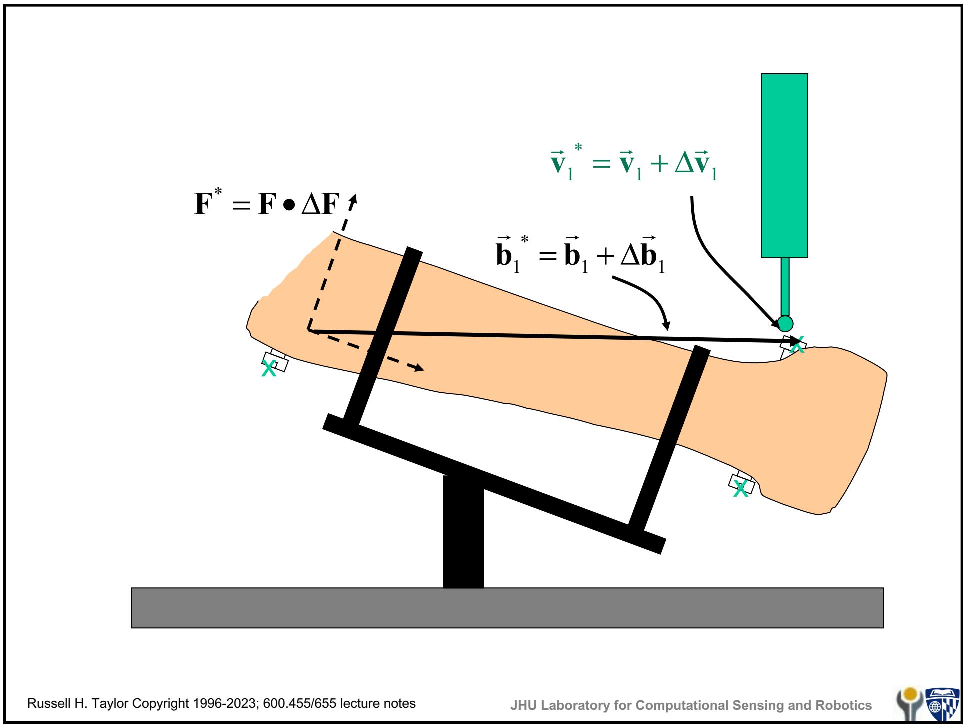
$$\mathbf{F}_{GH}^{\;*} = \mathbf{F}_{GH}\Delta\mathbf{F}_{GH} = (\mathbf{F}_{BG}^{\;*}\quad)^{-1}\mathbf{F}_{BE}^{\;*}\quad\mathbf{F}_{EH}$$

$$\Delta\mathbf{F}_{GH} = \mathbf{F}_{GH}^{\;-1}\left(\Delta\mathbf{F}_B\mathbf{F}_{BG}\Delta\mathbf{F}_{BG}\right)^{-1}\left(\Delta\mathbf{F}_B\mathbf{F}_{BE}\Delta\mathbf{F}_{BE}\right)\mathbf{F}_{EH}$$

$$\Delta\mathbf{F}_{GH} = \left(\mathbf{F}_{BG}^{\;-1}\mathbf{F}_{BE}\mathbf{F}_{EH}\right)^{-1}\Delta\mathbf{F}_{BG}^{\;-1}\mathbf{F}_{BG}^{\;-1}\Delta\mathbf{F}_B^{\;-1}\Delta\mathbf{F}_B\mathbf{F}_{BE}\Delta\mathbf{F}_{BE}\mathbf{F}_{EH}$$

$$= \mathbf{F}_{EH}^{\;-1}\mathbf{F}_{BE}^{\;-1}\mathbf{F}_{BG}\Delta\mathbf{F}_{BG}^{\;-1}\mathbf{F}_{BG}^{\;-1}\mathbf{F}_{BE}\Delta\mathbf{F}_{BE}\mathbf{F}_{EH}$$

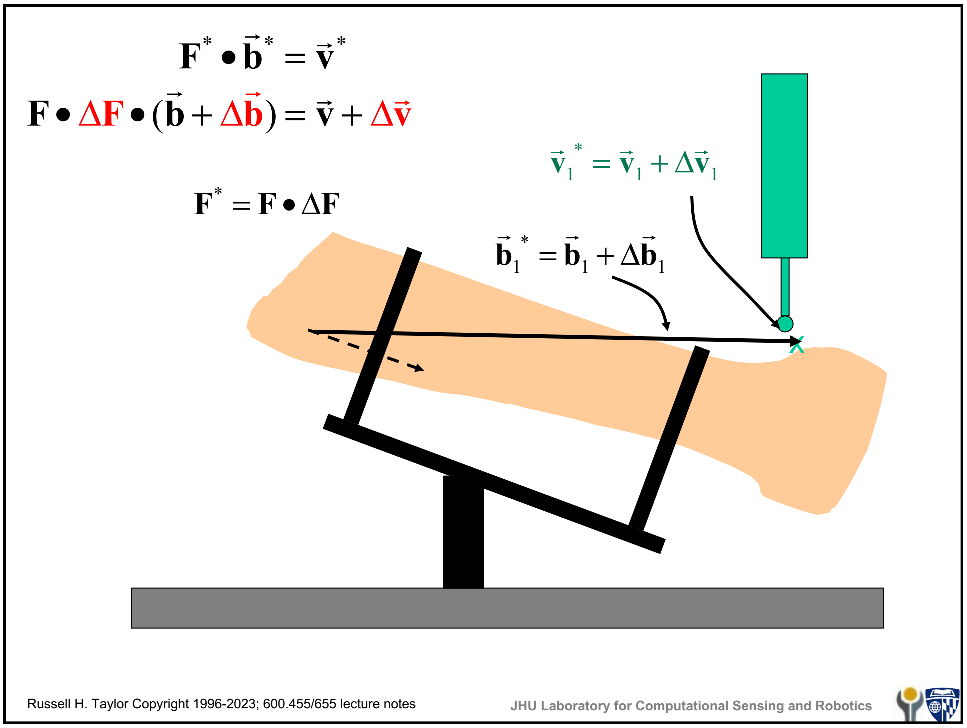




$$\vec{v}_1^{\,*} = \vec{v}_1 + \Delta\vec{v}_1$$

$$F^* = F \bullet \Delta F$$

$$\vec{b}_1^{\,*} = \vec{b}_1 + \Delta\vec{b}_1$$



$$F^* \bullet \vec{b}^{\,*} = \vec{v}^{\,*}$$

$$F \bullet \Delta F \bullet (\vec{b} + \Delta\vec{b}) = \vec{v} + \Delta\vec{v}$$

$$\vec{v}_1^{\,*} = \vec{v}_1 + \Delta\vec{v}_1$$

$$F^* = F \bullet \Delta F$$

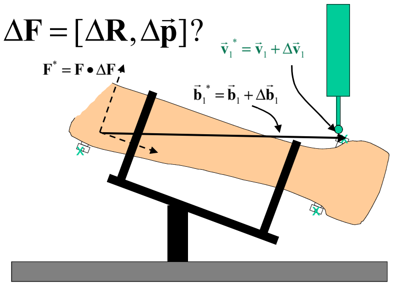
$$\vec{b}_1^{\,*} = \vec{b}_1 + \Delta\vec{b}_1$$

Errors & Sensitivity

Suppose that we know nominal values for $\mathbf{F}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{v}}$ and that

$$\left[-\varepsilon,-\varepsilon,-\varepsilon\right]^T \le \Delta\vec{\mathbf{v}}_1 \le \left[\varepsilon,\varepsilon,\varepsilon\right]^T \quad (i.e., \left\|\Delta\vec{\mathbf{v}}_1\right\|_\infty \le \varepsilon)$$

What does this tell us about $\Delta\mathbf{F} = [\Delta\mathbf{R}, \Delta\vec{\mathbf{p}}]$?



Errors & Sensitivity

$$\begin{aligned}
\vec{\mathbf{v}}^* &= \mathbf{F}^* \bullet \vec{\mathbf{b}}^* \\
&= \mathbf{F} \bullet \Delta\mathbf{F} \bullet (\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}}) \\
&= \mathbf{R} \bullet \left(\Delta\mathbf{R}(\vec{\alpha}) \bullet \left(\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}}\right) + \Delta\vec{\mathbf{p}}\right) + \vec{\mathbf{p}} \\
&\approx \mathbf{R} \bullet \left(\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \vec{\alpha} \times \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}\right) + \vec{\mathbf{p}} \\
&= \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}} + \mathbf{R} \bullet \left(\Delta\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \vec{\alpha} \times \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}\right) \\
&\approx \vec{\mathbf{v}} + \mathbf{R} \bullet \left(\Delta\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}\right)
\end{aligned}$$

if $\left\|\vec{\alpha} \times \Delta\vec{\mathbf{b}}\right\| \le \left\|\vec{\alpha}\right\| \left\|\Delta\vec{\mathbf{b}}\right\|$ is negligible (it usually is)

so

$$\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}^* - \vec{\mathbf{v}} \approx \mathbf{R} \bullet \left(\Delta\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}\right) = \mathbf{R} \bullet \Delta\vec{\mathbf{b}} + \mathbf{R} \bullet \vec{\alpha} \times \vec{\mathbf{b}} + \mathbf{R} \bullet \Delta\vec{\mathbf{p}}$$

Digression: “rotation triple product”

