

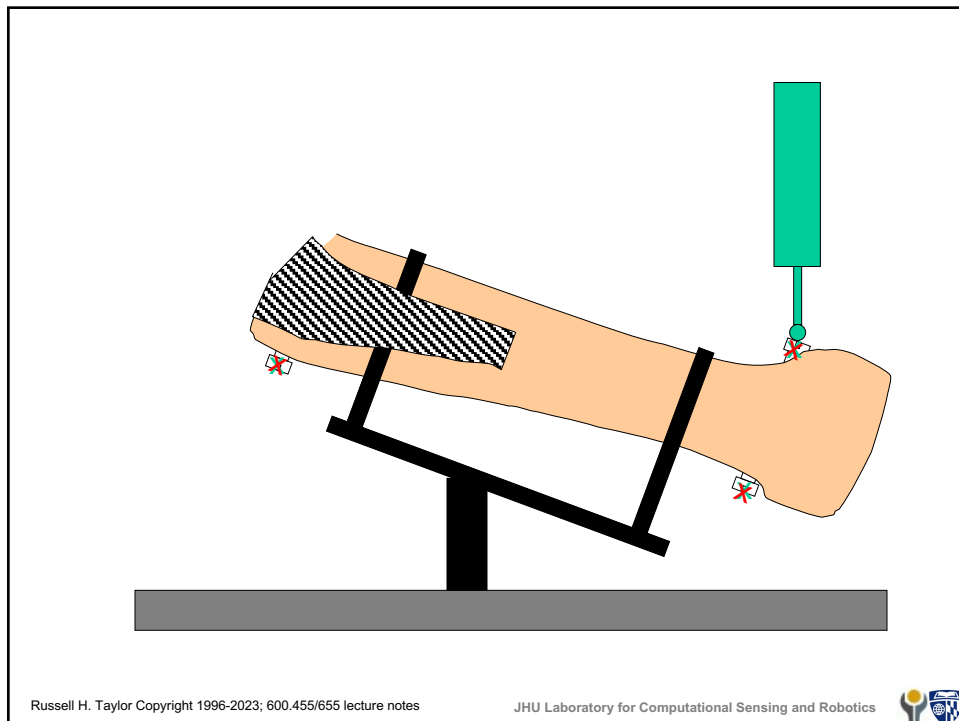
Cartesian Coordinates, Points, and Transformations

CIS - 600.455/655

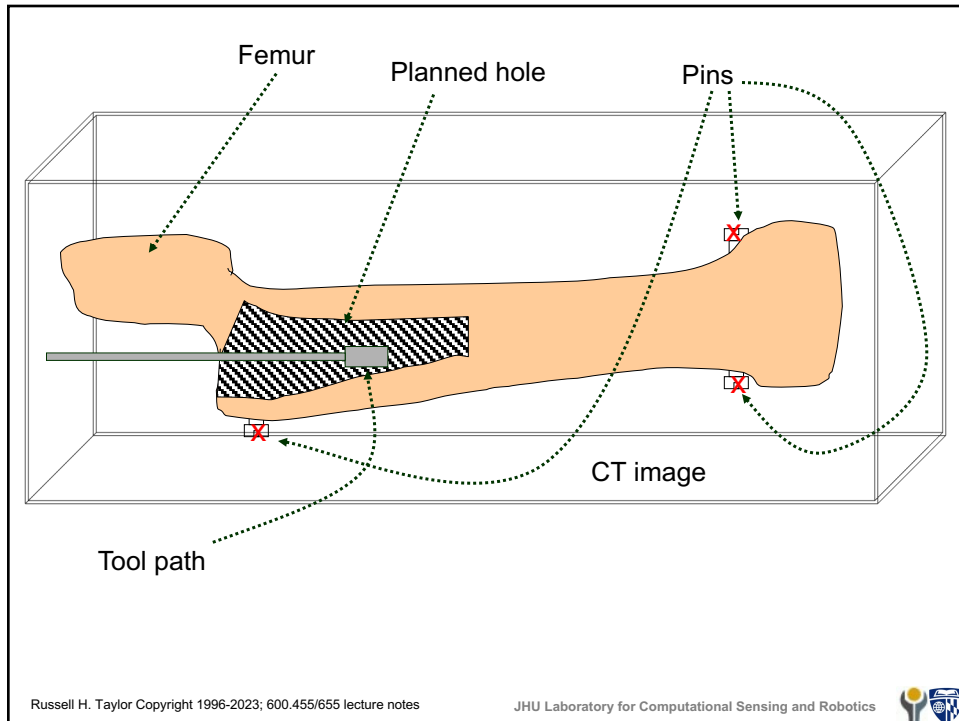
Russell Taylor



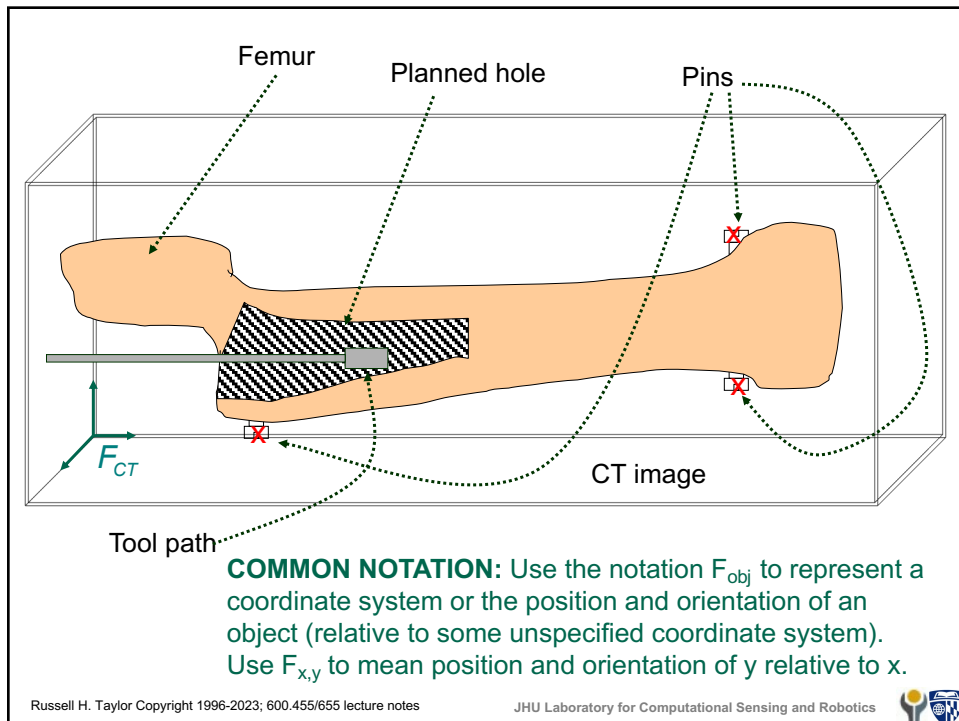
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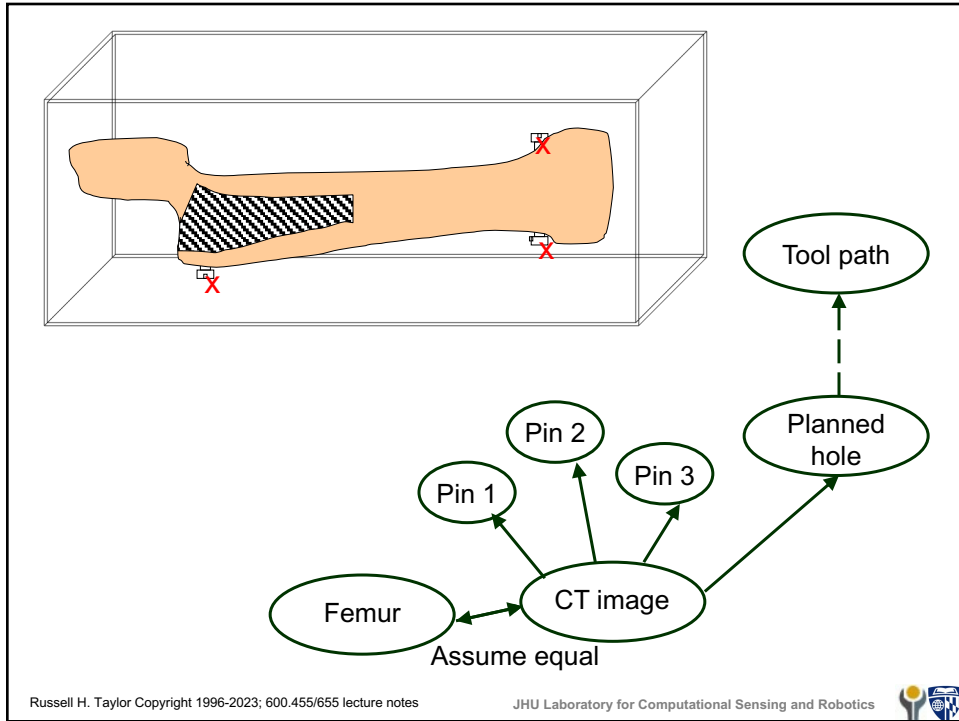
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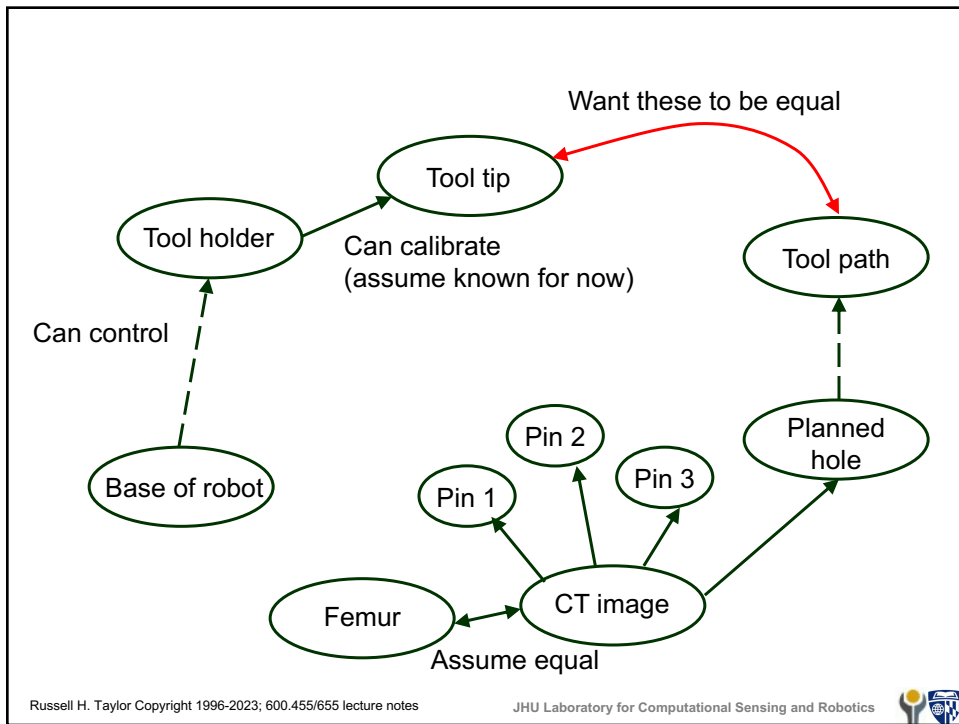
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More correctly, this would be $F_{\text{Base,Wrist}}$

Question: What value of F_{Wrist} will make $F_{\text{Tip}} = F_{\text{Target}}$?

Answer:

$$F_{\text{Wrist}} = F_{\text{Target}} \bullet I \bullet F_{\text{WT}}^{-1}$$

$$= F_{\text{Target}} \bullet F_{\text{WT}}^{-1}$$

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Notational Note
We use the notation $A \bullet B$ to represent composition or transformation. Where the context is clear, we may also use AB for the same thing.

Question: What value of F_{Wrist} will make $F_{\text{Tip}} = F_{\text{Target}}$?

Answer:

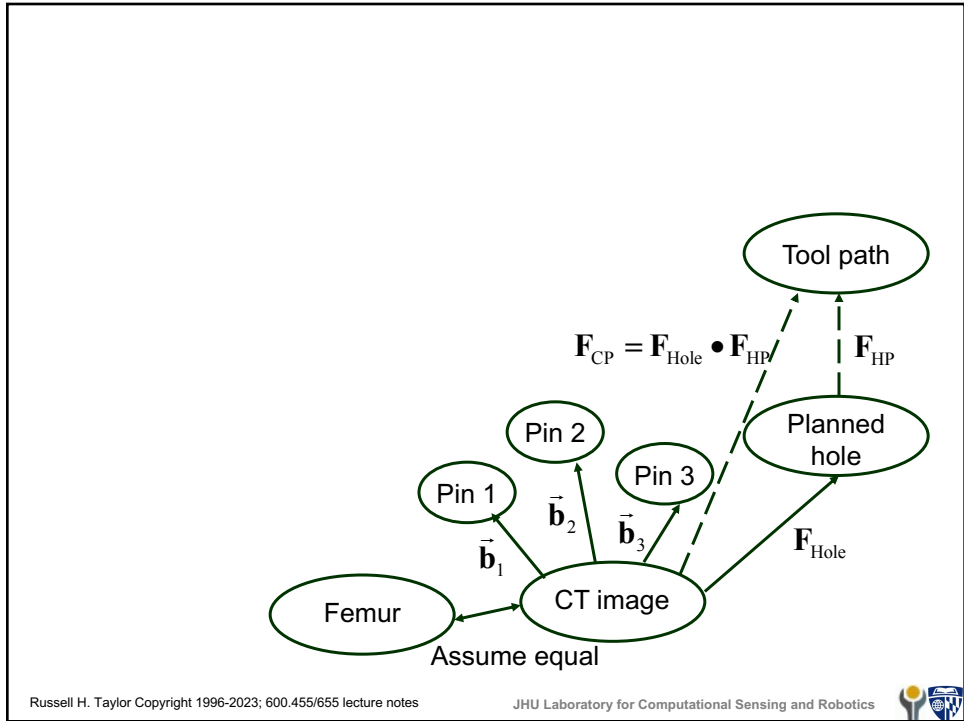
$$F_{\text{Wrist}} = F_{\text{Target}} \bullet I \bullet F_{\text{WT}}^{-1}$$

$$= F_{\text{Target}} \bullet F_{\text{WT}}^{-1}$$

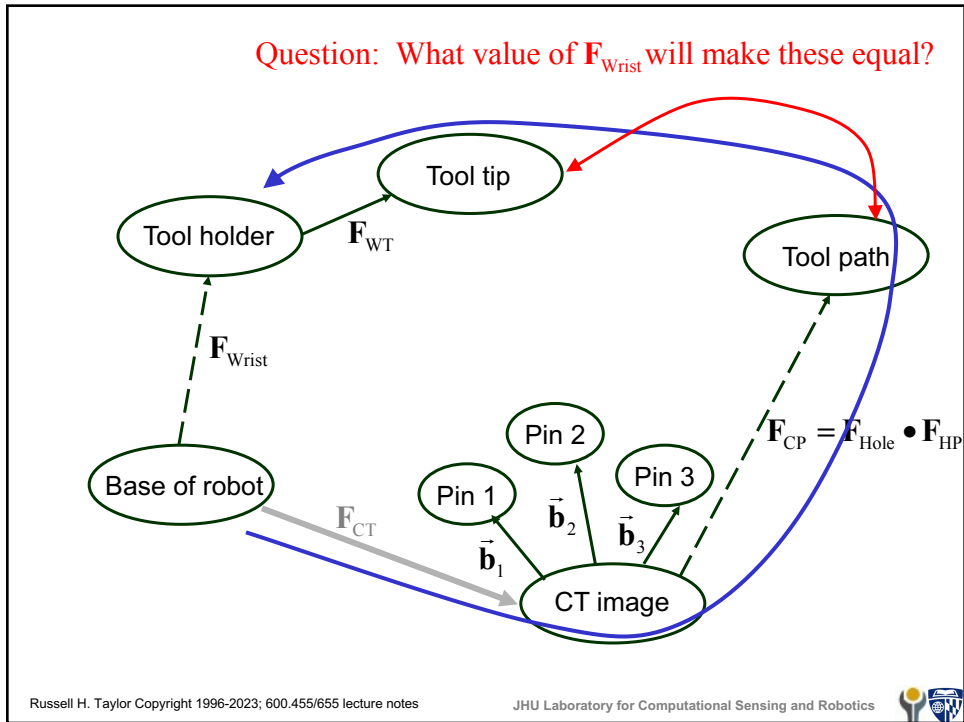
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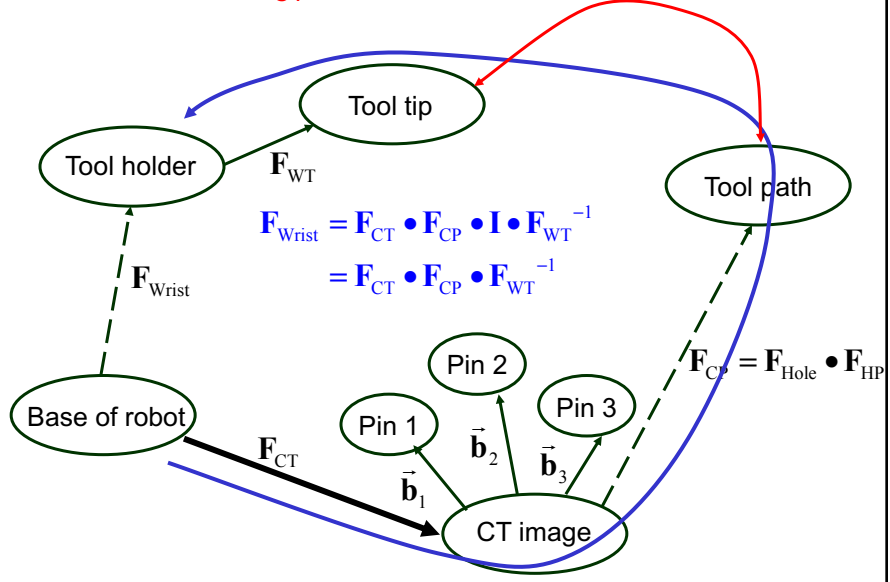


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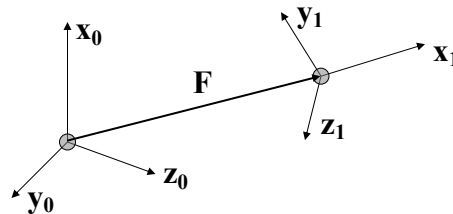
But: We must find F_{CT} ... Let's review some math



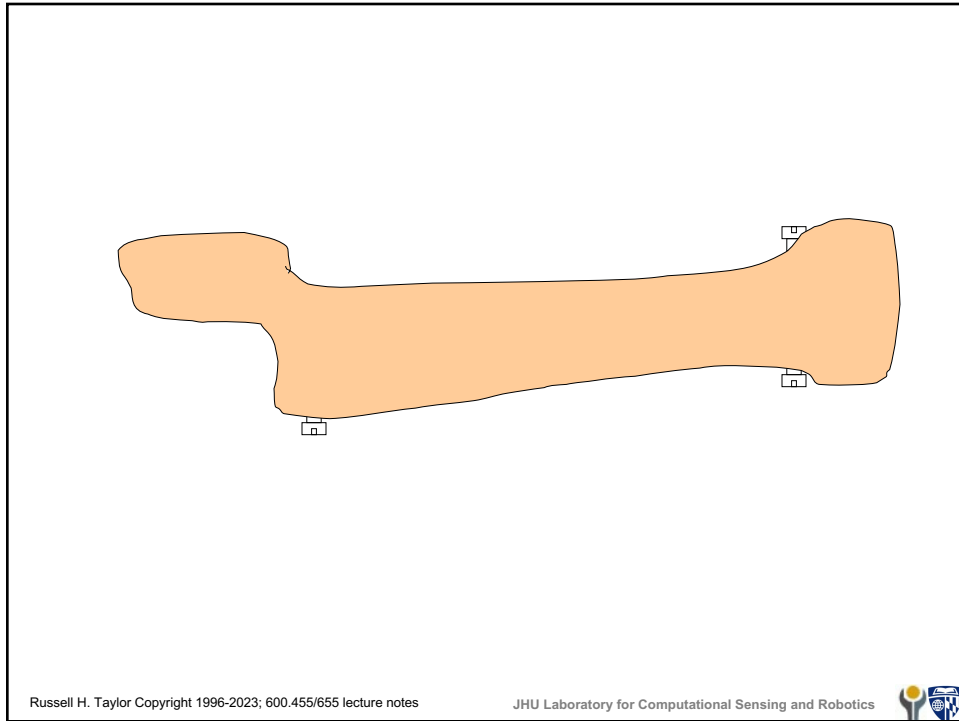
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Coordinate Frame Transformation

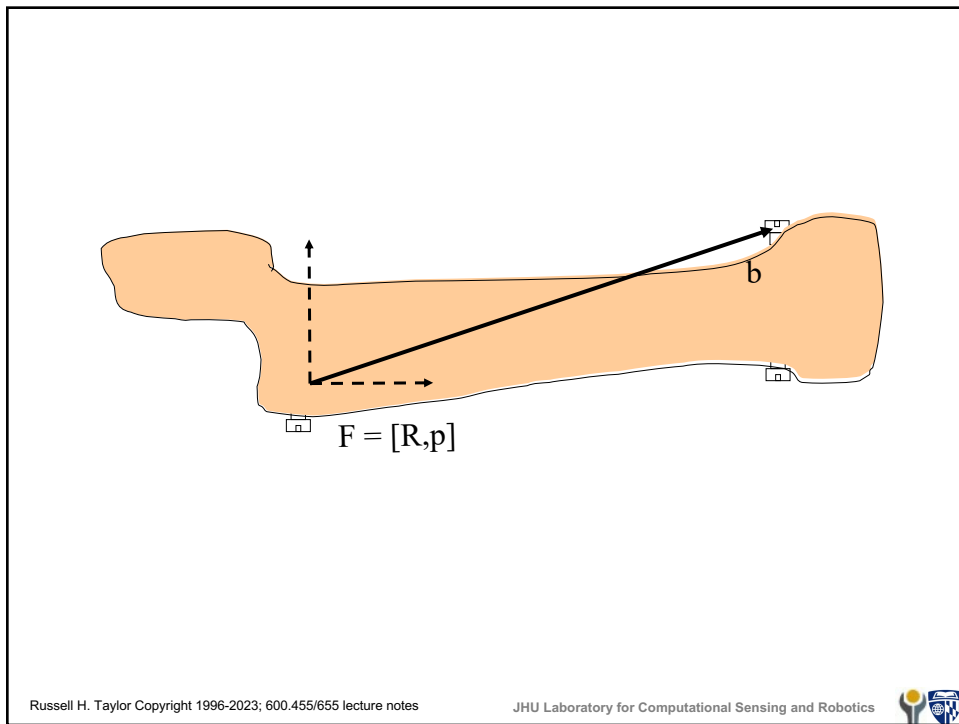
$$F = [R, p]$$



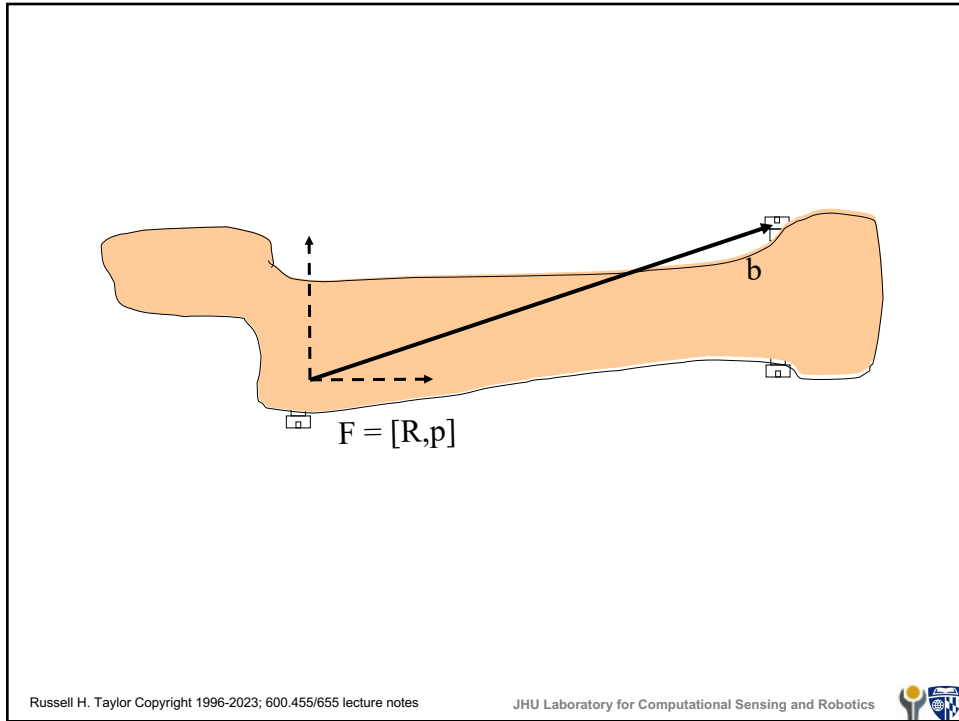
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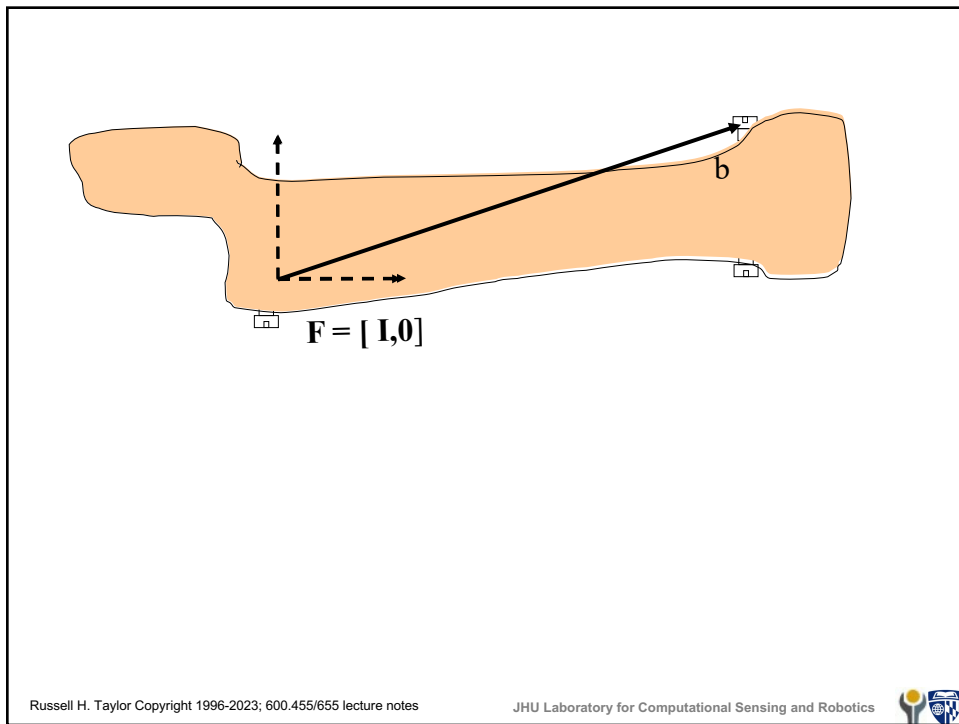
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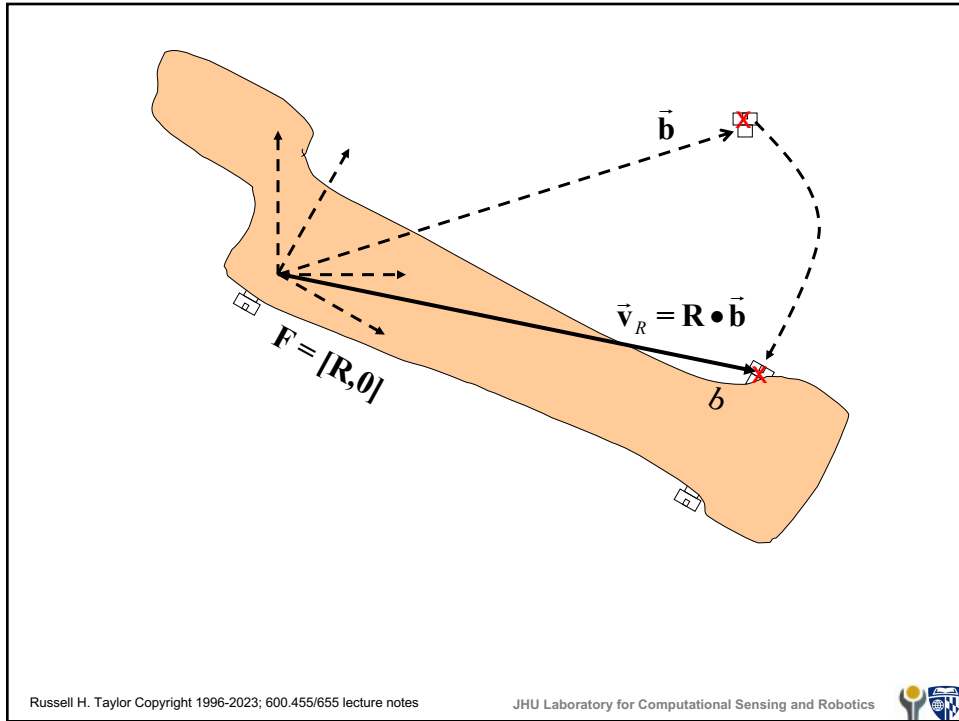
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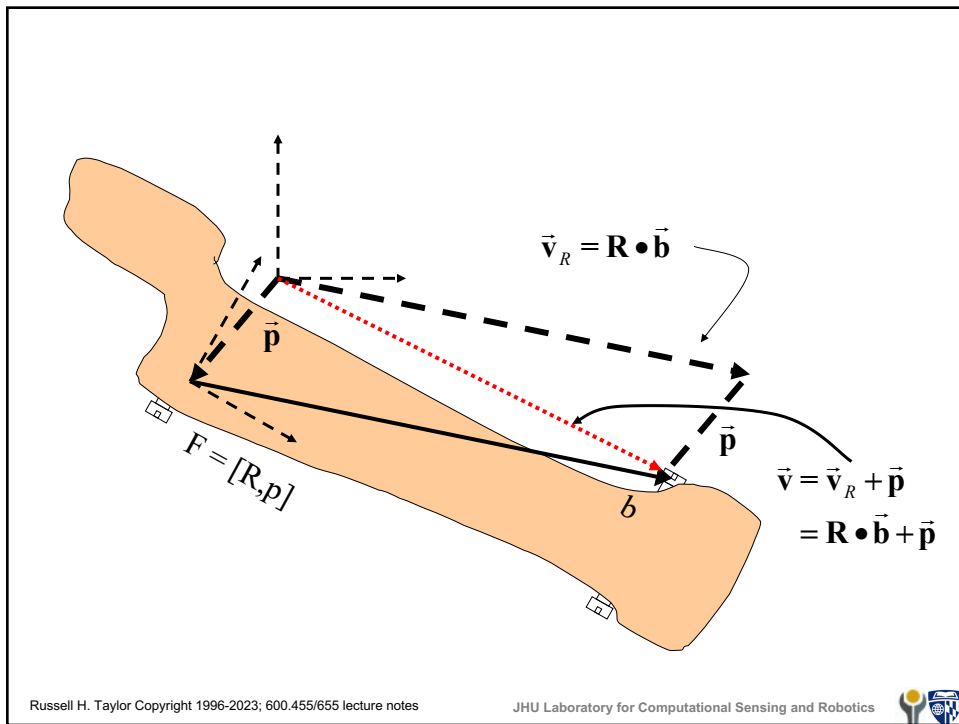
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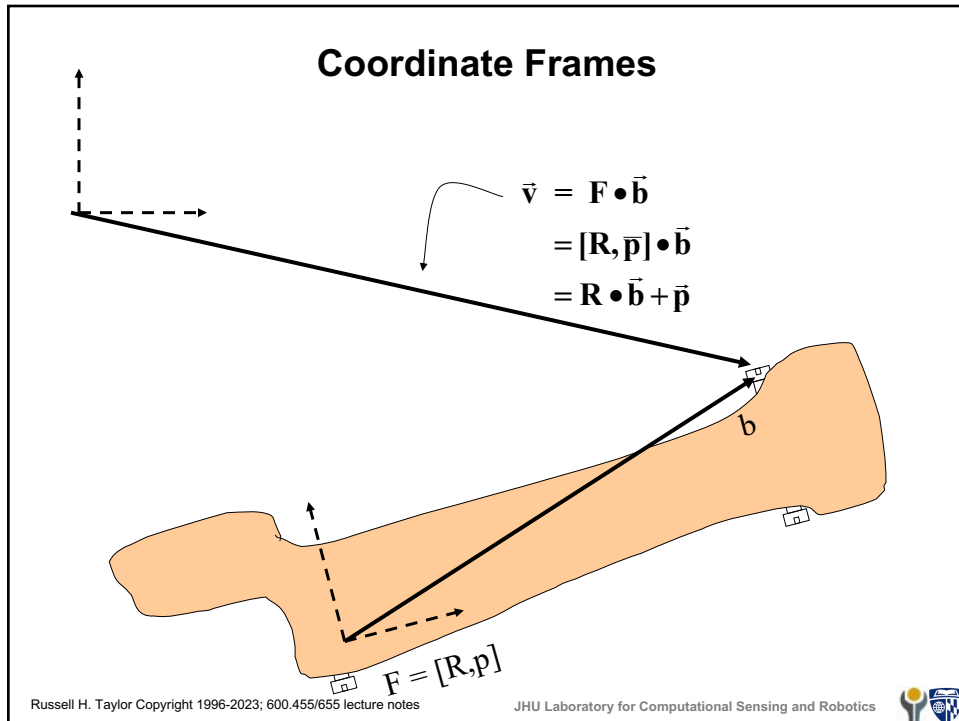
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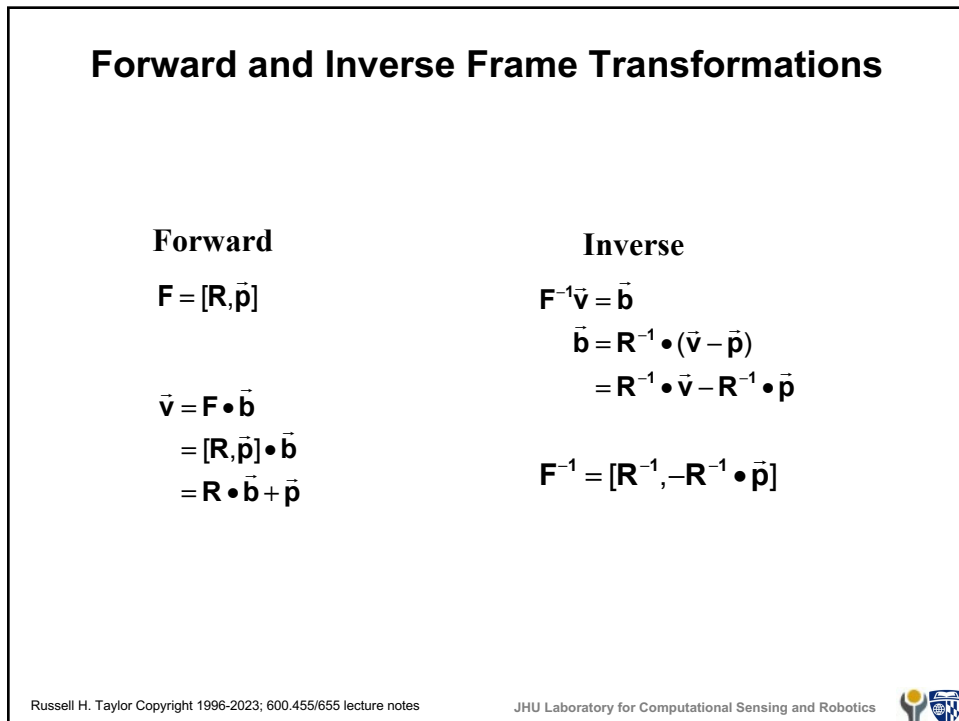
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Composition