Expressions like $\mathbf{R}\bullet\vec{\mathbf{a}}\times\vec{\mathbf{b}}$ are linear in $\vec{\mathbf{a}},$ but are not always convenient to work with. Often we would prefer something like $\mathbf{M}(\mathbf{R},\vec{\mathbf{b}})\bullet\vec{\mathbf{a}}.$

$$\begin{aligned}\mathbf{R}\bullet\vec{\mathbf{a}}\times\vec{\mathbf{b}} &= -\mathbf{R}\bullet\vec{\mathbf{b}}\times\vec{\mathbf{a}}\\ &= \mathbf{R}\bullet skew(-\vec{\mathbf{b}})\bullet\vec{\mathbf{a}}\\ &= \left[\mathbf{R}\bullet skew(\vec{\mathbf{b}})^T\right]\bullet\vec{\mathbf{a}}\end{aligned}$$

Digression: “rotation triple product”

Here are a few more useful facts:

$$\mathbf{R}\bullet\left(\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right)=\left(\mathbf{R}\bullet\vec{\mathbf{a}}\right)\times\left(\mathbf{R}\bullet\vec{\mathbf{b}}\right)$$

$$\vec{\mathbf{a}}\times\left(\mathbf{R}\bullet\vec{\mathbf{b}}\right)=\mathbf{R}\bullet\left(\left(\mathbf{R}^{-1}\bullet\vec{\mathbf{a}}\right)\times\vec{\mathbf{b}}\right)$$

Consequently

$$skew(\vec{\mathbf{a}})\bullet\mathbf{R}=\mathbf{R}\bullet skew(\mathbf{R}^{-1}\bullet\vec{\mathbf{a}})$$

$$\mathbf{R}^{-1}skew(\vec{\mathbf{a}})\bullet\mathbf{R}=skew(\mathbf{R}^{-1}\bullet\vec{\mathbf{a}})$$

A “standard form” for linearized error expressions

It is often convenient to use identities to rearrange expressions involving small error variables into sums of terms with the general form $\mathbf{M}_k \vec{\eta}_k$, where $\mathbf{M}_k$ involve things known to the computer, and the $\vec{\eta}_k$ are error variables.

For example,

$$\vec{\gamma} = \mathbf{R}\,sk(\vec{\alpha})\vec{\mathbf{a}} + sk(\vec{\beta})\vec{\mathbf{b}}$$

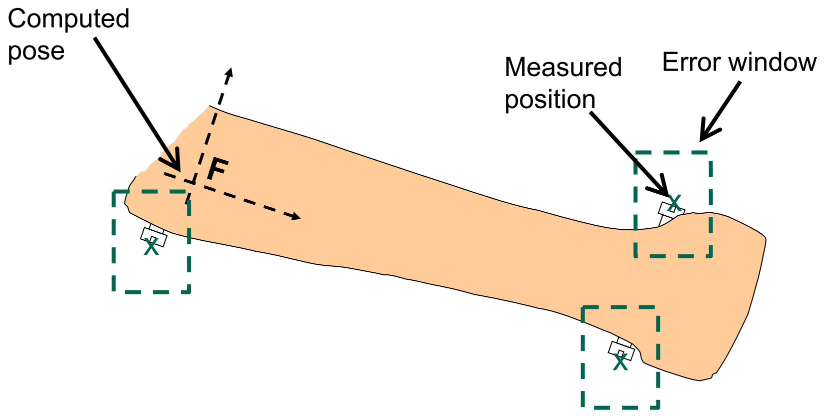
would be rewritten as

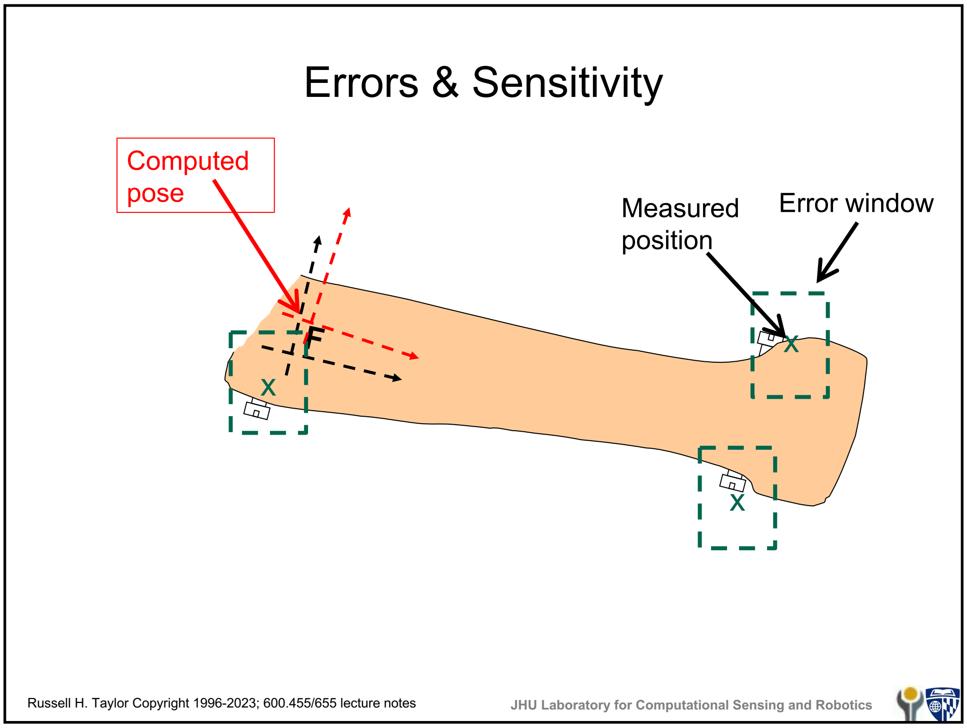
$$\vec{\gamma} = -\mathbf{R}\,sk(\vec{\mathbf{a}})\vec{\alpha} - sk(\vec{\mathbf{b}})\vec{\beta}$$

or

$$\vec{\gamma} = \mathbf{R}\,sk(-\vec{\mathbf{a}})\vec{\alpha} + sk(-\vec{\mathbf{b}})\vec{\beta}$$

Errors & Sensitivity





Errors & Sensitivity

Computed pose

Measured position

Error window

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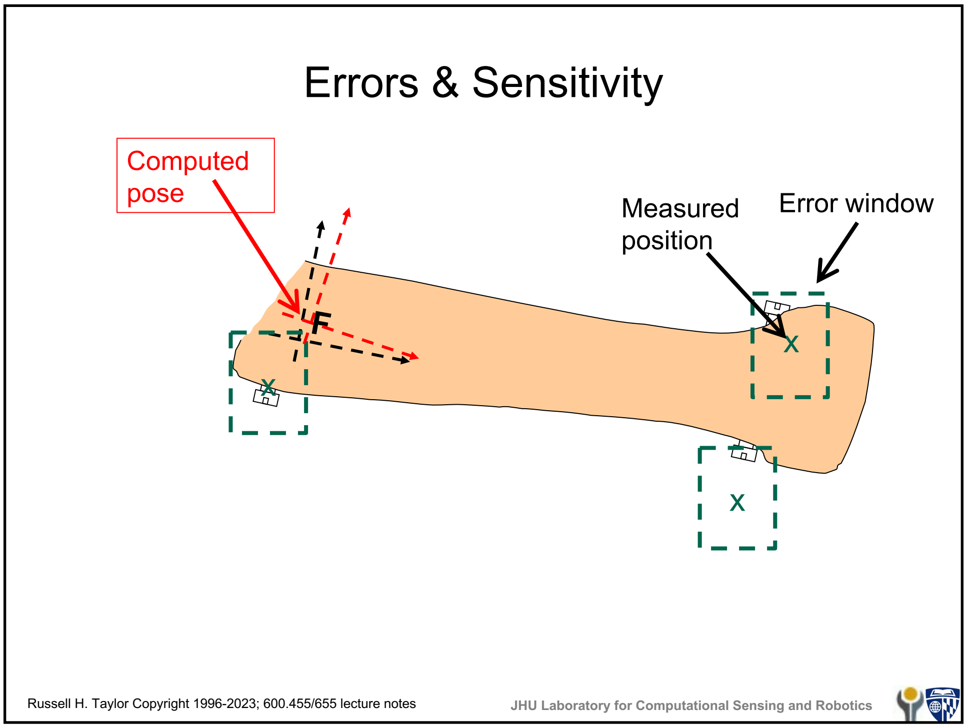

Errors & Sensitivity

Computed pose

Measured position

Error window

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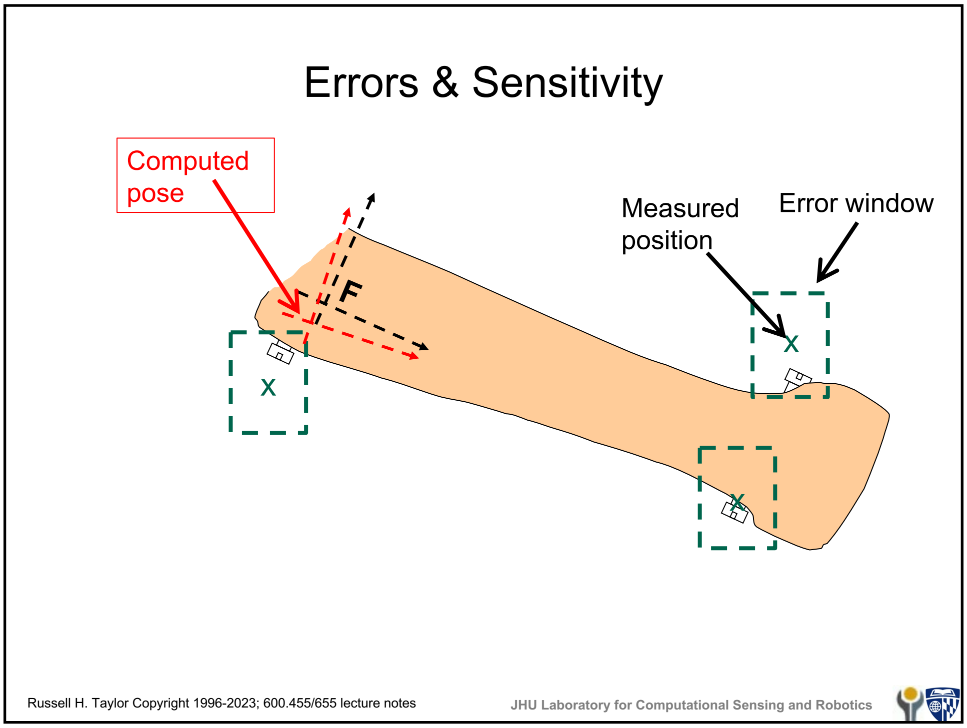
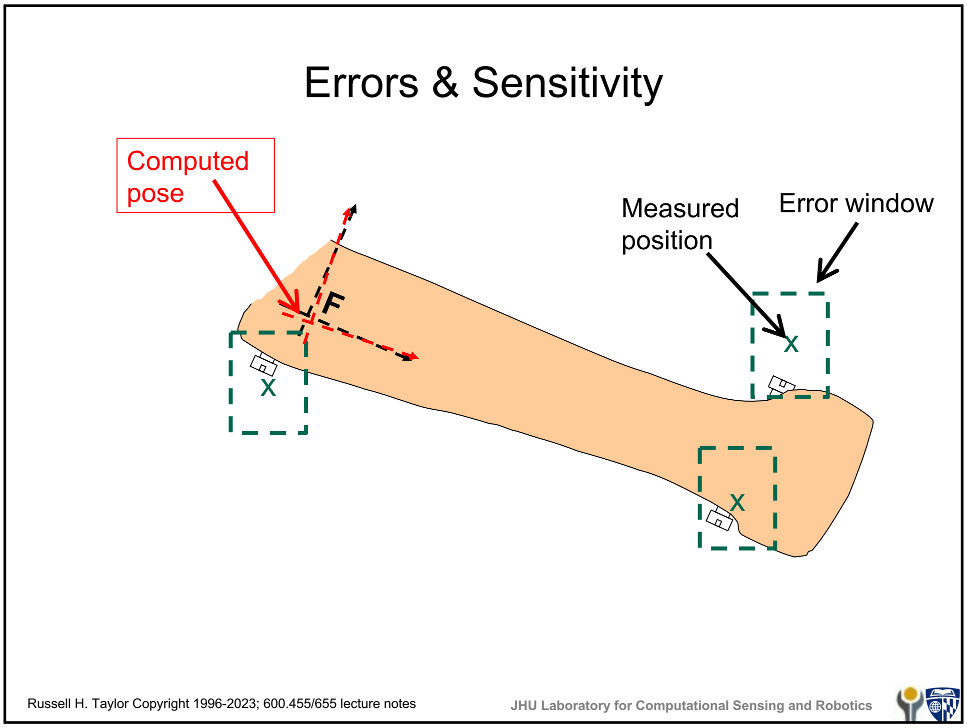