Assume $F_1 = [R_1, \vec{p}_1]$, $F_2 = [R_2, \vec{p}_2]$

Then

$$\begin{aligned}
 F_1 \bullet F_2 \bullet \vec{b} &= F_1 \bullet (F_2 \bullet \vec{b}) \\
 &= F_1 \bullet (R_2 \bullet \vec{b} + \vec{p}_2) \\
 &= [R_1, \vec{p}_1] \bullet (R_2 \bullet \vec{b} + \vec{p}_2) \\
 &= R_1 \bullet (R_2 \bullet \vec{b} + \vec{p}_2) + \vec{p}_1 \\
 &= R_1 \bullet R_2 \bullet \vec{b} + R_1 \bullet \vec{p}_2 + \vec{p}_1 \\
 &= [R_1 \bullet R_2, R_1 \bullet \vec{p}_2 + \vec{p}_1] \bullet \vec{b}
 \end{aligned}$$

So

$$\begin{aligned}
 F_1 \bullet F_2 &= [R_1, \vec{p}_1] \bullet [R_2, \vec{p}_2] \\
 &= [R_1 \bullet R_2, R_1 \vec{p}_2 + \vec{p}_1]
 \end{aligned}$$

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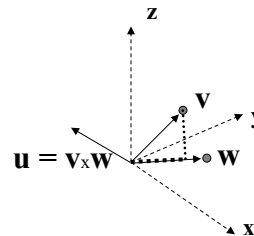


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Vectors

$$\vec{v}_{col} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\vec{v}_{row} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}$$



$$\text{length: } \|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$\text{dot product: } a = \vec{v} \cdot \vec{w} = (v_x w_x + v_y w_y + v_z w_z) = \|\vec{v}\| \|\vec{w}\| \cos \theta$$

$$\text{cross product: } \vec{u} = \vec{v} \times \vec{w} = \begin{bmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{bmatrix}, \|\vec{u}\| = \|\vec{v}\| \|\vec{w}\| \sin \theta$$

Slide acknowledgment: Sarah Graham and Andy Bzostek

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Matrix representation of cross product operator

Define

$$\hat{\vec{a}} = skew(\vec{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Then

$$\vec{a} \times \vec{v} = skew(\vec{a}) \bullet \vec{v}$$



Rotations: Some Notation

$Rot(\vec{a}, \alpha)$ = Rotation by angle α about axis \vec{a}

$\mathbf{R}_{\vec{a}}(\alpha)$ = Rotation by angle α about axis \vec{a}

$$\mathbf{R}(\vec{a}) = Rot(\vec{a}, \|\vec{a}\|)$$

$$\mathbf{R}_{xyz}(\alpha, \beta, \gamma) = \mathbf{R}(\vec{x}, \alpha) \bullet \mathbf{R}(\vec{y}, \beta) \bullet \mathbf{R}(\vec{z}, \gamma)$$

$$\mathbf{R}_{zyz}(\alpha, \beta, \gamma) = \mathbf{R}(\vec{z}, \alpha) \bullet \mathbf{R}(\vec{y}, \beta) \bullet \mathbf{R}(\vec{z}, \gamma)$$



Rotations: A few useful facts

$$Rot(s\vec{a}, \alpha) \bullet \vec{a} = \vec{a} \quad \text{and} \quad \|Rot(\vec{a}, \alpha) \bullet \vec{b}\| = \|\vec{b}\|$$

$$Rot(\vec{a}, \alpha) = Rot(\hat{\mathbf{a}}, \alpha) \quad \text{where} \quad \hat{\mathbf{a}} = \frac{\vec{a}}{\|\vec{a}\|}$$

NOTE: Unless otherwise stated, we will usually assume that \vec{a} in $Rot(\vec{a}, \theta)$ is a unit vector. I.e., $\|\vec{a}\|=1$.

$$Rot(\vec{a}, \alpha) \bullet Rot(\vec{a}, \beta) = Rot(\vec{a}, \alpha + \beta)$$

$$Rot(\vec{a}, \alpha)^{-1} = Rot(\vec{a}, -\alpha)$$

$$Rot(\vec{a}, 0) \bullet \vec{b} = \vec{b} \quad \text{i.e.,} \quad Rot(\vec{a}, 0) = \mathbf{I}_{Rot} = \text{the identity rotation}$$

$$Rot(\hat{\mathbf{a}}, \alpha) \bullet \vec{b} = (\hat{\mathbf{a}} \bullet \vec{b})\hat{\mathbf{a}} + Rot(\hat{\mathbf{a}}, \alpha) \bullet (\vec{b} - (\hat{\mathbf{a}} \bullet \vec{b})\hat{\mathbf{a}})$$

$$Rot(\hat{\mathbf{a}}, \alpha) \bullet Rot(\hat{\mathbf{b}}, \beta) = Rot(\hat{\mathbf{b}}, \beta) \bullet Rot(Rot(\hat{\mathbf{b}}, -\beta) \bullet \hat{\mathbf{a}}, \alpha)$$

$$Rot(\hat{\mathbf{a}}, \alpha) \bullet \mathbf{R}_\beta = \mathbf{R}_\beta \bullet Rot(\mathbf{R}_\beta^{-1} \bullet \hat{\mathbf{a}}, \alpha)$$

$$\mathbf{R}_\alpha \bullet Rot(\hat{\mathbf{b}}, \beta) = Rot(\mathbf{R}_\alpha \bullet \hat{\mathbf{b}}, \beta) \bullet \mathbf{R}_\alpha$$

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Rotations: more facts

If $\vec{v} = [v_x, v_y, v_z]^T$ then a rotation $\mathbf{R} \bullet \vec{v}$ may be described in terms of the effects of \mathbf{R} on orthogonal unit vectors, $\vec{e}_x = [1, 0, 0]^T$,

$$\vec{e}_y = [0, 1, 0]^T, \quad \vec{e}_z = [0, 0, 1]^T$$

$$\mathbf{R} \bullet \vec{v} = v_x \vec{r}_x + v_y \vec{r}_y + v_z \vec{r}_z$$

where

$$\vec{r}_x = \mathbf{R} \bullet \vec{e}_x$$

$$\vec{r}_y = \mathbf{R} \bullet \vec{e}_y$$

$$\vec{r}_z = \mathbf{R} \bullet \vec{e}_z$$

Note that rotation doesn't affect inner products

$$(\mathbf{R} \bullet \vec{b}) \bullet (\mathbf{R} \bullet \vec{c}) = \vec{b} \bullet \vec{c}$$

or lengths of vectors

$$\|\mathbf{R} \bullet \vec{v}\| = \|\vec{v}\|$$

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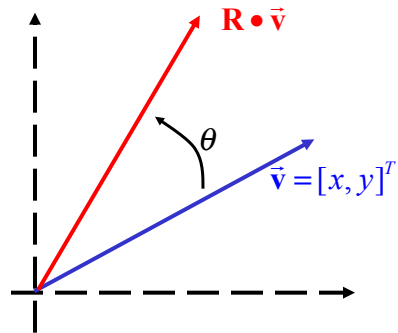


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Rotations in the plane

$$\mathbf{R} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$



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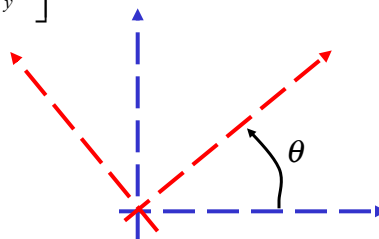
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Rotations in the plane

$$\mathbf{R} \cdot \begin{bmatrix} \vec{e}_x & \vec{e}_y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R} \cdot \vec{e}_x & \mathbf{R} \cdot \vec{e}_y \end{bmatrix}$$

$$= \begin{bmatrix} \vec{r}_x & \vec{r}_y \end{bmatrix}$$



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3D Rotation Matrices

$$\mathbf{R} \bullet \begin{bmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \end{bmatrix} = \begin{bmatrix} \mathbf{R} \bullet \vec{e}_x & \mathbf{R} \bullet \vec{e}_y & \mathbf{R} \bullet \vec{e}_z \end{bmatrix} \\ = \begin{bmatrix} \vec{r}_x & \vec{r}_y & \vec{r}_z \end{bmatrix}$$

$$\mathbf{R}^T \bullet \mathbf{R} = \begin{bmatrix} \hat{\mathbf{r}}_x^T \\ \hat{\mathbf{r}}_y^T \\ \hat{\mathbf{r}}_z^T \end{bmatrix} \bullet \begin{bmatrix} \vec{r}_x & \vec{r}_y & \vec{r}_z \end{bmatrix} \\ = \begin{bmatrix} \vec{r}_x^T \bullet \vec{r}_x & \vec{r}_x^T \bullet \vec{r}_y & \vec{r}_x^T \bullet \vec{r}_z \\ \vec{r}_y^T \bullet \vec{r}_x & \vec{r}_y^T \bullet \vec{r}_y & \vec{r}_y^T \bullet \vec{r}_z \\ \vec{r}_z^T \bullet \vec{r}_x & \vec{r}_z^T \bullet \vec{r}_y & \vec{r}_z^T \bullet \vec{r}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Properties of Rotation Matrices

Inverse of a Rotation Matrix equals its transpose:

$$\mathbf{R}^{-1} = \mathbf{R}^T \\ \mathbf{R}^T \mathbf{R} = \mathbf{R} \mathbf{R}^T = \mathbf{I}$$

The Determinant of a Rotation matrix is equal to +1:

$$\det(\mathbf{R}) = +1$$

Any Rotation can be described by consecutive rotations about the three primary axes, x, y, and z:

$$\mathbf{R} = \mathbf{R}_{z,\theta} \mathbf{R}_{y,\phi} \mathbf{R}_{x,\psi}$$



Canonical 3D Rotation Matrices

Note: Right-Handed Coordinate System

$$\mathbf{R}_{\bar{x}}(\theta) = \text{Rot}(\bar{x}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\bar{y}}(\theta) = \text{Rot}(\bar{y}, \theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\bar{z}}(\theta) = \text{Rot}(\bar{z}, \theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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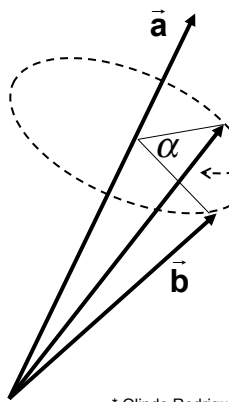
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Axis-angle Representations of Rotations

Rodrigues' Formula (1840)*

Rotation of a vector $\bar{\mathbf{b}}$ by angle α about axis $\bar{\mathbf{a}}$

(Assumes that $\bar{\mathbf{a}}$ is a unit vector, $\|\bar{\mathbf{a}}\| = 1$)



$$\begin{aligned} \bar{\mathbf{c}} &= \text{Rot}(\bar{\mathbf{a}}, \alpha) \cdot \bar{\mathbf{b}} \\ &= \bar{\mathbf{b}} \cos \alpha + \bar{\mathbf{a}} \times \bar{\mathbf{b}} \sin \alpha + \bar{\mathbf{a}} (\bar{\mathbf{a}} \cdot \bar{\mathbf{b}}) (1 - \cos \alpha) \end{aligned}$$

In matrix form this is $\bar{\mathbf{c}} = \mathbf{R} \cdot \bar{\mathbf{b}}$ where

$$\mathbf{R} = (\cos \alpha) \mathbf{I} + (\sin \alpha) \text{skew}(\bar{\mathbf{a}}) + (1 - \cos \alpha) \bar{\mathbf{a}} \cdot \bar{\mathbf{a}}^T$$

* Olinde Rodrigues, "Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendants des causes qui peuvent les produire", *Journal de Mathématiques Pures et Appliquées* 5 (1840), 380–440. (http://sites.mathdoc.fr/JMPA/PDF/JMPA_1840_1_5_A39_0.pdf)

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Exponential representation

Consider the matrix exponential of $\text{skew}(\theta\vec{n})$, where $\|\vec{n}\| = 1$. We have

$$e^{\text{skew}(\vec{n})\theta} = \mathbf{I} + \theta \text{skew}(\vec{n}) + \frac{\theta^2}{2!} \text{skew}(\vec{n})^2 + \dots + \frac{\theta^k}{k!} \text{skew}(\vec{n})^k + \dots$$

Since $\text{skew}(\vec{n})^3 = -\text{skew}(\vec{n})$, by doing some manipulation, you can show

$$\begin{aligned} e^{\text{skew}(\vec{n})\theta} &= \mathbf{I} + \text{skew}(\vec{n})\sin\theta + \text{skew}(\vec{n})^2(1 - \cos\theta) \\ &= \mathbf{I} + \text{skew}(\vec{n})\sin\theta + (\vec{n}\bullet\vec{n}^T - \mathbf{I})(1 - \cos\theta) \\ &= \mathbf{I}\cos\theta + \text{skew}(\vec{n})\sin\theta + \vec{n}\bullet\vec{n}^T(1 - \cos\theta) \end{aligned}$$

which is just Rodrigues' formula for $\text{Rot}(\vec{n}, \theta)$.

Note that for small θ , this reduces to

$$\text{Rot}(\vec{n}, \theta) \approx \mathbf{I} + \text{skew}(\theta\vec{n})$$



Cayley Transform Representation

Consider the rotation $\text{Rot}(\vec{n}, \theta)$ and define

$$\vec{a} = \left(\tan \frac{\theta}{2} \right) \vec{n}$$

$$\mathbf{A} = \text{skew}(\vec{a})$$

Then,

$$\mathbf{R} = (\mathbf{I} - \mathbf{A})^{-1}(\mathbf{I} + \mathbf{A}) = (\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A})^{-1}$$

gives the rotation matrix corresponding to $\text{Rot}(\vec{n}, \theta)$. Similarly, given \mathbf{R} ,

$$\mathbf{A} = (\mathbf{R} - \mathbf{I})(\mathbf{I} + \mathbf{R})^{-1}$$

gives the elements of $\text{skew}(\vec{a})$ and hence for \vec{a} .

Note: The above relations require that $\theta \neq \pm\pi$.



Note on the difference between two rotations

One often wants to consider the "difference" between two rotations

$$\mathbf{R}_{12} = \mathbf{R}_1^{-1} \cdot \mathbf{R}_2$$

In these cases, it is useful to consider an axis-angle representation

$$\mathbf{R}_{12} = \text{Rot}(\bar{\mathbf{n}}_{12}, \theta_{12})$$

Given \mathbf{R}_{12} , there are several ways to extract $\bar{\mathbf{n}}_{12}$ and θ_{12} . For example, you can use the Cayley formula to compute $\bar{\mathbf{a}}_{12} = \tan(\theta_{12}/2)\bar{\mathbf{n}}_{12}$ from

$$sk(\bar{\mathbf{a}}_{12}) = (\mathbf{R}_{12} - \mathbf{I})(\mathbf{I} + \mathbf{R}_{12})^{-1}$$

and then

$$\theta_{12} = 2 \arctan(\|\bar{\mathbf{a}}_{12}\|) \quad \text{and} \quad \bar{\mathbf{n}}_{12} = \bar{\mathbf{a}}_{12} / \|\bar{\mathbf{a}}_{12}\|$$

Also, if the rotations are very close to each other so that θ_{12} is small, then

$$\mathbf{R}_{12} \approx \mathbf{I} + sk(\bar{\alpha}_{12})$$

So

$$\theta_{12} \approx \|\bar{\alpha}_{12}\| \quad \text{and} \quad \bar{\mathbf{n}}_{12} \approx \bar{\alpha}_{12} / \theta_{12}$$

This last relationship is very useful in reporting things like registration error since the elements $\bar{\alpha}_{12}$ are small rotations about the x,y,z axes



Homogeneous Coordinates

- Widely used in graphics, geometric calculations
- Represent 3D vector as 4D quantity
- For our current purposes, we will keep the "scale" $s = 1$

$$\vec{\mathbf{V}} \equiv \begin{bmatrix} xS \\ yS \\ zS \\ S \end{bmatrix} \cong \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Representing Frame Transformations as Matrices

$$\mathbf{v} + \mathbf{p} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \mathbf{p}_x \\ 0 & 1 & 0 & \mathbf{p}_y \\ 0 & 0 & 1 & \mathbf{p}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ \mathbf{v}_z \\ 1 \end{bmatrix} = [\mathbf{I}, \bar{\mathbf{p}}] \bullet \mathbf{v}$$

$$\mathbf{R} \bullet \mathbf{v} \rightarrow \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}$$

$$\mathbf{P} \bullet \mathbf{R} \rightarrow \begin{bmatrix} \mathbf{I} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \bullet \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} = [\mathbf{R}, \mathbf{p}] = \mathbf{F}$$

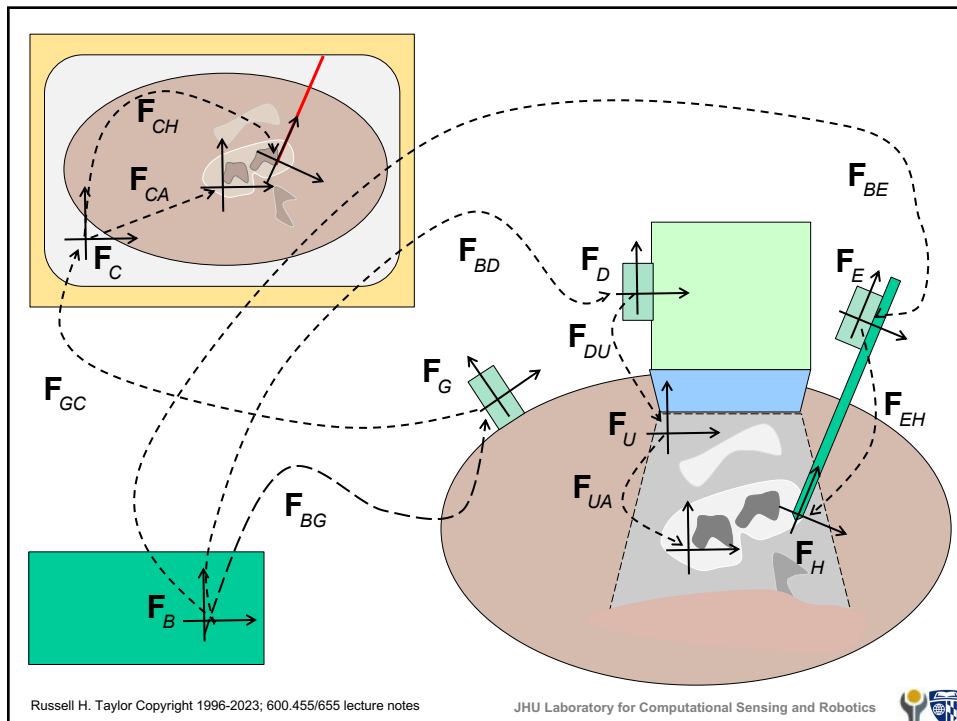
$$\mathbf{F} \bullet \mathbf{v} \rightarrow \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = \begin{bmatrix} (\mathbf{R} \bullet \mathbf{v}) + \mathbf{p} \\ 1 \end{bmatrix}$$

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Give a formula for computing the pose F_{GH} of the surgical tool coordinate system relative to the patient rigid body coordinate system F_G

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Give a formula for computing the pose F_{GH} of the surgical tool coordinate system relative to the patient rigid body coordinate system F_G ?

What are the components $F_{GH} = [R_{GH}, \vec{p}_{GH}]$?

$$F_{GH} = F_{BG}^{-1} F_{BE} F_{EH}$$

$$= F_{BG}^{-1} F_{BH}$$

$$R_{GH} = R_{BG}^{-1} R_{BE} R_{EH}$$

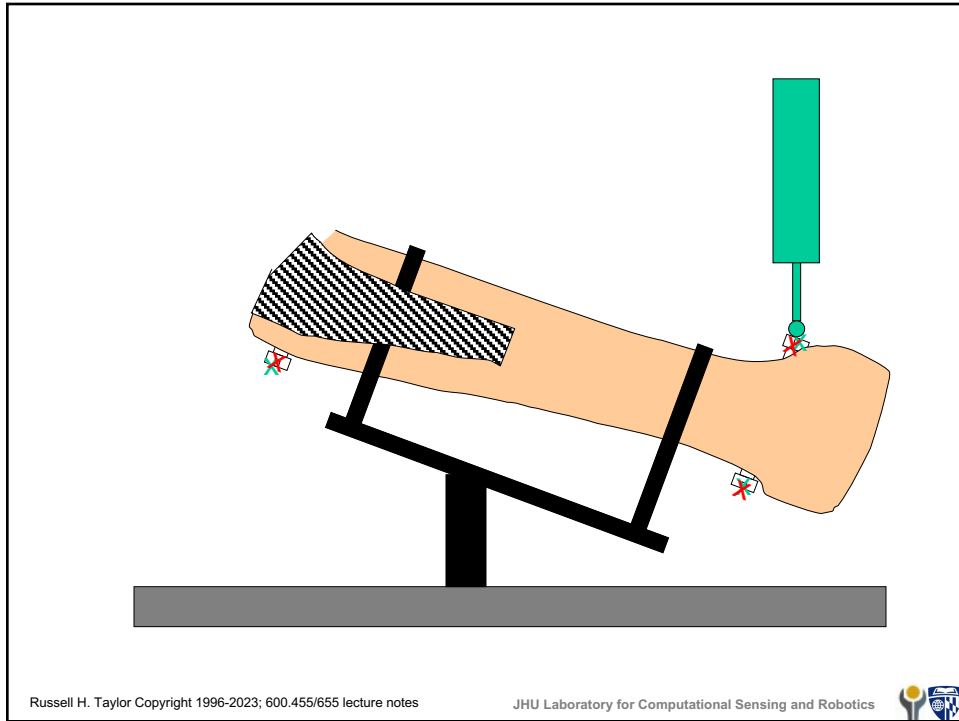
$$\vec{p}_{GH} = F_{BG}^{-1} \vec{p}_{BH} = F_{BG}^{-1} F_{BE} \vec{p}_{EH}$$

$$\vec{p}_{GH} = F_{BG}^{-1} (R_{BE} \vec{p}_{EH} + \vec{p}_{BE})$$

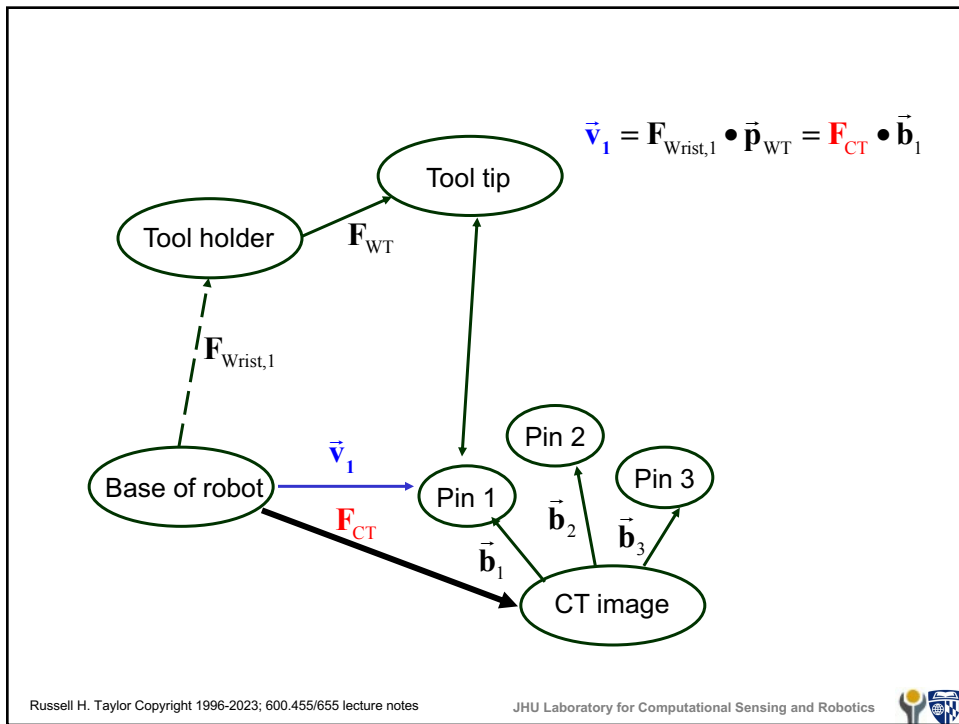
$$\vec{p}_{GH} = R_{BG}^{-1} (R_{BE} \vec{p}_{EH} + \vec{p}_{BE} - \vec{p}_{BG})$$

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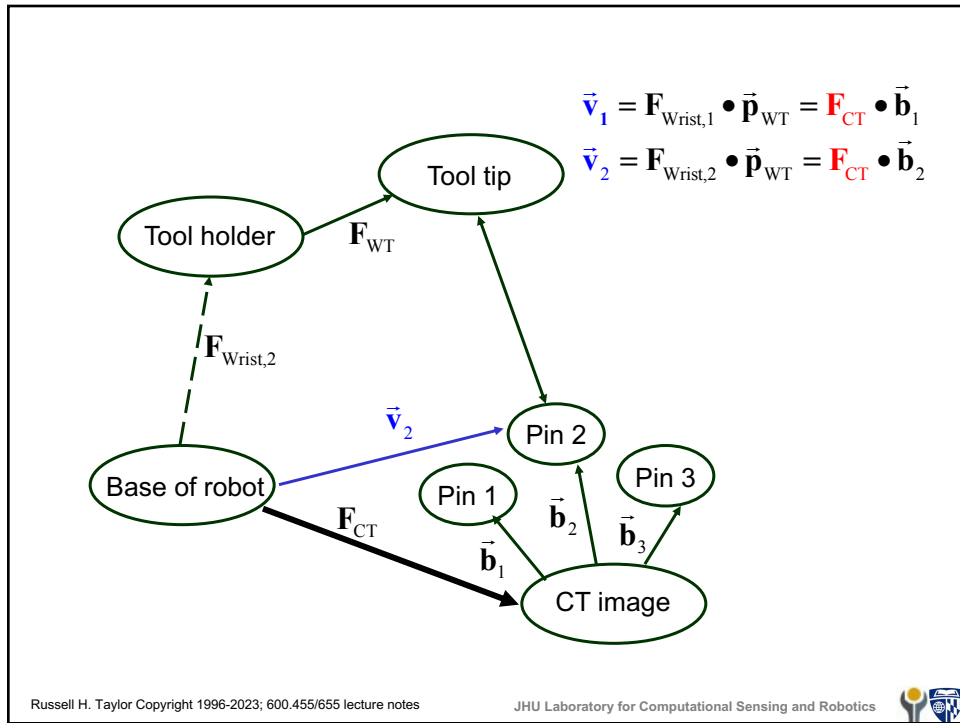
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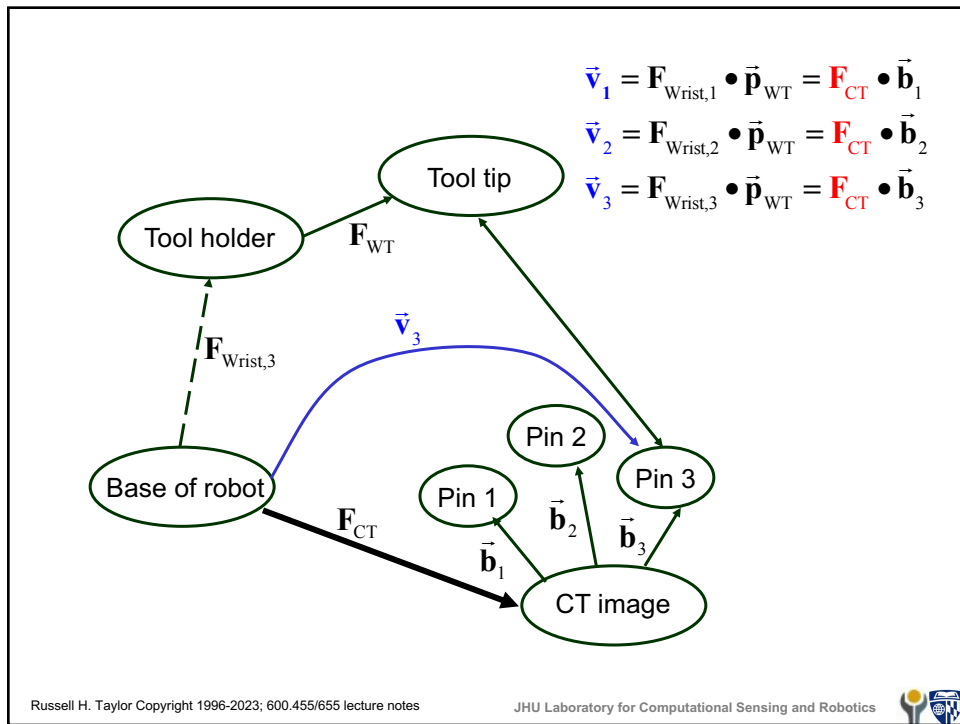
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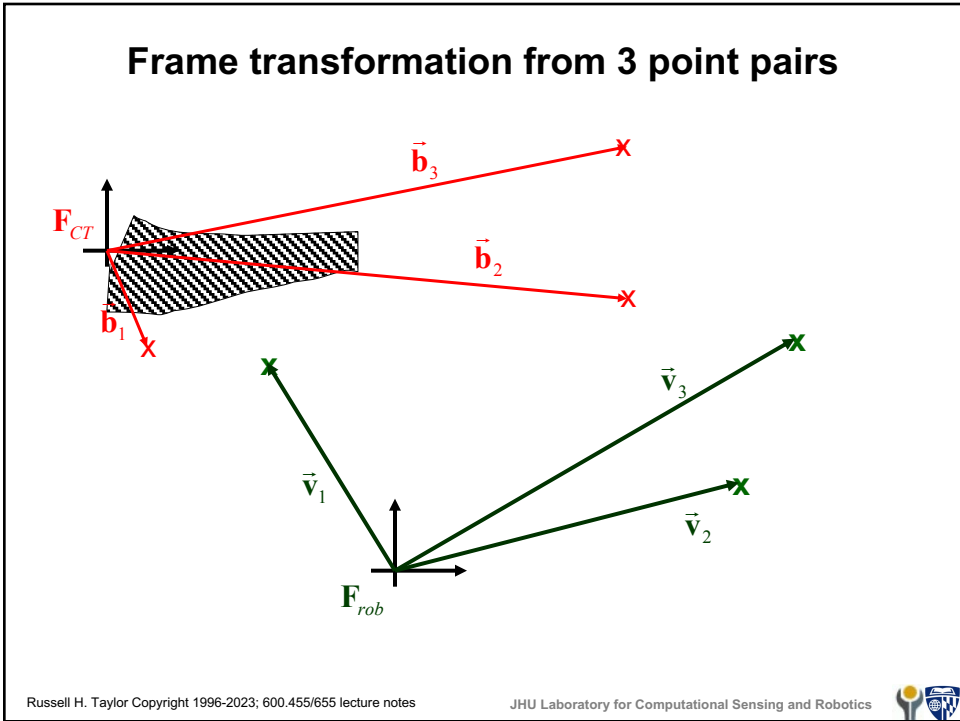
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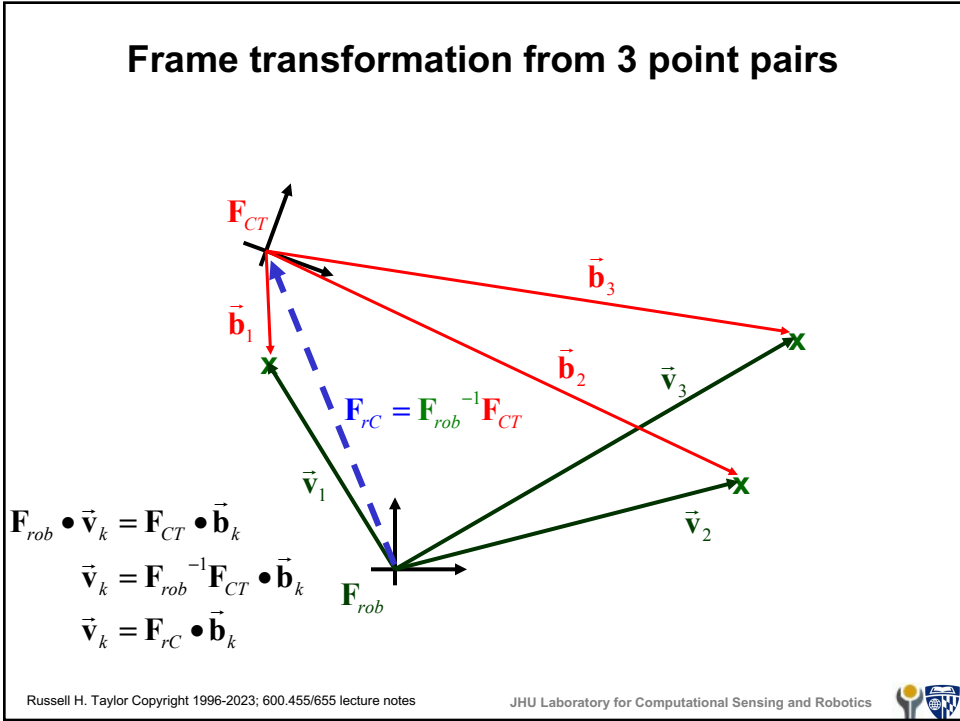
49