Errors & Sensitivity

Computed pose

Measured position

Error window



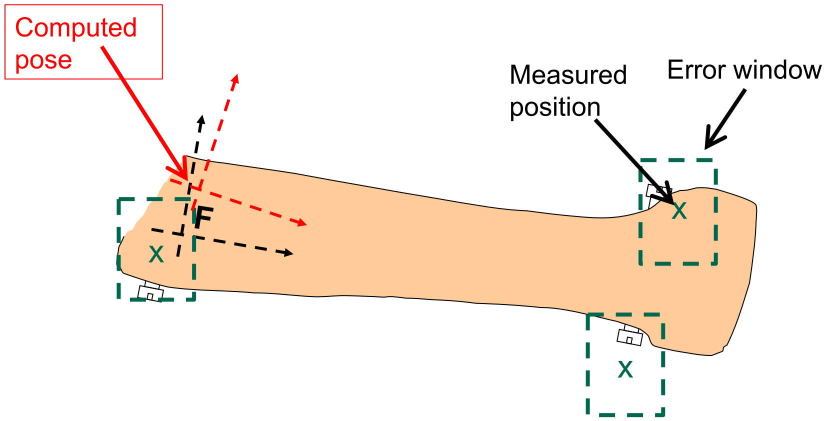
Errors & Sensitivity

Computed pose

Measured position

Error window

Errors & Sensitivity

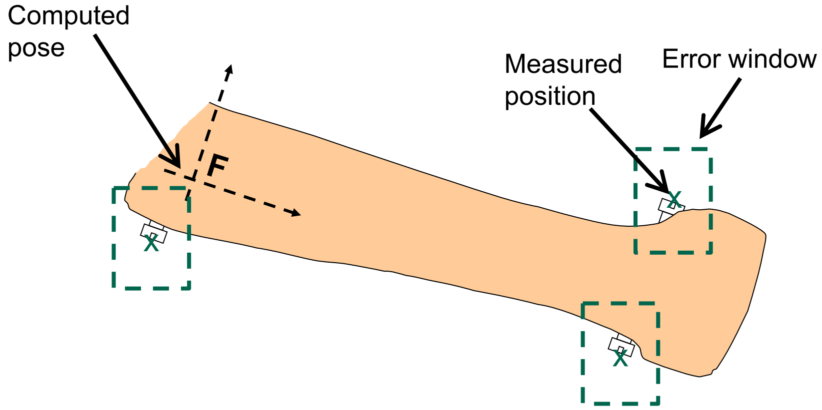


Computed pose

Measured position

Error window

Errors & Sensitivity

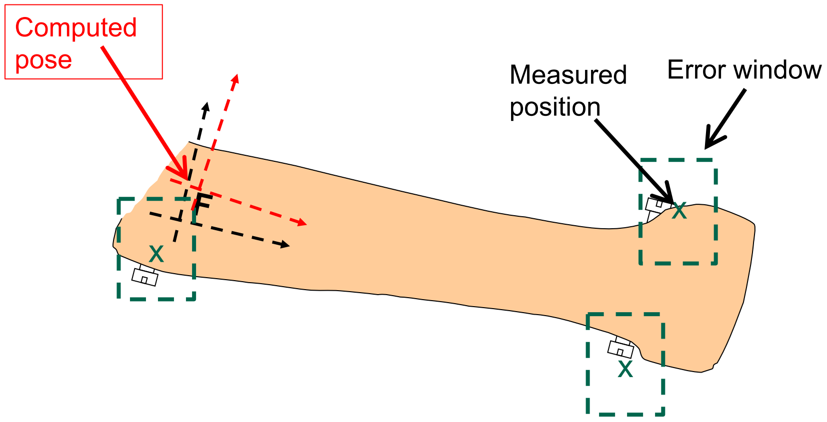


Computed pose

Measured position

Error window

Errors & Sensitivity

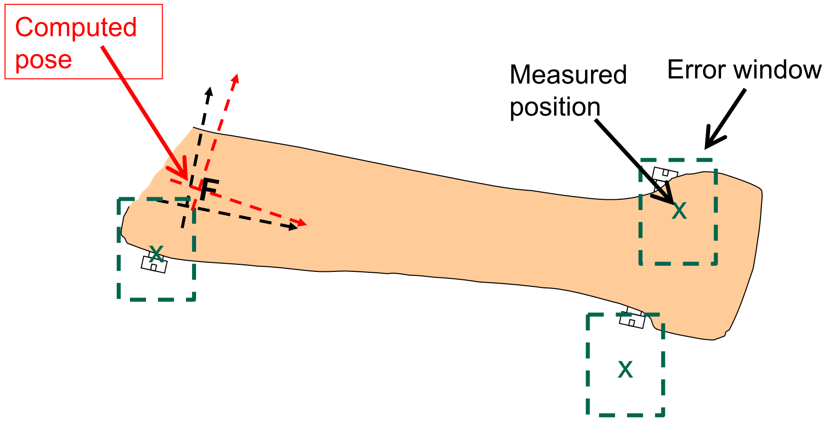


Computed pose

Measured position

Error window

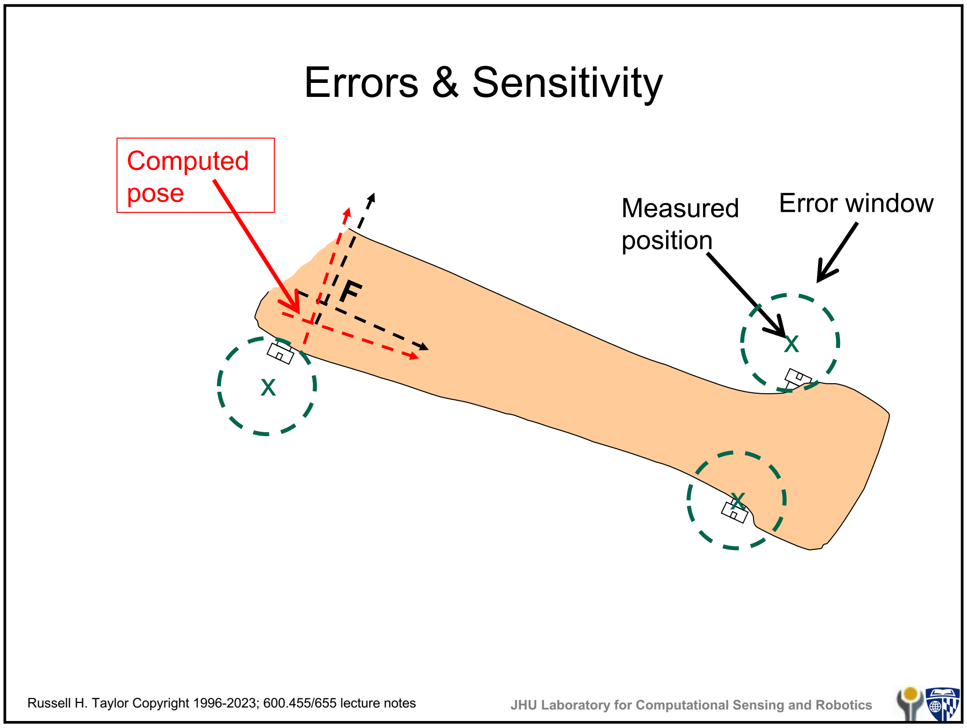
Errors & Sensitivity



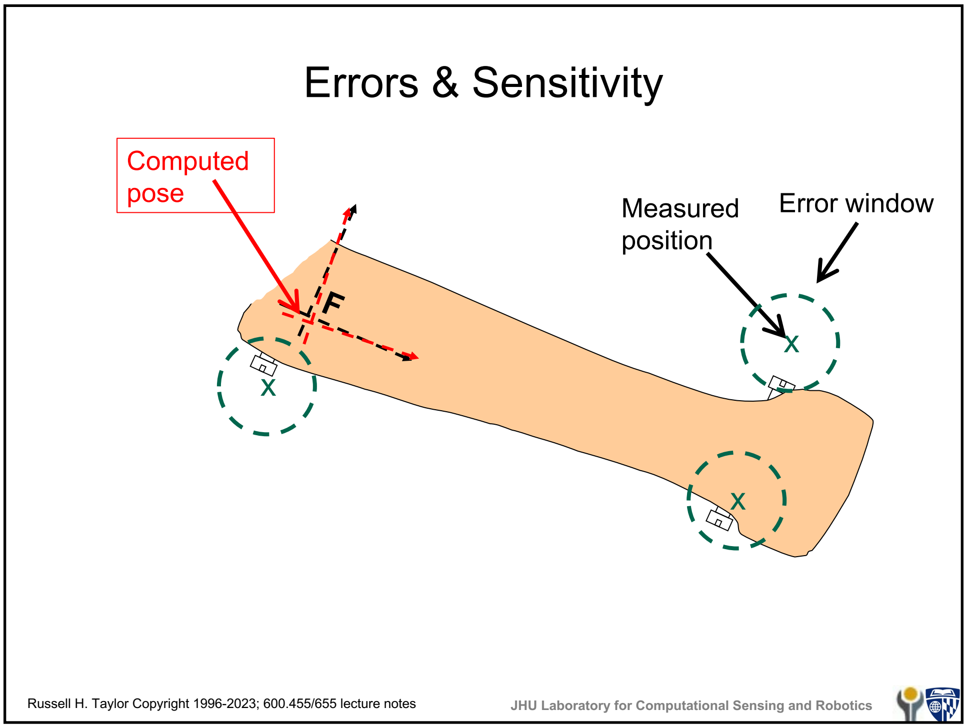
Computed pose

Measured position

Error window

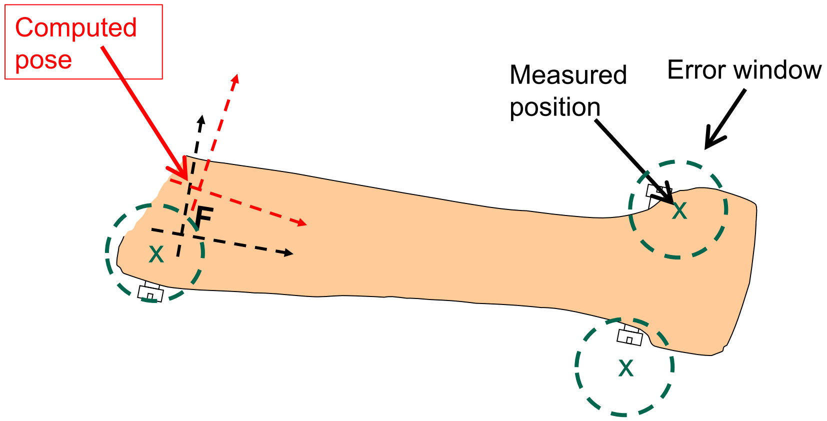


Errors & Sensitivity



Errors & Sensitivity

Errors & Sensitivity



Errors & Sensitivity

Previous expression was

$$\Delta\vec{\mathbf{v}}_1 \approx \mathbf{R}\bullet\left(\Delta\vec{\mathbf{b}}_1+\vec{\alpha}\times\vec{\mathbf{b}}+\Delta\vec{\mathbf{p}}_1\right)$$

Substituting triple product and rearranging gives

$$\Delta\vec{\mathbf{v}}_1 \approx \left[\begin{array}{c|c|c} \mathbf{R} & \mathbf{R} & \mathbf{R}\bullet skew(-\vec{\mathbf{b}}) \end{array}\right]\bullet\begin{bmatrix}\Delta\vec{\mathbf{b}}_1\\ \Delta\vec{\mathbf{p}}\\ \vec{\alpha}\end{bmatrix}$$

So

$$\begin{bmatrix}-\varepsilon\\ -\varepsilon\\ -\varepsilon\end{bmatrix}\le\left[\begin{array}{c|c|c} \mathbf{R} & \mathbf{R} & \mathbf{R}\bullet skew(-\vec{\mathbf{b}}) \end{array}\right]\begin{bmatrix}\Delta\vec{\mathbf{b}}_1\\ \Delta\vec{\mathbf{p}}\\ \vec{\alpha}\end{bmatrix}\le\begin{bmatrix}\varepsilon\\ \varepsilon\\ \varepsilon\end{bmatrix}$$

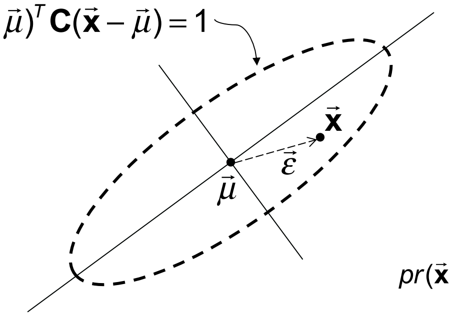
Errors & Sensitivity

Now, suppose we know that $\left|\Delta\vec{\mathbf{b}}_1\right| \le \beta$, this will give us a system of linear constraints

$$\left[\begin{array}{c} -\varepsilon \\ -\varepsilon \\ -\varepsilon \\ \hdashline -\beta \\ -\beta \\ -\beta \end{array}\right] \le \left[\begin{array}{c:c:c} \mathbf{R} & \mathbf{R} & \mathbf{R}\bullet skew(-\vec{\mathbf{b}}) \\ \hdashline \mathbf{I} & \mathbf{0} & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \Delta\vec{\mathbf{b}}_1 \\ \Delta\vec{\mathbf{p}}_1 \\ \vec{\alpha} \end{array}\right] \le \left[\begin{array}{c} \varepsilon \\ \varepsilon \\ \varepsilon \\ \hdashline \beta \\ \beta \\ \beta \end{array}\right]$$

Probabilistic Error Modeling: Multivariable Gaussian

$(\vec{\mathbf{x}}-\vec{\mu})^T\mathbf{C}(\vec{\mathbf{x}}-\vec{\mu}) = 1$



$$\vec{\mathbf{x}} \sim \mathrm{N}(\vec{\mu},\mathbf{C})$$
$$\vec{\mu} = E\left[\vec{\mathbf{x}}\right]$$
$$\mathbf{C} = E\left[\left(\vec{\mathbf{x}}-\vec{\mu}\right)^T\left(\vec{\mathbf{x}}-\vec{\mu}\right)\right] = E\left[\vec{\mathbf{x}}^T\vec{\mathbf{x}}\right] - \vec{\mu}^T\vec{\mu}$$

$$pr(\vec{\mathbf{x}}) = \frac{\exp(-\left(\vec{\mathbf{x}}-\vec{\mu}\right)^T\mathbf{C}\left(\vec{\mathbf{x}}-\vec{\mu}\right)/2)}{\sqrt{(2\pi)^n\left|\mathbf{C}\right|}}$$

If $\vec{\mathbf{x}} \sim \mathrm{N}(\vec{\mu}_x,\mathbf{C}_{xx})$, then

$\vec{\varepsilon} = \vec{\mathbf{x}} - \vec{\mu} \sim \mathrm{N}(\vec{\mathbf{0}},\mathbf{C})$

$\mathbf{A}\vec{\mathbf{x}}+\vec{\mathbf{c}} \sim \mathrm{N}(\mathbf{A}\vec{\mu}_x+\vec{\mathbf{c}},\ \mathbf{A}\mathbf{C}_{xx}\mathbf{A}^T)$ for constants $\mathbf{A}$, $\vec{\mathbf{c}}$

Also there will be a random vector $\vec{\theta}$ with independent elements $\vec{\theta}_i \sim \mathrm{N}(0,1)$ and matrix $\mathbf{M}$ such that $\vec{\mathbf{x}} = \vec{\mu} + \mathbf{M}\vec{\theta}$, and $\mathbf{C}_{xx} = \mathbf{M}\mathbf{M}^T = \mathbf{Q}\Lambda^2\mathbf{Q}^{-1}$ where Λ=diag($\vec{\lambda}$), **M**=**Q**diag($\vec{\lambda}$)