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Frame transformation from 3 point pairs

$$\vec{v}_k = \mathbf{F}_{rC} \vec{b}_k = \mathbf{R}_{rC} \vec{b}_k + \vec{p}_{rC}$$

Define

$$\vec{v}_m = \frac{1}{3} \sum_1^3 \vec{v}_k \quad \vec{b}_m = \frac{1}{3} \sum_1^3 \vec{b}_k$$

$$\vec{u}_k = \vec{v}_k - \vec{v}_m \quad \vec{a}_k = \vec{b}_k - \vec{b}_m$$

$$\mathbf{F}_{rC} \vec{a}_k = \mathbf{R}_{rC} \vec{a}_k + \vec{p}_{rC}$$

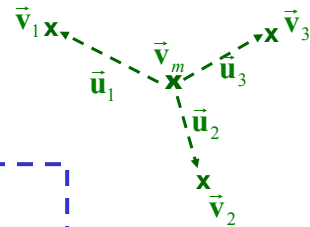
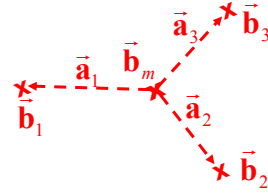
$$\mathbf{R}_{rC} \vec{a}_k + \vec{p}_{rC} = \mathbf{R}_{rC} (\vec{b}_k - \vec{b}_m) + \vec{p}_{rC}$$

$$\mathbf{R}_{rC} \vec{a}_k = \mathbf{R}_{rC} \vec{b}_k + \vec{p}_{rC} - \mathbf{R}_{rC} \vec{b}_m - \vec{p}_{rC}$$

$$\mathbf{R}_{rC} \vec{a}_k = \vec{v}_k - \vec{v}_m = \vec{u}_k$$

$$\vec{p}_{rC} = \vec{v}_m - \mathbf{R}_{rC} \vec{b}_m$$

**Solve
These!!**



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Rotation from multiple vector pairs

Given a system $\mathbf{R}\vec{a}_k = \vec{u}_k$ for $k=1, \dots, n$ the problem is to estimate \mathbf{R} . This will require at least three such point pairs. Later in the course we will cover some good ways to solve this system. Here is a not-so-good way that will produce roughly correct answers:

Step 1: Form matrices $\mathbf{U} = [\vec{u}_1 \ \dots \ \vec{u}_n]$ and $\mathbf{A} = [\vec{a}_1 \ \dots \ \vec{a}_n]$

Step 2: Solve the system $\mathbf{R}\mathbf{A} = \mathbf{U}$ for \mathbf{R} . E.g., by $\mathbf{R} = \mathbf{U}\mathbf{A}^{-1}$

Step 3: Renormalize \mathbf{R} to guarantee $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

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Renormalizing Rotation Matrix

Given "rotation" matrix $\mathbf{R} = [\vec{r}_x \mid \vec{r}_y \mid \vec{r}_z]$, modify it so $\mathbf{R}^T \mathbf{R} = \mathbf{I}$.

Step 1: $\vec{a} = \vec{r}_y \times \vec{r}_z$

Step 2: $\vec{b} = \vec{r}_z \times \vec{a}$

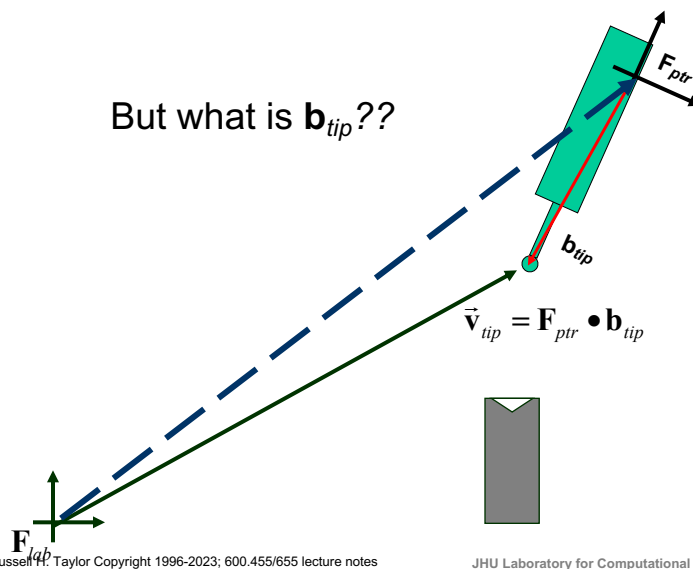
Step 3: $\mathbf{R}_{normalized} = \left[\begin{array}{c|c|c} \frac{\vec{a}}{\|\vec{a}\|} & \frac{\vec{b}}{\|\vec{b}\|} & \frac{\vec{r}_z}{\|\vec{r}_z\|} \end{array} \right]$



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Calibrating a pointer

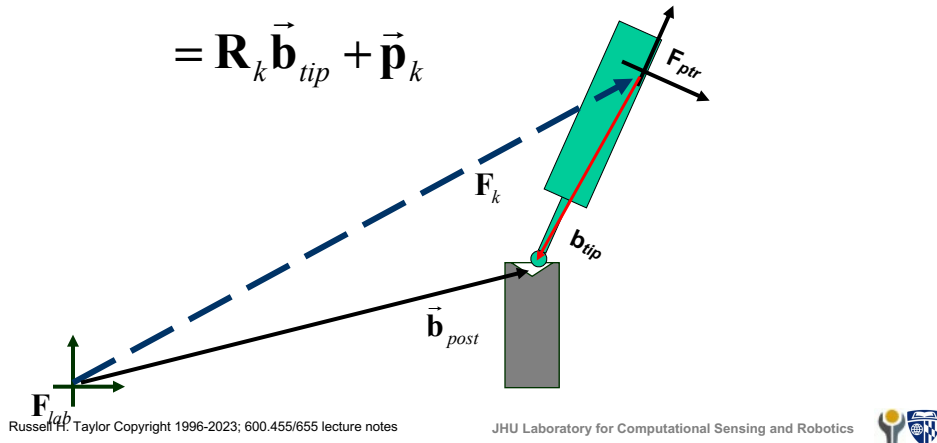
But what is \mathbf{b}_{tip} ??



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Calibrating a pointer

$$\begin{aligned}\vec{\mathbf{b}}_{post} &= \mathbf{F}_k \vec{\mathbf{b}}_{tip} \\ &= \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k\end{aligned}$$



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Calibrating a pointer

For each measurement k , we have

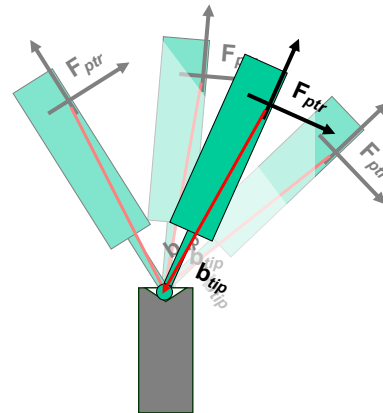
$$\vec{\mathbf{b}}_{post} = \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k$$

i. e.,

$$\mathbf{R}_k \vec{\mathbf{b}}_{tip} - \vec{\mathbf{b}}_{post} = -\vec{\mathbf{p}}_k$$

Set up a least squares problem

$$\begin{bmatrix} \vdots & \vdots \\ \mathbf{R}_k & -\mathbf{I} \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vec{\mathbf{b}}_{tip} \\ \vec{\mathbf{b}}_{post} \end{bmatrix} \cong \begin{bmatrix} \vdots \\ -\vec{\mathbf{p}}_k \\ \vdots \end{bmatrix}$$



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"Registration Transformations"

Given a coordinate system F_C and another coordinate system F_G (e.g., a CT scan and a tracked "rigid body" attached to the patient), and points \vec{c}_i in the coordinate system F_C and points \vec{g}_i in the coordinate system F_G , then the "registration transformation" F_{GC} between F_G and F_C is one in which $F_{GC} \vec{c}_i = \vec{g}_i$ if and only if \vec{c}_i and \vec{g}_i refer to the same or corresponding points.

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Use in surgical navigation

If an anatomic structure is identified at pose F_{UA} in ultrasound image coordinates give the formula for computing the corresponding pose F_{CA} in CT coordinates

NOTE: In cases where there is a reference body (here F_G) affixed to the patient, it is sometimes convenient to assume that the navigation software routinely transforms all tracked markers to the coordinate system of F_G (i.e., $F_{Gx} = F_{BG}^{-1} F_{Bx}$).

$$F_{BA} = F_{BD} F_{DU} F_{UA}$$

$$F_{BA} = F_{BG} F_{GC} F_{CA}$$

$$F_{BG} F_{GC} F_{CA} = F_{BD} F_{DU} F_{UA}$$

$$F_{CA} = (F_{BG} F_{GC})^{-1} F_{BD} F_{DU} F_{UA}$$

$$F_{GC} = F_{BG}^{-1} F_{BD} F_{DU} F_{UA} F_{CA}^{-1}$$

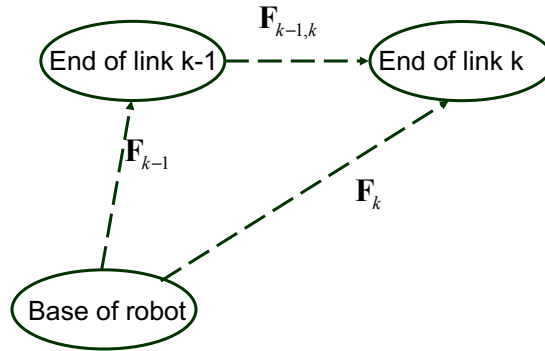
$$F_{GC} = F_{GD} F_{DU} F_{UA} F_{CA}^{-1}$$

where $F_{GD} = F_{BG}^{-1} F_{BD}$

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Kinematic Links



$$\mathbf{F}_k = \mathbf{F}_{k-1} \bullet \mathbf{F}_{k-1,k}$$

$$\left[\mathbf{R}_k, \bar{\mathbf{p}}_k \right] = \left[\mathbf{R}_{k-1}, \mathbf{p}_{k-1} \right] \bullet \left[\mathbf{R}_{k-1,k}, \mathbf{p}_{k-1,k} \right]$$

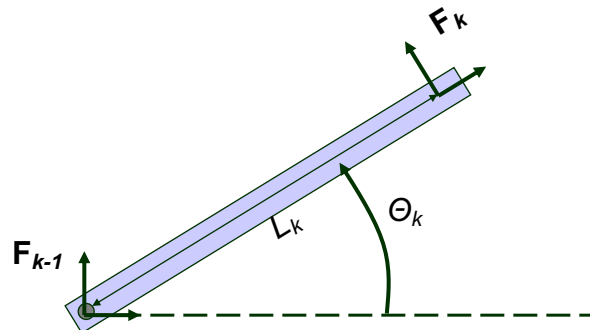
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Kinematic Links



$$\mathbf{F}_k = \mathbf{F}_{k-1} \bullet \mathbf{F}_{k-1,k}$$

$$\left[\mathbf{R}_k, \bar{\mathbf{p}}_k \right] = \left[\mathbf{R}_{k-1}, \mathbf{p}_{k-1} \right] \bullet \left[\mathbf{R}_{k-1,k}, \mathbf{p}_{k-1,k} \right]$$

$$= \left[\mathbf{R}_{k-1}, \mathbf{p}_{k-1} \right] \bullet \left[\text{Rot}(\vec{\mathbf{r}}_k, \theta_k), \vec{\mathbf{0}} \right] \bullet \left[\mathbf{I}, L_k \vec{\mathbf{x}} \right]$$

$$= \left[\mathbf{R}_{k-1}, \mathbf{p}_{k-1} \right] \bullet \left[\text{Rot}(\vec{\mathbf{r}}_k, \theta_k), L_k \text{Rot}(\vec{\mathbf{r}}_k, \theta_k) \bullet \vec{\mathbf{x}} \right]$$

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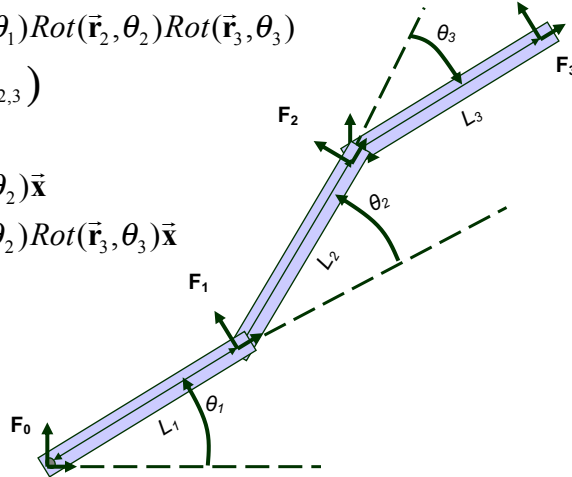
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Kinematic Chains

$$\begin{aligned} \mathbf{F}_0 &= [\mathbf{I}, \vec{\mathbf{0}}] \\ \mathbf{R}_3 &= \mathbf{R}_{0,1} \mathbf{R}_{1,2} \mathbf{R}_{2,3} = \text{Rot}(\vec{\mathbf{r}}_1, \theta_1) \text{Rot}(\vec{\mathbf{r}}_2, \theta_2) \text{Rot}(\vec{\mathbf{r}}_3, \theta_3) \\ \vec{\mathbf{p}}_3 &= \vec{\mathbf{p}}_{0,1} + \mathbf{R}_{0,1} (\vec{\mathbf{p}}_{1,2} + \mathbf{R}_{1,2} \vec{\mathbf{p}}_{2,3}) \\ &= L_1 \text{Rot}(\vec{\mathbf{r}}_1, \theta_1) \vec{\mathbf{x}} \\ &\quad + L_2 \text{Rot}(\vec{\mathbf{r}}_1, \theta_1) \text{Rot}(\vec{\mathbf{r}}_2, \theta_2) \vec{\mathbf{x}} \\ &\quad + L_3 \text{Rot}(\vec{\mathbf{r}}_1, \theta_1) \text{Rot}(\vec{\mathbf{r}}_2, \theta_2) \text{Rot}(\vec{\mathbf{r}}_3, \theta_3) \vec{\mathbf{x}} \end{aligned}$$