Probabilistic Error Modeling: Multivariable Gaussian

Suppose $\vec{\mathbf{x}} \sim \mathrm{N}(\vec{\mu}_x, \mathbf{C}_{xx})$, $\vec{\mathbf{y}} \sim \mathrm{N}(\vec{\mu}_y, \mathbf{C}_{yy})$, and $\vec{\mathbf{z}} = \begin{bmatrix} \vec{\mathbf{x}} \\ \vec{\mathbf{y}} \end{bmatrix}$

Then $\mathbf{C}_{zz} = \begin{bmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xy} \\ \mathbf{C}_{yx} & \mathbf{C}_{yy} \end{bmatrix}$

If we know that $\vec{\mathbf{y}} = \vec{\mathbf{a}}$ then $[\vec{\mathbf{x}} | \vec{\mathbf{y}} = \vec{\mathbf{a}}] \sim \mathrm{N}(\vec{\mu}_{[x|y=a]}, \mathbf{C}_{[x|y=a]})$, where

$$\vec{\mu}_{[x|y=a]} = \vec{\mu}_x + \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}(\vec{\mathbf{a}} - \vec{\mu}_y)$$

$$\mathbf{C}_{[x|y=a]} = \mathbf{C}_{xx} - \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}\mathbf{C}_{yx}$$

Also $\vec{\mathbf{y}}$ and $\vec{\mathbf{x}} - \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}\vec{\mathbf{y}}$ are independent.

Error from frame composition

Consider $\mathbf{R}_1^*\mathbf{R}_2^* = \mathbf{R}_3^*$ where $\mathbf{R}_1^* = \mathbf{R}_1\Delta\mathbf{R}_1, \mathbf{R}_2^* = \mathbf{R}_2\Delta\mathbf{R}_2$, $\mathbf{R}_3^* = \mathbf{R}_3\Delta\mathbf{R}_3$
and $\Delta\mathbf{R}_1 \approx \mathbf{I} + sk(\vec{\alpha}_1)$, $\Delta\mathbf{R}_2 \approx \mathbf{I} + sk(\vec{\alpha}_2)$, estimate $\Delta\mathbf{R}_3 \approx \mathbf{I} + sk(\vec{\alpha}_3)$

$$\mathbf{R}_1\Delta\mathbf{R}_1\mathbf{R}_2\Delta\mathbf{R}_2 = \mathbf{R}_1\mathbf{R}_2\Delta\mathbf{R}_3$$

$$\mathbf{R}_1(\mathbf{I} + sk(\vec{\alpha}_1))\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_2)) \approx \mathbf{R}_1\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_3))$$

$$\left(\mathbf{R}_1\mathbf{R}_2\right)^{-1}\mathbf{R}_1(\mathbf{I} + sk(\vec{\alpha}_1))\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_2)) \approx \mathbf{I} + sk(\vec{\alpha}_3)$$

$$\mathbf{R}_2^{-1}\cancel{\mathbf{R}_1^{-1}\mathbf{R}_1}(\mathbf{I} + sk(\vec{\alpha}_1))\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_2)) \approx \mathbf{I} + sk(\vec{\alpha}_3)$$

$$\mathbf{I} + \mathbf{R}_2^{-1}sk(\vec{\alpha}_1)\mathbf{R}_2 + sk(\vec{\alpha}_2) + \cancel{\mathbf{R}_2^{-1}sk(\vec{\alpha}_1)\mathbf{R}_2 sk(\vec{\alpha}_2)} \approx \mathbf{I} + sk(\vec{\alpha}_3)$$

$$\mathbf{R}_2^{-1}sk(\vec{\alpha}_1)\mathbf{R}_2 + sk(\vec{\alpha}_2) \approx sk(\vec{\alpha}_3)$$

Since $\mathbf{R}^{-1} \bullet (\vec{\mathbf{a}} \times \mathbf{R}\vec{\mathbf{b}}) = (\mathbf{R}^{-1}\vec{\mathbf{a}}) \times \vec{\mathbf{b}}$ for all $\mathbf{R}, \vec{\mathbf{a}}, \vec{\mathbf{b}}$ we get $\mathbf{R}_2^{-1}sk(\vec{\alpha}_1)\mathbf{R}_2 = sk(\mathbf{R}_2^{-1}\vec{\alpha}_1)$

$$sk(\vec{\alpha}_3) \approx sk(\mathbf{R}_2^{-1}\vec{\alpha}_1) + sk(\vec{\alpha}_2) = sk(\mathbf{R}_2^{-1}\vec{\alpha}_1 + \vec{\alpha}_2)$$

$$\vec{\alpha}_3 \approx \mathbf{R}_2^{-1}\vec{\alpha}_1 + \vec{\alpha}_2$$

Error from frame composition

Consider $\mathbf{F}_1^*\mathbf{F}_2^* = \mathbf{F}_3^*$ where $\mathbf{F}_1^* = \mathbf{F}_1\Delta\mathbf{F}_1, \mathbf{F}_2^* = \mathbf{F}_2\Delta\mathbf{F}_2,\ \mathbf{F}_3^* = \mathbf{F}_3\Delta\mathbf{F}_3$
and $\Delta\mathbf{F}_1 \approx \left[\mathbf{I} + sk\left(\vec{\alpha}_1\right), \vec{\varepsilon}_1\right],\ \Delta\mathbf{F}_2 \approx \left[\mathbf{I} + sk\left(\vec{\alpha}_2\right), \vec{\varepsilon}_2\right]$,
estimate $\Delta\mathbf{F}_3 \approx \left[\mathbf{I} + sk\left(\vec{\alpha}_3\right), \vec{\varepsilon}_3\right]$

From before, we have $\vec{\alpha}_3 \approx \mathbf{R}_2^{-1}\vec{\alpha}_1 + \vec{\alpha}_2$. So now we just need $\vec{\varepsilon}_3$.

$$
\begin{aligned}
\mathbf{F}_3\Delta\mathbf{F}_3 &= [\mathbf{R}_3\Delta\mathbf{R}_3, \mathbf{R}_3\Delta\vec{\mathbf{p}}_3 + \vec{\mathbf{p}}_3] \\
\vec{\mathbf{p}}_3 + \mathbf{R}_3\Delta\vec{\mathbf{p}}_3 &= \mathbf{R}_1(\Delta\mathbf{R}_1(\vec{\mathbf{p}}_2 + \mathbf{R}_2\vec{\varepsilon}_2) + \varepsilon_1) + \vec{\mathbf{p}}_1 \\
\vec{\mathbf{p}}_3 + \mathbf{R}_3\vec{\varepsilon}_3 &\approx \mathbf{R}_1\left(\mathbf{I} + sk(\vec{\alpha}_1)\right)\left(\vec{\mathbf{p}}_2 + \mathbf{R}_2\vec{\varepsilon}_2\right) + \mathbf{R}_1\vec{\varepsilon}_1 + \vec{\mathbf{p}}_1 \\
&= \mathbf{R}_1\vec{\mathbf{p}}_2 + \mathbf{R}_1\mathbf{R}_2\vec{\varepsilon}_2 + \mathbf{R}_1 \bullet \left(\vec{\alpha}_1 \times \vec{\mathbf{p}}_2 + \vec{\alpha}_1 \times \mathbf{R}_2\vec{\varepsilon}_2\right) + \vec{\mathbf{p}}_1 + \mathbf{R}_1\vec{\varepsilon}_1 \\
&= \vec{\mathbf{p}}_3 + \mathbf{R}_1\mathbf{R}_2\vec{\varepsilon}_2 + \mathbf{R}_1 \bullet \left(\vec{\alpha}_1 \times \vec{\mathbf{p}}_2 + \vec{\alpha}_1 \times \mathbf{R}_2\vec{\varepsilon}_2\right) + \mathbf{R}_1\vec{\varepsilon}_1 \\
\mathbf{R}_3\vec{\varepsilon}_3 &\approx \mathbf{R}_1\mathbf{R}_2\vec{\varepsilon}_2 + \mathbf{R}_1 \bullet \vec{\alpha}_1 \times \vec{\mathbf{p}}_2 + \mathbf{R}_1\vec{\varepsilon}_1 \\
\vec{\varepsilon}_3 &\approx \mathbf{R}_2^{-1}\mathbf{R}_1^{-1}\mathbf{R}_1\mathbf{R}_2\vec{\varepsilon}_2 + \mathbf{R}_2^{-1}\mathbf{R}_1^{-1}\mathbf{R}_1 \bullet \vec{\alpha}_1 \times \vec{\mathbf{p}}_2 + \mathbf{R}_2^{-1}\mathbf{R}_1^{-1}\mathbf{R}_1\vec{\varepsilon}_1 \\
&= \vec{\varepsilon}_2 + \mathbf{R}_2^{-1} \bullet \vec{\alpha}_1 \times \vec{\mathbf{p}}_2 + \mathbf{R}_2^{-1}\vec{\varepsilon}_1 \\
&= \vec{\varepsilon}_2 - \mathbf{R}_2^{-1}sk(\vec{\mathbf{p}}_2)\vec{\alpha}_1 + \mathbf{R}_2^{-1}\vec{\varepsilon}_1
\end{aligned}
$$