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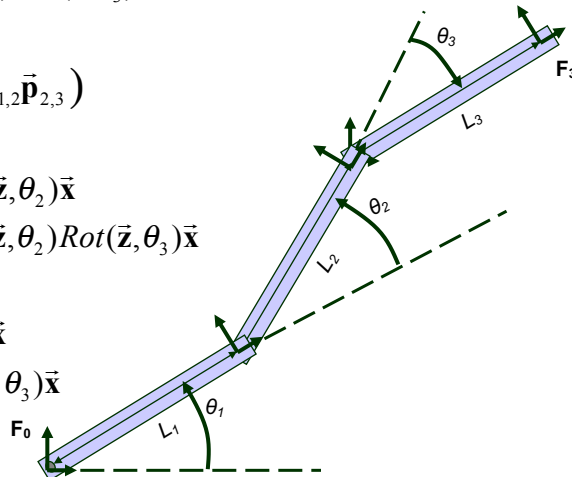
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Kinematic Chains

$$\begin{aligned} \text{If } \vec{\mathbf{r}}_1 &= \vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_3 = \vec{\mathbf{z}}, \\ \mathbf{R}_3 &= \text{Rot}(\vec{\mathbf{z}}, \theta_1) \text{Rot}(\vec{\mathbf{z}}, \theta_2) \text{Rot}(\vec{\mathbf{z}}, \theta_3) \\ &= \text{Rot}(\vec{\mathbf{z}}, \theta_1 + \theta_2 + \theta_3) \\ \vec{\mathbf{p}}_3 &= \vec{\mathbf{p}}_{0,1} + \mathbf{R}_{0,1} (\vec{\mathbf{p}}_{1,2} + \mathbf{R}_{1,2} \vec{\mathbf{p}}_{2,3}) \\ &= L_1 \text{Rot}(\vec{\mathbf{z}}, \theta_1) \vec{\mathbf{x}} \\ &\quad + L_2 \text{Rot}(\vec{\mathbf{z}}, \theta_1) \text{Rot}(\vec{\mathbf{z}}, \theta_2) \vec{\mathbf{x}} \\ &\quad + L_3 \text{Rot}(\vec{\mathbf{z}}, \theta_1) \text{Rot}(\vec{\mathbf{z}}, \theta_2) \text{Rot}(\vec{\mathbf{z}}, \theta_3) \vec{\mathbf{x}} \\ &= L_1 \text{Rot}(\vec{\mathbf{z}}, \theta_1) \vec{\mathbf{x}} \\ &\quad + L_2 \text{Rot}(\vec{\mathbf{z}}, \theta_1 + \theta_2) \vec{\mathbf{x}} \\ &\quad + L_3 \text{Rot}(\vec{\mathbf{z}}, \theta_1 + \theta_2 + \theta_3) \vec{\mathbf{x}} \end{aligned}$$



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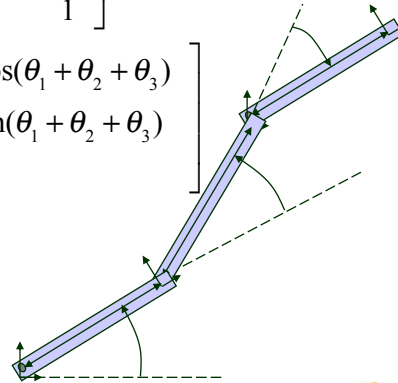
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Kinematic Chains

If $\bar{\mathbf{r}}_1 = \bar{\mathbf{r}}_2 = \bar{\mathbf{r}}_3 = \bar{\mathbf{z}}$,

$$\mathbf{R}_3 = \begin{bmatrix} \cos(\theta_1 + \theta_2 + \theta_3) & -\sin(\theta_1 + \theta_2 + \theta_3) & 0 \\ \sin(\theta_1 + \theta_2 + \theta_3) & \cos(\theta_1 + \theta_2 + \theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{\mathbf{p}}_3 = \begin{bmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ 0 \end{bmatrix}$$



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“Small” Transformations

- A great deal of CIS is concerned with computing and using geometric information based on imprecise knowledge
- Similarly, one is often concerned with the effects of relatively small rotations and displacements
- Essentially, we will be using fairly straightforward linearizations to model these situations, but a specialized notation is often useful

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“Small” Frame Transformations

Represent a "small" pose shift consisting of a small rotation $\Delta\mathbf{R}$ followed by a small displacement $\Delta\vec{\mathbf{p}}$ as

$$\Delta\mathbf{F} = [\Delta\mathbf{R}, \Delta\vec{\mathbf{p}}]$$

Then

$$\Delta\mathbf{F} \bullet \vec{\mathbf{v}} = \Delta\mathbf{R} \bullet \vec{\mathbf{v}} + \Delta\vec{\mathbf{p}}$$



Small Rotations

$\Delta\mathbf{R}$ = a small rotation

$\mathbf{R}_{\vec{\mathbf{a}}}(\Delta\alpha)$ = a rotation by a small angle $\Delta\alpha$ about axis $\vec{\mathbf{a}}$

$\text{Rot}(\vec{\mathbf{a}}, \|\vec{\mathbf{a}}\|) \bullet \vec{\mathbf{b}} \approx \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{b}}$ for $\|\vec{\mathbf{a}}\|$ sufficiently small

$\Delta\mathbf{R}(\vec{\mathbf{a}})$ = a rotation that is small enough so that any error introduced by this approximation is negligible

$\Delta\mathbf{R}(\lambda \vec{\mathbf{a}}) \bullet \Delta\mathbf{R}(\mu \vec{\mathbf{b}}) \cong \Delta\mathbf{R}(\lambda \vec{\mathbf{a}} + \mu \vec{\mathbf{b}})$ (Linearity for small rotations)

Exercise: Work out the linearity proposition by substitution



Approximations to “Small” Frames

$$\begin{aligned}\Delta\mathbf{F}(\bar{\mathbf{a}}, \Delta\bar{\mathbf{p}}) &\triangleq [\Delta\mathbf{R}(\bar{\mathbf{a}}), \Delta\bar{\mathbf{p}}] \\ \Delta\mathbf{F}(\bar{\mathbf{a}}, \Delta\bar{\mathbf{p}}) \bullet \bar{\mathbf{v}} &= \Delta\mathbf{R}(\bar{\mathbf{a}}) \bullet \bar{\mathbf{v}} + \Delta\bar{\mathbf{p}} \\ &\approx \bar{\mathbf{v}} + \bar{\mathbf{a}} \times \bar{\mathbf{v}} + \Delta\bar{\mathbf{p}}\end{aligned}$$

$$\begin{aligned}\bar{\mathbf{a}} \times \bar{\mathbf{v}} &= \text{skew}(\bar{\mathbf{a}}) \bullet \bar{\mathbf{v}} \\ &= \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \bullet \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \\ \text{skew}(\bar{\mathbf{a}}) \bullet \bar{\mathbf{a}} &= \bar{\mathbf{a}} \times \bar{\mathbf{a}} = \bar{\mathbf{0}}\end{aligned}$$

$$\begin{aligned}\Delta\mathbf{R}(\bar{\mathbf{a}}) &\approx \mathbf{I} + \text{skew}(\bar{\mathbf{a}}) \\ \Delta\mathbf{R}(\bar{\mathbf{a}})^{-1} &\approx \mathbf{I} - \text{skew}(\bar{\mathbf{a}}) = \mathbf{I} + \text{skew}(-\bar{\mathbf{a}})\end{aligned}$$



Approximations to “Small” Frames

Notational NOTE:

We often use $\bar{\alpha}$ to represent a vector of small angles
and $\bar{\epsilon}$ to represent a vector of small displacements

In using these approximations, we typically ignore second order terms. I.e.,

$$\bar{\alpha}_A \bar{\alpha}_B \approx \bar{\mathbf{0}}, \bar{\alpha}_A \bar{\epsilon}_B \approx \bar{\mathbf{0}}, \bar{\epsilon}_A \bar{\epsilon}_B \approx \bar{\mathbf{0}}, \text{ etc.}$$



Errors & sensitivity

Often, we do not have an accurate value for a transformation, so we need to model the error. We model this as a composition of a "nominal" frame and a small displacement

$$\mathbf{F}_{\text{actual}} = \mathbf{F}_{\text{nominal}} \bullet \Delta \mathbf{F}$$

Often, we will use the notation \mathbf{F}^* for $\mathbf{F}_{\text{actual}}$ and will just use \mathbf{F} for $\mathbf{F}_{\text{nominal}}$. Thus we may write something like

$$\mathbf{F}^* = \mathbf{F} \bullet \Delta \mathbf{F}$$

or (less often) $\mathbf{F}^* = \Delta \mathbf{F} \bullet \mathbf{F}$. We also use $\vec{\mathbf{v}}^* = \vec{\mathbf{v}} + \Delta \vec{\mathbf{v}}$, etc.

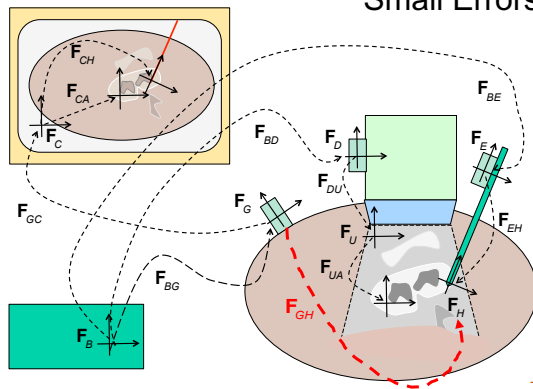
Thus, if we use the former form (error on the right), and have nominal relationship $\vec{\mathbf{v}} = \mathbf{F} \bullet \vec{\mathbf{b}}$, we get

$$\begin{aligned} \vec{\mathbf{v}}^* &= \mathbf{F}^* \bullet \vec{\mathbf{b}}^* \\ &= \mathbf{F} \bullet \Delta \mathbf{F} \bullet (\vec{\mathbf{b}} + \Delta \vec{\mathbf{b}}) = \mathbf{F} \bullet (\Delta \mathbf{R} \bullet \vec{\mathbf{b}} + \Delta \mathbf{R} \bullet \Delta \vec{\mathbf{b}} + \Delta \vec{\mathbf{p}}) \\ &\approx \mathbf{R} \bullet ((\mathbf{I} + sk(\vec{\alpha})) \bullet (\vec{\mathbf{b}} + \Delta \vec{\mathbf{b}})) + \Delta \vec{\mathbf{p}} + \vec{\mathbf{p}} = \mathbf{R} \bullet (\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \Delta \vec{\mathbf{b}} + \Delta \vec{\mathbf{p}}) + \vec{\mathbf{p}} \\ &\approx \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta \vec{\mathbf{b}} + \Delta \vec{\mathbf{p}}) + \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}} = \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta \vec{\mathbf{b}} + \Delta \vec{\mathbf{p}}) + \vec{\mathbf{v}} \\ \Delta \vec{\mathbf{v}} &\approx \mathbf{R} \bullet (\vec{\alpha} \times \vec{\mathbf{b}} + \Delta \vec{\mathbf{b}} + \Delta \vec{\mathbf{p}}) \end{aligned}$$



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"Small Errors"



Suppose that there is a small systematic error in the tracking system so that

$$\mathbf{F}_{B^*}^* = \Delta \mathbf{F}_{B^*} \mathbf{F}_{B^*}$$

for \mathbf{F}_{BG} , \mathbf{F}_{BD} , \mathbf{F}_{BE} . How does this affect the calculation of \mathbf{F}_{GH} ?

$$\begin{aligned} \mathbf{F}_{GH}^* &= (\mathbf{F}_{BG}^*)^{-1} \mathbf{F}_{BE}^* \mathbf{F}_{EH} \\ \mathbf{F}_{GH} \Delta \mathbf{F}_{GH} &= (\Delta \mathbf{F}_{B^*} \mathbf{F}_{BG})^{-1} \Delta \mathbf{F}_{B^*} \mathbf{F}_{BE} \mathbf{F}_{EH} \\ \Delta \mathbf{F}_{GH} &= \mathbf{F}_{GH}^{-1} (\Delta \mathbf{F}_{B^*} \mathbf{F}_{BG})^{-1} \Delta \mathbf{F}_{B^*} \mathbf{F}_{BE} \mathbf{F}_{EH} \\ &= \mathbf{F}_{GH}^{-1} \mathbf{F}_{BG}^{-1} \Delta \mathbf{F}_{B^*}^{-1} \Delta \mathbf{F}_{B^*} \mathbf{F}_{BE} \mathbf{F}_{EH} \\ &= \mathbf{F}_{GH}^{-1} \mathbf{F}_{BG}^{-1} \mathbf{F}_{BE} \mathbf{F}_{EH} = \mathbf{F}_{GH}^{-1} \mathbf{F}_{GH} = \mathbf{I} \end{aligned}$$



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"Small Errors"

Suppose that there are additional errors in the tracking of each tracker body so that

$$\mathbf{F}_{Bx}^* = \Delta \mathbf{F}_B \mathbf{F}_{Bx} \Delta \mathbf{F}_{Bx}$$

for $\mathbf{F}_{BG}, \mathbf{F}_{BD}, \mathbf{F}_{BE}$. How does this affect the calculation of \mathbf{F}_{GH} ?

$$\mathbf{F}_{GH}^* = \mathbf{F}_{GH} \Delta \mathbf{F}_{GH} = (\mathbf{F}_{BG}^*)^{-1} \mathbf{F}_{BE}^* \mathbf{F}_{EH}$$

$$\Delta \mathbf{F}_{GH} = \mathbf{F}_{GH}^{-1} (\Delta \mathbf{F}_B \mathbf{F}_{BG} \Delta \mathbf{F}_{BG})^{-1} (\Delta \mathbf{F}_B \mathbf{F}_{BE} \Delta \mathbf{F}_{BE}) \mathbf{F}_{EH}$$

$$\Delta \mathbf{F}_{GH} = (\mathbf{F}_{BG}^{-1} \mathbf{F}_{BE} \mathbf{F}_{EH})^{-1} \Delta \mathbf{F}_{BG}^{-1} \mathbf{F}_{BG}^{-1} \Delta \mathbf{F}_B^{-1} \Delta \mathbf{F}_B \mathbf{F}_{BE} \Delta \mathbf{F}_{BE} \mathbf{F}_{EH}$$

$$= \mathbf{F}_{EH}^{-1} \mathbf{F}_{BE}^{-1} \mathbf{F}_{BG} \Delta \mathbf{F}_{BG}^{-1} \mathbf{F}_{BG}^{-1} \mathbf{F}_{BE} \Delta \mathbf{F}_{BE} \mathbf{F}_{EH}$$

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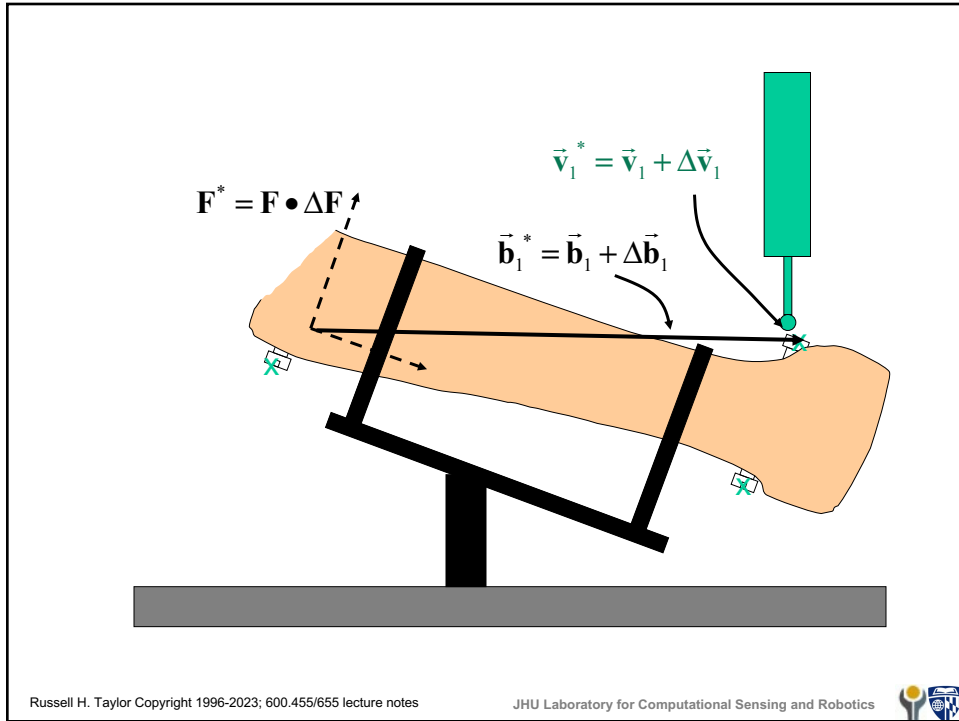
$\mathbf{F} = [\mathbf{R}, \mathbf{p}]$