Inverse of frame transformation with errors

For $\mathbf{F}^* = \mathbf{F}\Delta\mathbf{F}$, if we want $\Delta\mathbf{F}_i$ such that $\mathbf{F}_i^* = \mathbf{F}_i\Delta\mathbf{F}_i$, we have

$$
\begin{aligned}
\mathbf{F}_i &= \mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1}\vec{\mathbf{p}}] \\
\mathbf{F}_i^* &= \mathbf{F}_i\Delta\mathbf{F}_i = (\mathbf{F}\Delta\mathbf{F})^{-1} = \Delta\mathbf{F}^{-1}\mathbf{F}^{-1} = [\Delta\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}}] \bullet [\mathbf{R}^{-1}, -\mathbf{R}^{-1}\vec{\mathbf{p}}] \\
\mathbf{F}_i\Delta\mathbf{F}_i &= [\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}} - \Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}}] \\
\Delta\mathbf{F}_i &= \mathbf{F}_i^{-1}[\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}} - \Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}}] \\
&= \left(\mathbf{F}^{-1}\right)^{-1}[\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}} - \Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}}] \\
&= [\mathbf{R}, \vec{\mathbf{p}}] \bullet [\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}} - \Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}}] \\
&= [\mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}} - \mathbf{R}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} + \vec{\mathbf{p}}]
\end{aligned}
$$

Inverse of frame transformation with errors

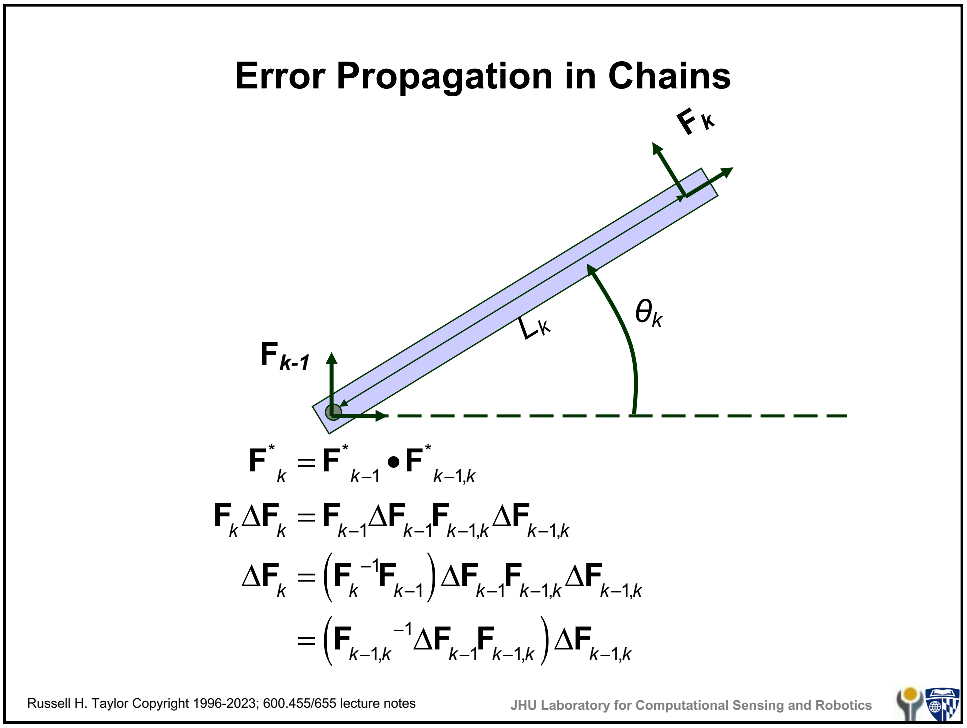
For $\mathbf{F}^* = \Delta\mathbf{F}\mathbf{F}$, if we want $\mathbf{F}_i^* = \Delta\mathbf{F}_i\mathbf{F}_i$, we have

$$\begin{aligned}
\mathbf{F}_i &= \mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1}\vec{\mathbf{p}}] \\
\mathbf{F}_i^* &= \Delta\mathbf{F}_i\mathbf{F}_i = (\Delta\mathbf{F}\mathbf{F})^{-1} = \mathbf{F}^{-1}\Delta\mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1}\vec{\mathbf{p}}] \bullet [\Delta\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}}] \\
\Delta\mathbf{F}_i\mathbf{F}_i &= [\mathbf{R}^{-1}\Delta\mathbf{R}^{-1}, -\mathbf{R}^{-1}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \mathbf{R}^{-1}\vec{\mathbf{p}}] \\
\Delta\mathbf{F}_i &= [\mathbf{R}^{-1}\Delta\mathbf{R}^{-1}, -\mathbf{R}^{-1}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \mathbf{R}^{-1}\vec{\mathbf{p}}] \bullet [\mathbf{R}, \vec{\mathbf{p}}] \\
&= [\mathbf{R}^{-1}\Delta\mathbf{R}^{-1}\mathbf{R}, \mathbf{R}^{-1}\Delta\mathbf{R}^{-1}\vec{\mathbf{p}} - \mathbf{R}^{-1}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \mathbf{R}^{-1}\vec{\mathbf{p}}]
\end{aligned}$$

Inverse of frame transformation with errors

Suppose we know that $\Delta\mathbf{R}$ is "small", i.e., $\Delta\mathbf{R} \approx \mathbf{I} + sk(\vec{\alpha})$, and for notational convenience we write $\Delta\vec{\mathbf{p}} = \vec{\varepsilon}$, we get

$$\begin{aligned}
\Delta\mathbf{R}_i &= \mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1} \approx \mathbf{R}\left(\mathbf{I} + sk(\vec{\alpha})\right)^{-1}\mathbf{R}^{-1} \\
&\approx \mathbf{R}\left(\mathbf{I} - sk(\vec{\alpha})\right)\mathbf{R}^{-1} \\
&= \mathbf{R}\mathbf{R}^{-1} - \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1} \\
&= \mathbf{I} - \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1} \\
&= \mathbf{I} - sk\left(\left(\mathbf{R}^{-1}\right)^{-1}\vec{\alpha}\right) = \mathbf{I} - sk(\mathbf{R}\vec{\alpha}) \\
\Delta\vec{\mathbf{p}}_i &= -\mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}} - \mathbf{R}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} + \vec{\mathbf{p}} \\
&\approx -\mathbf{R}\left(\mathbf{I} - sk(\vec{\alpha})\right)\mathbf{R}^{-1}\vec{\mathbf{p}} - \mathbf{R}\left(\mathbf{I} - sk(\vec{\alpha})\right)\vec{\varepsilon} + \vec{\mathbf{p}} \\
&= -\vec{\mathbf{p}} + \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1}\vec{\mathbf{p}} - \mathbf{R}\vec{\varepsilon} + \mathbf{R}sk(\vec{\alpha})\vec{\varepsilon} + \vec{\mathbf{p}} \\
&\approx \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1}\vec{\mathbf{p}} - \mathbf{R}\vec{\varepsilon} = \mathbf{R}\left(\vec{\alpha} \times \mathbf{R}^{-1}\vec{\mathbf{p}}\right) - \mathbf{R}\vec{\varepsilon} \\
\Delta\vec{\mathbf{p}} &\approx -\mathbf{R}\vec{\varepsilon} - \mathbf{R}sk(\mathbf{R}^{-1}\vec{\mathbf{p}})\vec{\alpha}
\end{aligned}$$



Error Propagation in Chains

$$\mathbf{F}^*_k = \mathbf{F}^*_{k-1} \bullet \mathbf{F}^*_{k-1,k}$$

$$\mathbf{F}_k \Delta\mathbf{F}_k = \mathbf{F}_{k-1} \Delta\mathbf{F}_{k-1} \mathbf{F}_{k-1,k} \Delta\mathbf{F}_{k-1,k}$$

$$\begin{aligned}\Delta\mathbf{F}_k &= \left(\mathbf{F}_k^{-1}\mathbf{F}_{k-1}\right)\Delta\mathbf{F}_{k-1}\mathbf{F}_{k-1,k}\Delta\mathbf{F}_{k-1,k} \\ &= \left(\mathbf{F}_{k-1,k}{}^{-1}\Delta\mathbf{F}_{k-1}\mathbf{F}_{k-1,k}\right)\Delta\mathbf{F}_{k-1,k}\end{aligned}$$

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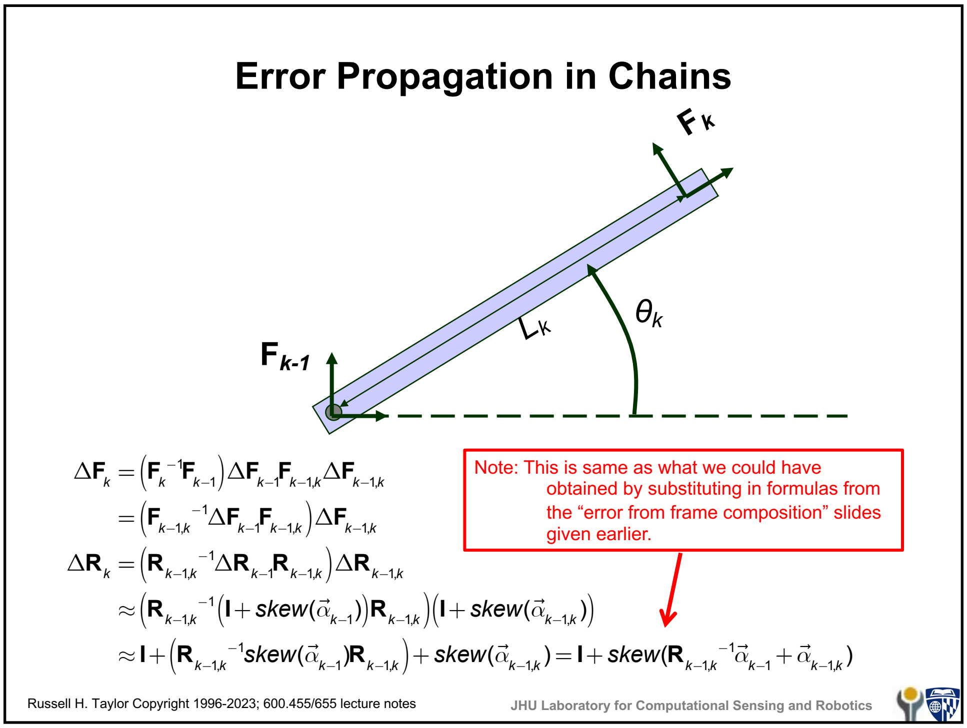