\vec{b}_1

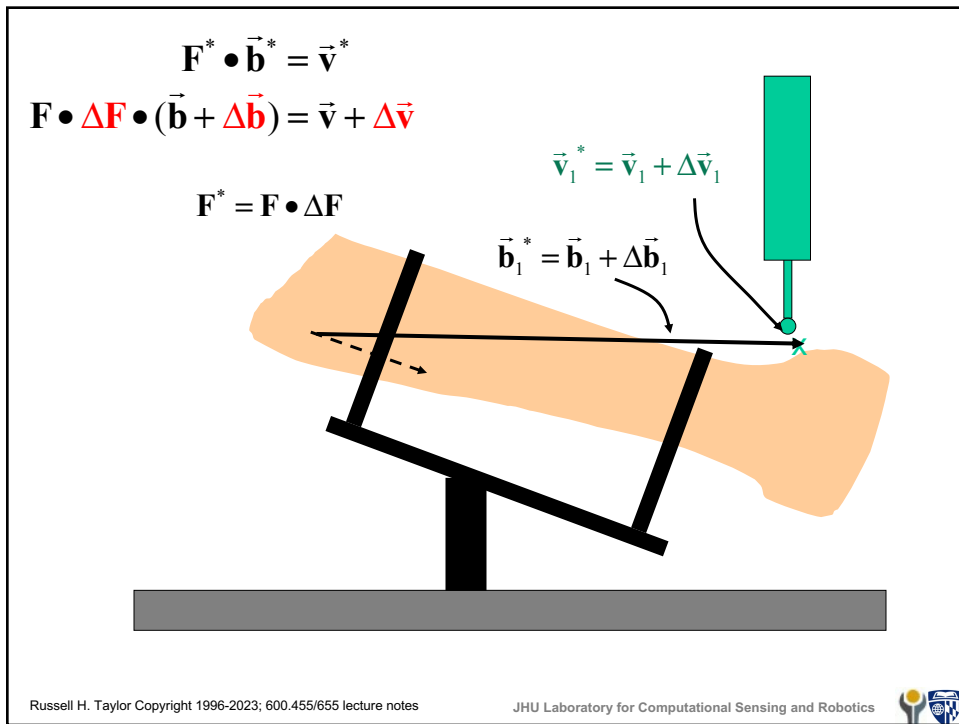
\vec{v}_1

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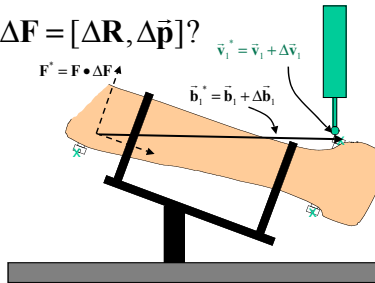
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Errors & Sensitivity

Suppose that we know nominal values for \mathbf{F} , $\bar{\mathbf{b}}$, and $\bar{\mathbf{v}}$ and that

$$[-\varepsilon, -\varepsilon, -\varepsilon]^T \leq \Delta \bar{\mathbf{v}}_1 \leq [\varepsilon, \varepsilon, \varepsilon]^T \quad (\text{i.e., } \|\Delta \bar{\mathbf{v}}_1\|_\infty \leq \varepsilon)$$

What does this tell us about $\Delta \mathbf{F} = [\Delta \mathbf{R}, \Delta \bar{\mathbf{p}}]$?



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Errors & Sensitivity

$$\begin{aligned} \bar{\mathbf{v}}^* &= \mathbf{F}^* \bullet \bar{\mathbf{b}}^* \\ &= \mathbf{F} \bullet \Delta \mathbf{F} \bullet (\bar{\mathbf{b}} + \Delta \bar{\mathbf{b}}) \\ &= \mathbf{R} \bullet (\Delta \mathbf{R}(\bar{\boldsymbol{\alpha}}) \bullet (\bar{\mathbf{b}} + \Delta \bar{\mathbf{b}}) + \Delta \bar{\mathbf{p}}) + \bar{\mathbf{p}} \\ &\approx \mathbf{R} \bullet (\bar{\mathbf{b}} + \Delta \bar{\mathbf{b}} + \bar{\boldsymbol{\alpha}} \times \bar{\mathbf{b}} + \bar{\boldsymbol{\alpha}} \times \Delta \bar{\mathbf{b}} + \Delta \bar{\mathbf{p}}) + \bar{\mathbf{p}} \\ &= \mathbf{R} \bullet \bar{\mathbf{b}} + \bar{\mathbf{p}} + \mathbf{R} \bullet (\Delta \bar{\mathbf{b}} + \bar{\boldsymbol{\alpha}} \times \bar{\mathbf{b}} + \bar{\boldsymbol{\alpha}} \times \Delta \bar{\mathbf{b}} + \Delta \bar{\mathbf{p}}) \\ &\approx \bar{\mathbf{v}} + \mathbf{R} \bullet (\Delta \bar{\mathbf{b}} + \bar{\boldsymbol{\alpha}} \times \bar{\mathbf{b}} + \Delta \bar{\mathbf{p}}) \end{aligned}$$

if $\|\bar{\boldsymbol{\alpha}} \times \Delta \bar{\mathbf{b}}\| \leq \|\bar{\boldsymbol{\alpha}}\| \|\Delta \bar{\mathbf{b}}\|$ is negligible (it usually is)

SO

$$\Delta \bar{\mathbf{v}} = \bar{\mathbf{v}}^* - \bar{\mathbf{v}} \approx \mathbf{R} \bullet (\Delta \bar{\mathbf{b}} + \bar{\boldsymbol{\alpha}} \times \bar{\mathbf{b}} + \Delta \bar{\mathbf{p}}) = \mathbf{R} \bullet \Delta \bar{\mathbf{b}} + \mathbf{R} \bullet \bar{\boldsymbol{\alpha}} \times \bar{\mathbf{b}} + \mathbf{R} \bullet \Delta \bar{\mathbf{p}}$$

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Digression: “rotation triple product”

Expressions like $\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}}$ are linear in $\vec{\mathbf{a}}$, but are not always convenient to work with. Often we would prefer something like $\mathbf{M}(\mathbf{R}, \vec{\mathbf{b}}) \bullet \vec{\mathbf{a}}$.

$$\begin{aligned}\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}} &= -\mathbf{R} \bullet \vec{\mathbf{b}} \times \vec{\mathbf{a}} \\ &= \mathbf{R} \bullet \text{skew}(-\vec{\mathbf{b}}) \bullet \vec{\mathbf{a}} \\ &= \left[\mathbf{R} \bullet \text{skew}(\vec{\mathbf{b}})^T \right] \bullet \vec{\mathbf{a}}\end{aligned}$$



Digression: “rotation triple product”

Here are a few more useful facts:

$$\begin{aligned}\mathbf{R} \bullet (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) &= (\mathbf{R} \bullet \vec{\mathbf{a}}) \times (\mathbf{R} \bullet \vec{\mathbf{b}}) \\ \vec{\mathbf{a}} \times (\mathbf{R} \bullet \vec{\mathbf{b}}) &= \mathbf{R} \bullet \left((\mathbf{R}^{-1} \bullet \vec{\mathbf{a}}) \times \vec{\mathbf{b}} \right)\end{aligned}$$

Consequently

$$\begin{aligned}\text{skew}(\vec{\mathbf{a}}) \bullet \mathbf{R} &= \mathbf{R} \bullet \text{skew}(\mathbf{R}^{-1} \bullet \vec{\mathbf{a}}) \\ \mathbf{R}^{-1} \text{skew}(\vec{\mathbf{a}}) \bullet \mathbf{R} &= \text{skew}(\mathbf{R}^{-1} \bullet \vec{\mathbf{a}})\end{aligned}$$



A “standard form” for linearized error expressions

It is often convenient to use identities to rearrange expressions involving small error variables into sums of terms with the general form $\mathbf{M}_k \bar{\eta}_k$, where \mathbf{M}_k involve things known to the computer, and the $\bar{\eta}_k$ are error variables.

For example,

$$\bar{\gamma} = \mathbf{R} \text{sk}(\bar{\alpha}) \bar{\mathbf{a}} + \text{sk}(\bar{\beta}) \bar{\mathbf{b}}$$

would be rewritten as

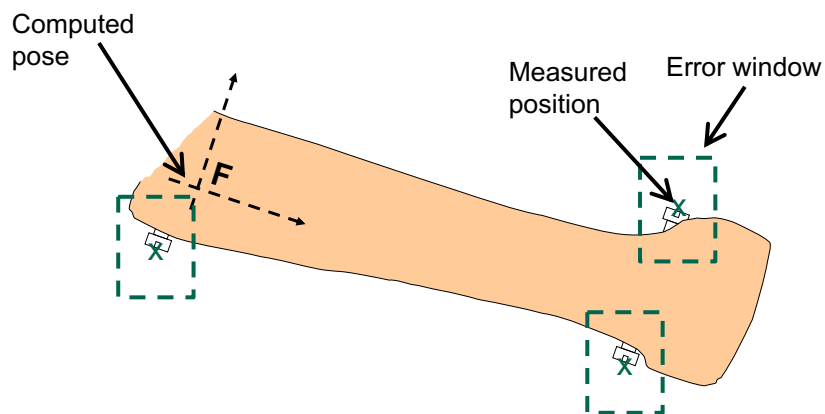
$$\bar{\gamma} = -\mathbf{R} \text{sk}(\bar{\mathbf{a}}) \bar{\alpha} - \text{sk}(\bar{\mathbf{b}}) \bar{\beta}$$

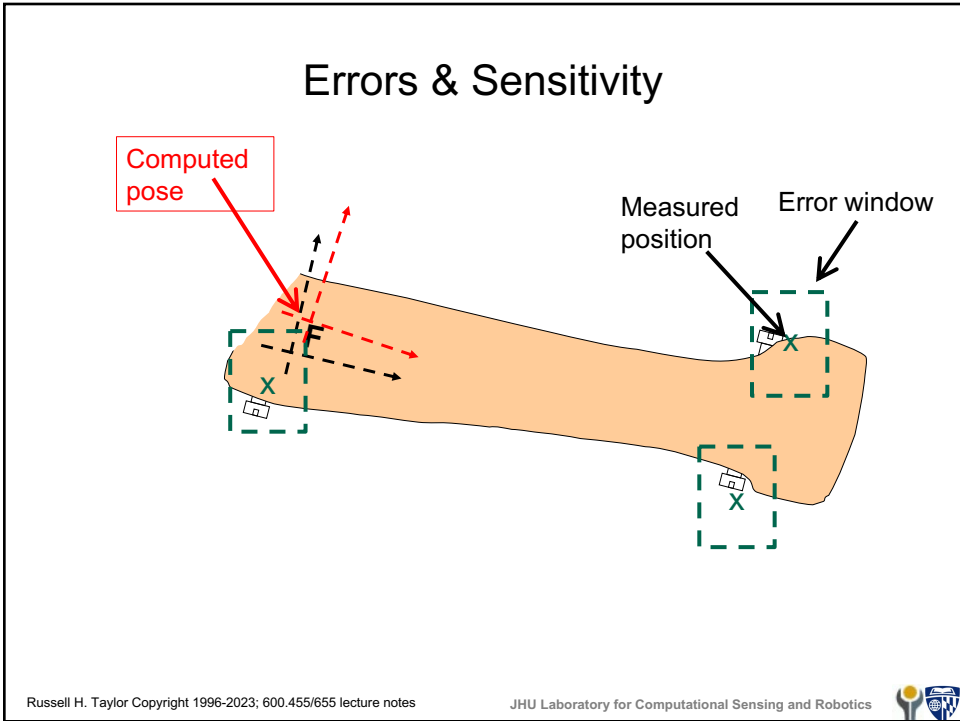
or

$$\bar{\gamma} = \mathbf{R} \text{sk}(-\bar{\mathbf{a}}) \bar{\alpha} + \text{sk}(-\bar{\mathbf{b}}) \bar{\beta}$$