Error Propagation in Chains

$$\begin{aligned}\Delta\mathbf{F}_k &= \left(\mathbf{F}_k^{-1}\mathbf{F}_{k-1}\right)\Delta\mathbf{F}_{k-1}\mathbf{F}_{k-1,k}\Delta\mathbf{F}_{k-1,k} \\ &= \left(\mathbf{F}_{k-1,k}{}^{-1}\Delta\mathbf{F}_{k-1}\mathbf{F}_{k-1,k}\right)\Delta\mathbf{F}_{k-1,k} \\ \Delta\mathbf{R}_k &= \left(\mathbf{R}_{k-1,k}{}^{-1}\Delta\mathbf{R}_{k-1}\mathbf{R}_{k-1,k}\right)\Delta\mathbf{R}_{k-1,k} \\ &\approx \left(\mathbf{R}_{k-1,k}{}^{-1}\left(\mathbf{I}+skew(\vec{\alpha}_{k-1})\right)\mathbf{R}_{k-1,k}\right)\left(\mathbf{I}+skew(\vec{\alpha}_{k-1,k})\right) \\ &\approx \mathbf{I}+\left(\mathbf{R}_{k-1,k}{}^{-1}skew(\vec{\alpha}_{k-1})\mathbf{R}_{k-1,k}\right)+skew(\vec{\alpha}_{k-1,k}) = \mathbf{I}+skew(\mathbf{R}_{k-1,k}{}^{-1}\vec{\alpha}_{k-1}+\vec{\alpha}_{k-1,k})\end{aligned}$$

Note: This is same as what we could have obtained by substituting in formulas from the “error from frame composition” slides given earlier.

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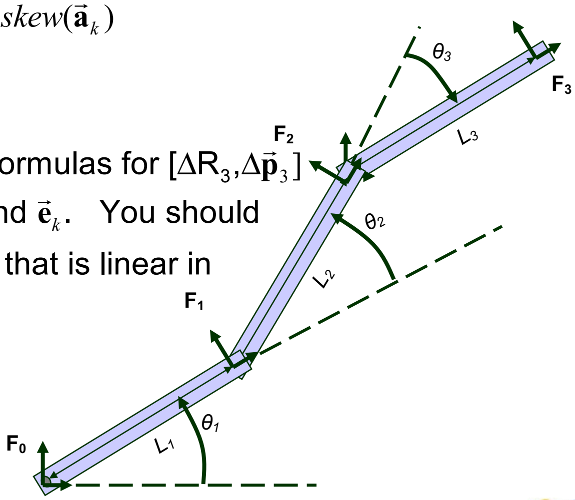
Exercise

Suppose that you have

$$\Delta \mathbf{R}_{k-1,k} = \Delta \mathbf{R}(\vec{\mathbf{a}}_k) \cong \mathbf{I} + skew(\vec{\mathbf{a}}_k)$$
$$\Delta \vec{\mathbf{p}}_{k-1,k} = \vec{\mathbf{e}}_k$$

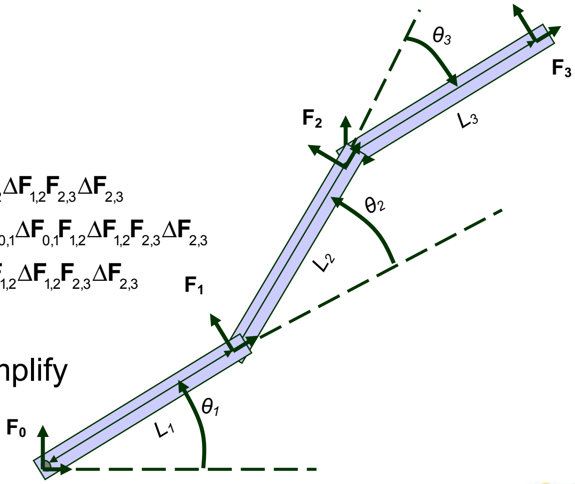
Work out approximate formulas for $[\Delta \mathrm{R}_3, \Delta \vec{\mathbf{p}}_3]$ in terms of $L_k, \vec{\mathbf{r}}_{k,} \theta_{k,}$ $\vec{\mathbf{a}}_k$ and $\vec{\mathbf{e}}_k$. You should come up with a formula that is linear in L_k, $\vec{\mathbf{a}}_k$, and $\vec{\mathbf{e}}_k$.



Exercise

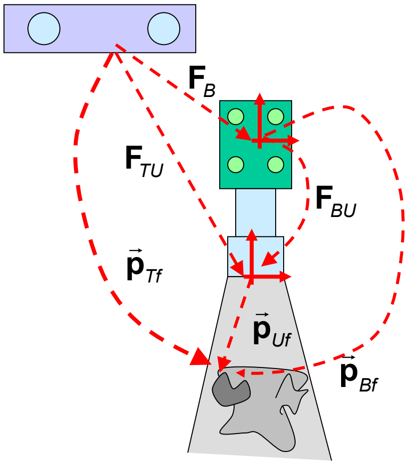
Suppose we want to know error in $\mathbf{F}_{0,3} = \mathbf{F}_0^{-1}\mathbf{F}_3$

$$\begin{aligned}
\mathbf{F}_{0,3} &= \mathbf{F}_0^{-1}\mathbf{F}_0\mathbf{F}_{0,1}\mathbf{F}_{1,2}\mathbf{F}_{2,3} \\
\mathbf{F}_{0,3}^{*} &= \mathbf{F}_{0,1}^{*}\mathbf{F}_{1,2}^{*}\mathbf{F}_{2,3}^{*} \\
\mathbf{F}_{0,3}\Delta\mathbf{F}_{0,3} &= \mathbf{F}_{0,1}^{*}\mathbf{F}_{1,2}^{*}\mathbf{F}_{2,3}^{*} \\
\Delta\mathbf{F}_{0,3} &= \mathbf{F}_{0,3}^{-1}\mathbf{F}_{0,1}\Delta\mathbf{F}_{0,1}\mathbf{F}_{1,2}\Delta\mathbf{F}_{1,2}\mathbf{F}_{2,3}\Delta\mathbf{F}_{2,3} \\
&= \mathbf{F}_{2,3}^{-1}\mathbf{F}_{1,2}^{-1}\mathbf{F}_{0,1}^{-1}\mathbf{F}_{0,1}\Delta\mathbf{F}_{0,1}\mathbf{F}_{1,2}\Delta\mathbf{F}_{1,2}\mathbf{F}_{2,3}\Delta\mathbf{F}_{2,3} \\
&= \mathbf{F}_{2,3}^{-1}\mathbf{F}_{1,2}^{-1}\Delta\mathbf{F}_{0,1}\mathbf{F}_{1,2}\Delta\mathbf{F}_{1,2}\mathbf{F}_{2,3}\Delta\mathbf{F}_{2,3}
\end{aligned}$$

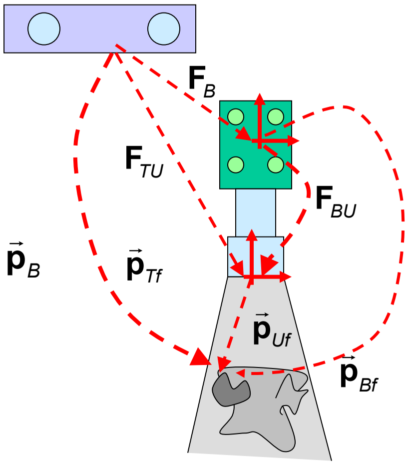


Now substitute and simplify

Another Example



Another Example

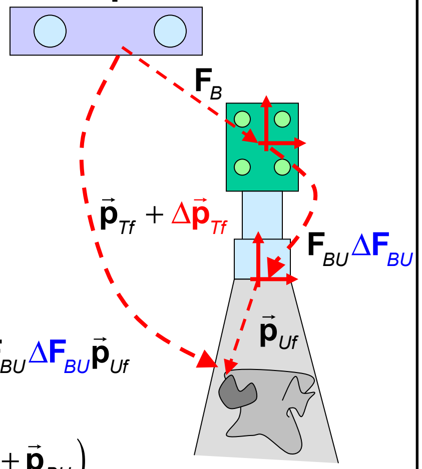


$$
\begin{aligned}
\vec{\mathbf{p}}_{Tf} &= \mathbf{F}_{TU} \bullet \vec{\mathbf{p}}_{Uf} \\
\mathbf{F}_{TU} &= \mathbf{F}_{B} \bullet \mathbf{F}_{BU} \\
&= [\mathbf{R}_B \bullet \mathbf{R}_{BU}, \mathbf{R}_B \bullet \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_B] \\
\vec{\mathbf{p}}_{Tf} &= \mathbf{R}_B \bullet \mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} + \mathbf{R}_B \bullet \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_B
\end{aligned}
$$

Also

$$
\begin{aligned}
\vec{\mathbf{p}}_{Tf} &= \mathbf{F}_{B} \bullet \vec{\mathbf{p}}_{Bf} \\
\vec{\mathbf{p}}_{Bf} &= \mathbf{F}_{BU} \bullet \vec{\mathbf{p}}_{Uf} \\
&= \mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} + \vec{\mathbf{p}}_{BU} \\
\vec{\mathbf{p}}_{Tf} &= \mathbf{R}_B \bullet \mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} + \mathbf{R}_B \bullet \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_B
\end{aligned}
$$

Another Example



Suppose that the track body
to US calibration is not perfect

$$\begin{aligned}\mathbf{F}_{BU}^{\ *} &= \mathbf{F}_{BU}\Delta\mathbf{F}_{BU}\\ &= [\mathbf{R}_{BU}\Delta\mathbf{R}_{BU},\mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}+\vec{\mathbf{p}}_{BU}]\end{aligned}$$

$$\vec{\mathbf{p}}_{Bf}{}^{*} = \mathbf{F}_{BU}{}^{*}\bullet\vec{\mathbf{p}}_{Uf} \quad \text{I.e.,} \quad \vec{\mathbf{p}}_{Bf}+\Delta\vec{\mathbf{p}}_{BF} = \mathbf{F}_{BU}\Delta\mathbf{F}_{BU}\vec{\mathbf{p}}_{Uf}$$

$$\begin{aligned}\Delta\vec{\mathbf{p}}_{Bf} &= \mathbf{F}_{BU}\Delta\mathbf{F}_{BU}\vec{\mathbf{p}}_{Uf}-\vec{\mathbf{p}}_{Bf}\\ &= \mathbf{F}_{BU}(\Delta\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}+\Delta\vec{\mathbf{p}}_{BU})-\left(\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}+\vec{\mathbf{p}}_{BU}\right)\\ &= \mathbf{R}_{BU}\Delta\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}+\mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}+\vec{\mathbf{p}}_{BU}-\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}-\vec{\mathbf{p}}_{BU}\\ &= \mathbf{R}_{BU}\Delta\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}+\mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}-\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}\end{aligned}$$

Another Example

Continuing …

$$\Delta\vec{\mathbf{p}}_{Bf} = \mathbf{R}_{BU}\Delta\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}+\mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}-\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}$$

$$\begin{aligned}&\approx \mathbf{R}_{BU}\overbrace{\left(\mathbf{I}+\mathbf{skew}(\vec{\alpha}_{BU})\right)}\vec{\mathbf{p}}_{Uf}+\mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}-\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}\\ &= \cancel{\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}}+\mathbf{R}_{BU}\bullet\vec{\alpha}_{BU}\times\vec{\mathbf{p}}_{Uf}+\mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}-\cancel{\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}}\\ &= \mathbf{R}_{BU}\bullet\vec{\alpha}_{BU}\times\vec{\mathbf{p}}_{Uf}+\mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}\\ &= -\mathbf{R}_{BU}\bullet\vec{\mathbf{p}}_{Uf}\times\vec{\alpha}_{BU}+\mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}\\ &= \mathbf{R}_{BU}\,skew(-\vec{\mathbf{p}}_{Uf})\vec{\alpha}_{BU}+\mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}\end{aligned}$$

Another Example

$$\vec{\mathbf{p}}_{Tf} + \Delta\vec{\mathbf{p}}_{Tf} = \mathbf{F}_B \Delta\mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right)$$

$$\Delta\vec{\mathbf{p}}_{Tf} = \mathbf{F}_B \Delta\mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) - \mathbf{F}_B\vec{\mathbf{p}}_{Bf}$$

$$\Delta\mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) = \Delta\mathbf{R}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \Delta\vec{\mathbf{p}}_B$$

$$\approx \left(\mathbf{I} + skew(\vec{\alpha}_B)\right)\left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \Delta\vec{\mathbf{p}}_B$$

$$= \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \cancel{\vec{\alpha}_B \times \Delta\vec{\mathbf{p}}_{Bf}} + \Delta\vec{\mathbf{p}}_B$$

$$\approx \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B$$

$$\Delta\vec{\mathbf{p}}_{Tf} \approx \mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right) - \mathbf{F}_B\vec{\mathbf{p}}_{Bf}$$

$$= \mathbf{R}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right) + \vec{\mathbf{p}}_B - \left(\mathbf{R}_B\vec{\mathbf{p}}_{Bf} + \vec{\mathbf{p}}_B\right)$$

$$= \mathbf{R}_B \left(\Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right)$$

$$\Delta\vec{\mathbf{p}}_{Bf} \approx \mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}$$

Another Example

$$\Delta\vec{\mathbf{p}}_{Tf} \approx \mathbf{R}_B \left(\Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right)$$

$$\Delta\vec{\mathbf{p}}_{Bf} \approx \mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}$$

$$\Delta\vec{\mathbf{p}}_{Tf} \approx \mathbf{R}_B \left(\mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right)$$

$$= \left(\begin{array}{c} \mathbf{R}_B\mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_B\mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} \\ +\mathbf{R}_B skew(-\vec{\mathbf{p}}_{Bf})\vec{\alpha}_B + \mathbf{R}_B\Delta\vec{\mathbf{p}}_B \end{array}\right)$$

$$= \left[\mathbf{R}_B\mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU}) \;\middle|\; \mathbf{R}_B\mathbf{R}_{BU} \;\middle|\; \mathbf{R}_B skew(-\vec{\mathbf{p}}_{Bf}) \;\middle|\; \mathbf{R}_B\right] \begin{bmatrix} \vec{\alpha}_{BU} \\ \Delta\vec{\mathbf{p}}_{BU} \\ \vec{\alpha}_B \\ \Delta\vec{\mathbf{p}}_B \end{bmatrix}$$

Parametric Sensitivity

Suppose you have an explicit formula like

$$\vec{\mathbf{p}}_3 = \begin{bmatrix} L_1\cos(\theta_1)+L_2\cos(\theta_1+\theta_2)+L_3\cos(\theta_1+\theta_2+\theta_3) \\ L_1\sin(\theta_1)+L_2\sin(\theta_1+\theta_2)+L_3\sin(\theta_1+\theta_2+\theta_3) \\ 0 \end{bmatrix}$$

and know that the only variation is in parameters like L_k and θ_k. Then you can estimate the variation in $\vec{\mathbf{p}}_3$ as a function of variation in L_k and θ_k by remembering your calculus.

$$\Delta\vec{\mathbf{p}}_3 \cong \begin{bmatrix} \frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{L}} & \frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{\theta}} \end{bmatrix} \begin{bmatrix} \Delta\vec{L} \\ \Delta\vec{\theta} \end{bmatrix}$$

Parametric Sensitivity

Grinding this out gives:

$$\Delta\vec{\mathbf{p}}_3 \cong \begin{bmatrix} \frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{L}} & \frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{\theta}} \end{bmatrix} \begin{bmatrix} \Delta\vec{L} \\ \Delta\vec{\theta} \end{bmatrix}$$

where

$$\vec{L} = [L_1, L_2, L_3]^T$$

$$\vec{\theta} = [\theta_1, \theta_2, \theta_3]^T$$

$$\frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{L}} = \begin{bmatrix} \cos(\theta_1) & \cos(\theta_1+\theta_2) & \cos(\theta_1+\theta_2+\theta_3) \\ \sin(\theta_1) & \sin(\theta_1+\theta_2) & \sin(\theta_1+\theta_2+\theta_3) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{\theta}} = \begin{bmatrix} -L_1\sin(\theta_1)-L_2\sin(\theta_1+\theta_2)-L_3\sin(\theta_1+\theta_2+\theta_3) & -L_2\sin(\theta_1+\theta_2)-L_3\sin(\theta_1+\theta_2+\theta_3) & -L_3\sin(\theta_1+\theta_2+\theta_3) \\ L_1\cos(\theta_1)+L_2\cos(\theta_1+\theta_2)+L_3\cos(\theta_1+\theta_2+\theta_3) & L_2\cos(\theta_1+\theta_2)+L_3\cos(\theta_1+\theta_2+\theta_3) & L_3\cos(\theta_1+\theta_2+\theta_3) \\ 0 & 0 & 0 \end{bmatrix}$$

More generally …

Suppose that we have a vector function

$$\vec{\mathbf{v}} = \vec{\mathbf{g}}(\vec{\mathbf{q}}) \quad = [g_1(\vec{\mathbf{q}}),\ldots,g_m(\vec{\mathbf{q}})]^T$$

of parameters $\vec{\mathbf{q}} = [q_1,\ldots,q_n]$. Then we can estimate the value of

$$\vec{\mathbf{v}} + \Delta\vec{\mathbf{v}} = \vec{\mathbf{g}}(\vec{\mathbf{q}} + \Delta\vec{\mathbf{q}})$$

by

$$\vec{\mathbf{v}} + \Delta\vec{\mathbf{v}} \approx \vec{\mathbf{g}}(\vec{\mathbf{q}}) + \mathbf{J}_{\mathbf{G}}(\vec{\mathbf{q}}) \bullet \Delta\vec{\mathbf{q}}$$

where

$$\mathbf{J}_{\mathbf{G}}(\vec{\mathbf{q}}) = \begin{bmatrix} \frac{\partial g_1}{\partial q_1} & & \frac{\partial g_1}{\partial q_j} & & \frac{\partial g_1}{\partial q_n} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_i}{\partial q_1} & \cdots & \frac{\partial g_i}{\partial q_j} & \cdots & \frac{\partial g_i}{\partial q_n} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_m}{\partial q_1} & & \frac{\partial g_m}{\partial q_j} & & \frac{\partial g_m}{\partial q_n} \end{bmatrix}$$