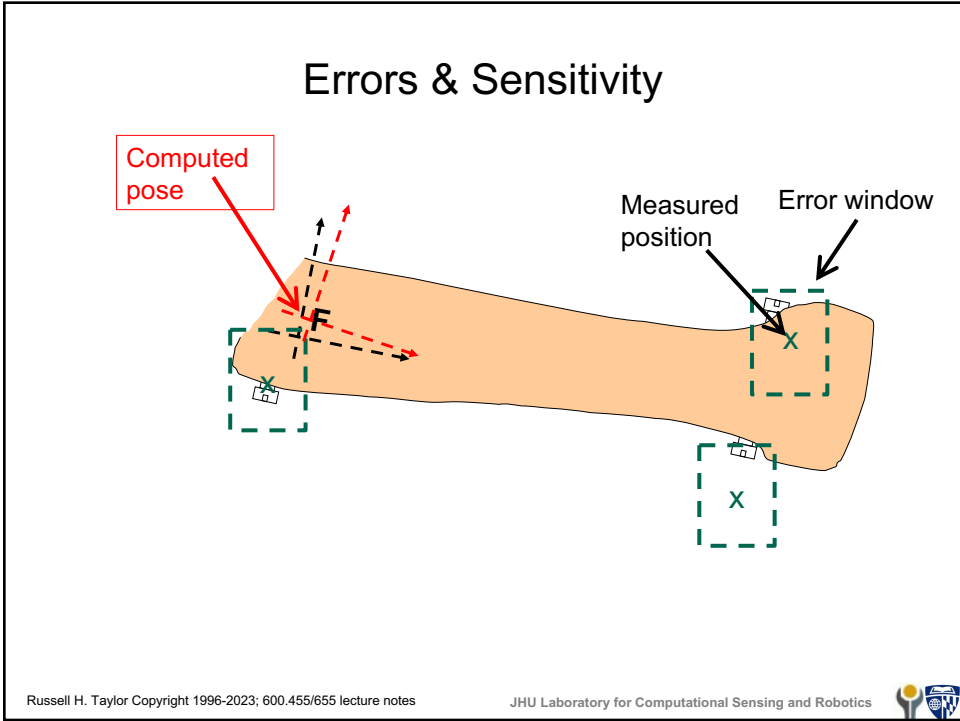


Errors & Sensitivity

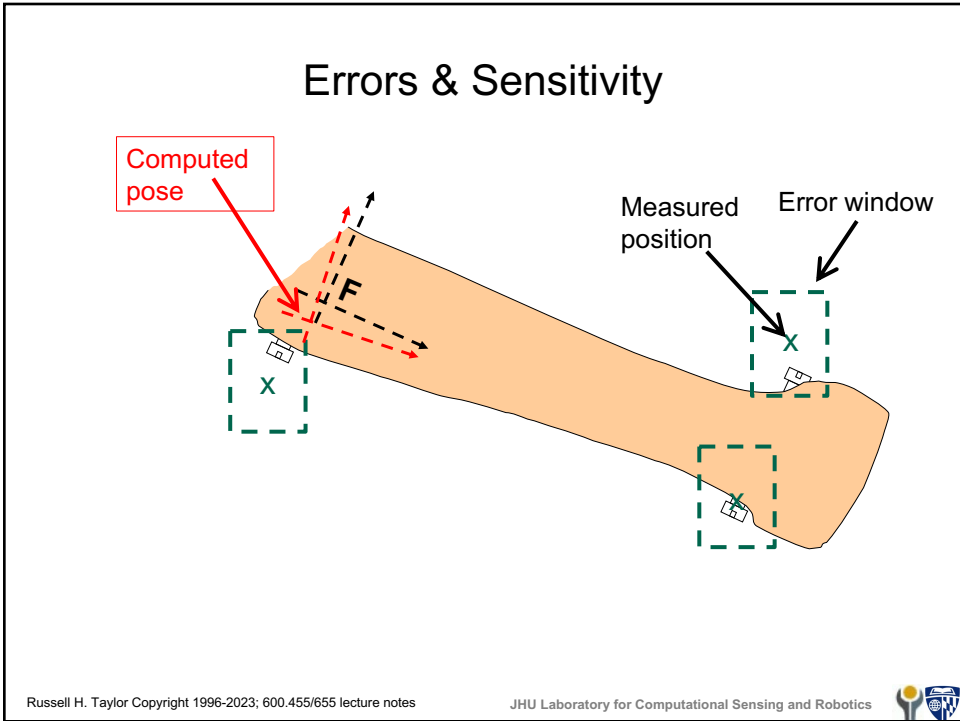




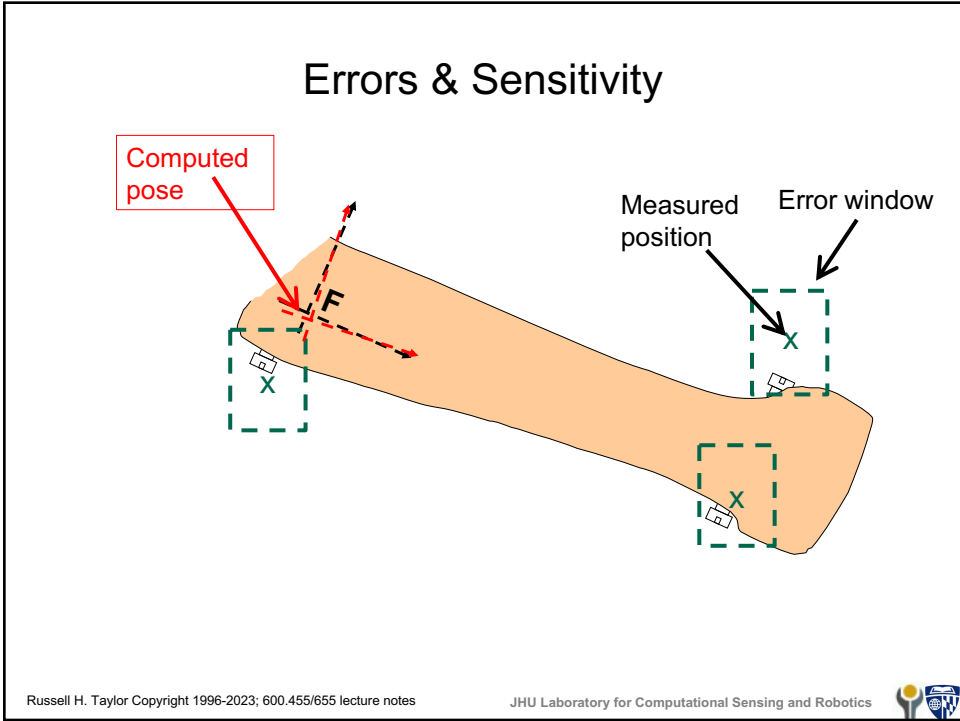
84



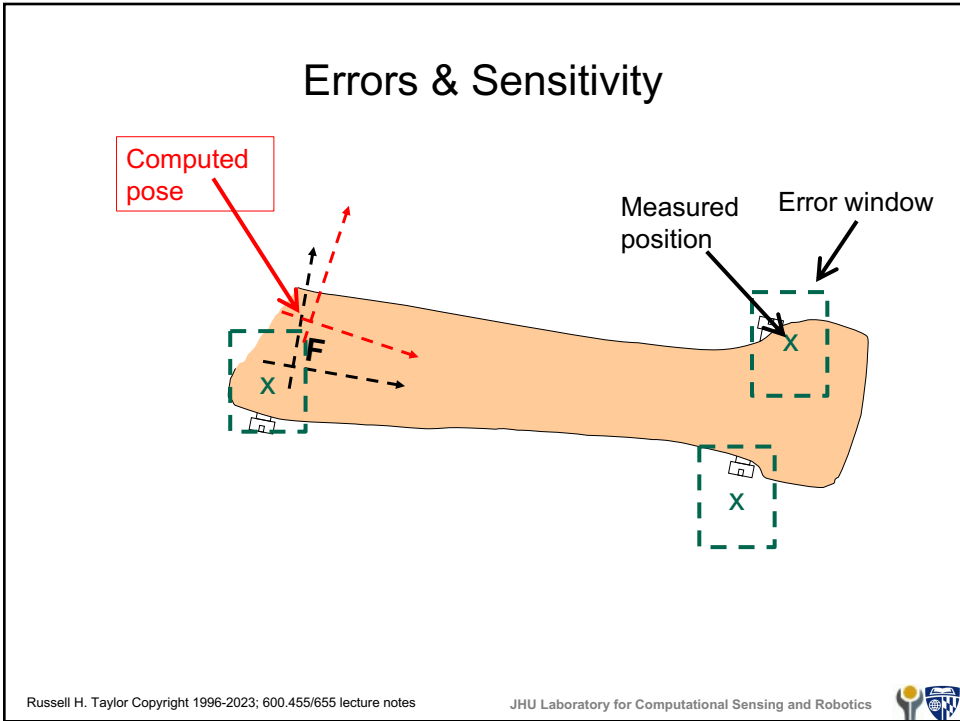
85



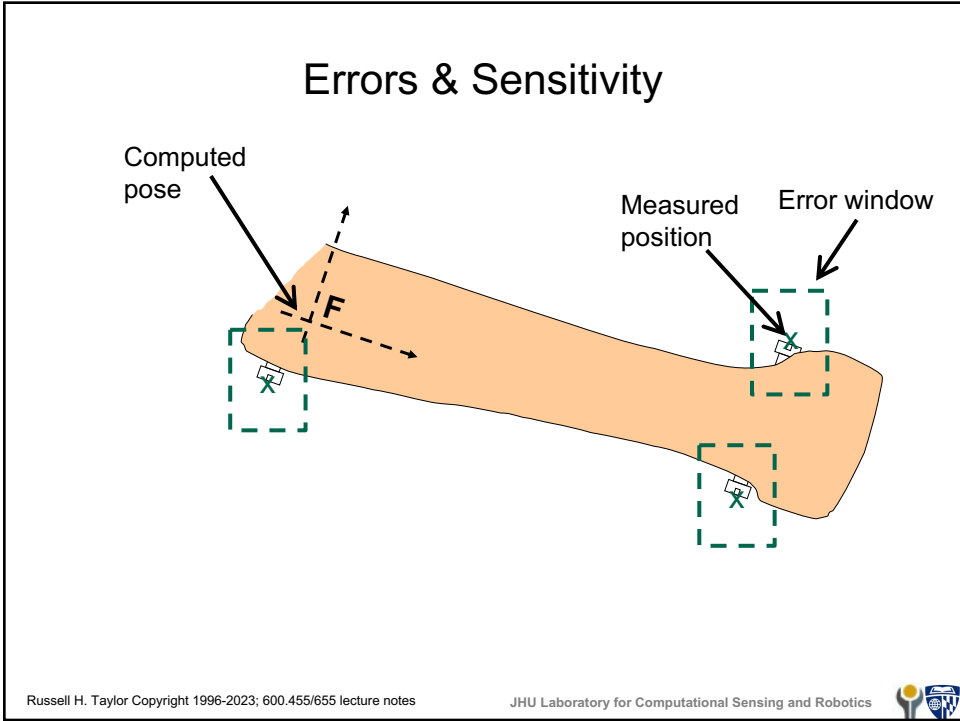
86



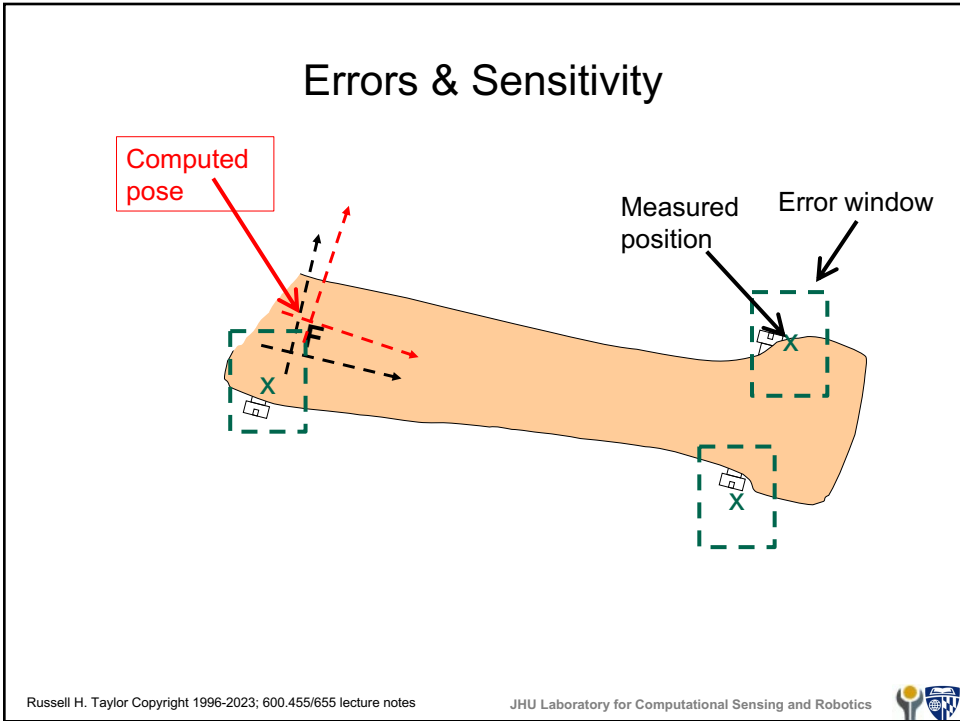
87



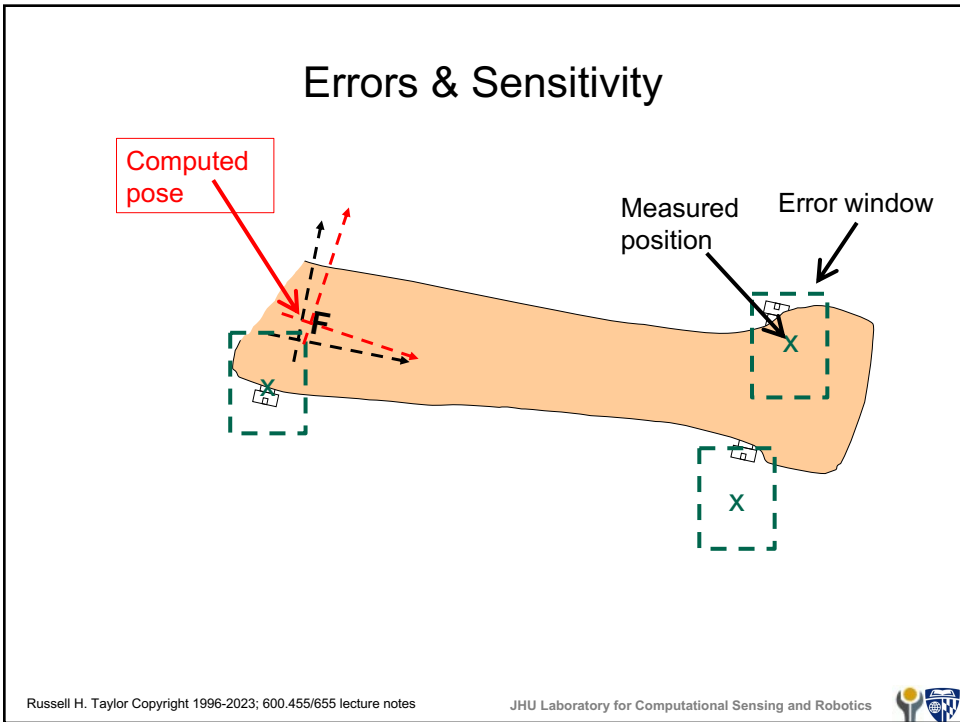
88



89



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Errors & Sensitivity

Previous expression was

$$\Delta \vec{v}_1 \approx \mathbf{R} \bullet (\Delta \vec{b}_1 + \vec{\alpha} \times \vec{b} + \Delta \vec{p}_1)$$

Substituting triple product and rearranging gives

$$\Delta \vec{v}_1 \approx \left[\mathbf{R} \mid \mathbf{R} \mid \mathbf{R} \bullet \text{skew}(-\vec{b}) \right] \bullet \begin{bmatrix} \Delta \vec{b}_1 \\ \Delta \vec{p} \\ \vec{\alpha} \end{bmatrix}$$

So

$$\begin{bmatrix} -\varepsilon \\ -\varepsilon \\ -\varepsilon \end{bmatrix} \leq \left[\mathbf{R} \mid \mathbf{R} \mid \mathbf{R} \bullet \text{skew}(-\vec{b}) \right] \begin{bmatrix} \Delta \vec{b}_1 \\ \Delta \vec{p} \\ \vec{\alpha} \end{bmatrix} \leq \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix}$$

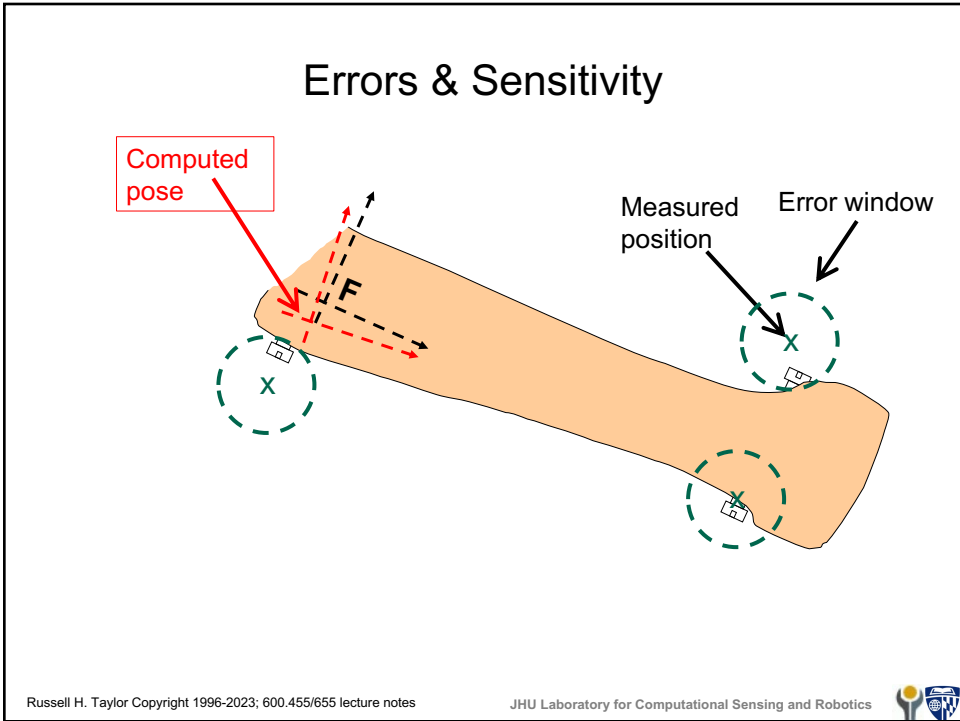


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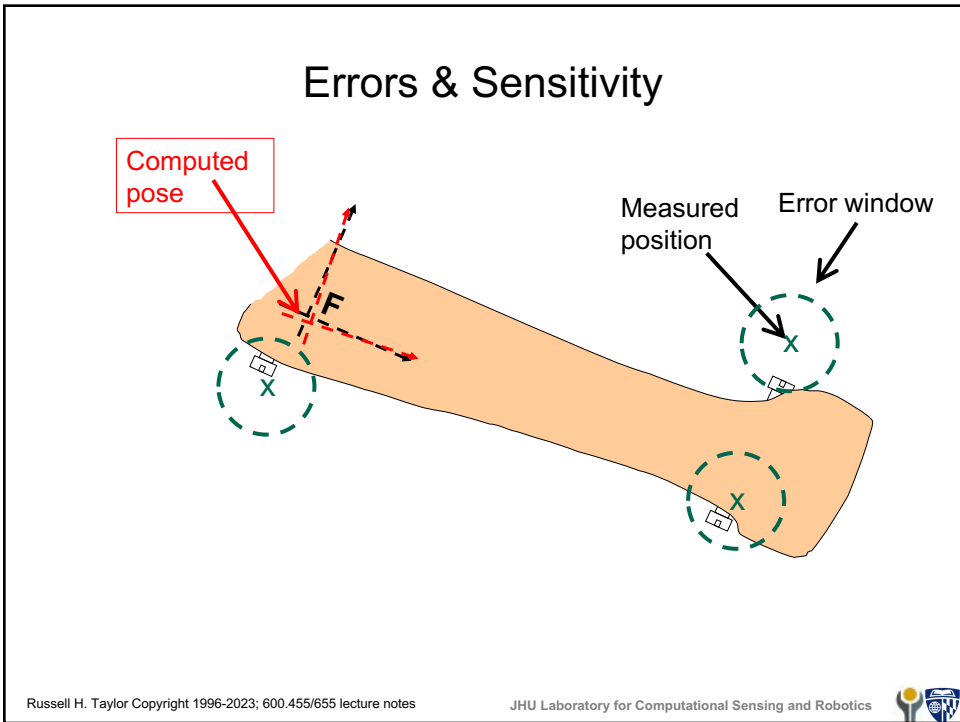
Now, suppose we know that $|\Delta \vec{b}_1| \leq \beta$, this will give us a system of linear constraints

$$\begin{bmatrix} -\varepsilon \\ -\varepsilon \\ -\varepsilon \\ -\beta \\ -\beta \\ -\beta \end{bmatrix} \leq \left[\begin{array}{c|c|c} \mathbf{R} & \mathbf{R} & \mathbf{R} \bullet \text{skew}(-\vec{b}) \\ \hline \mathbf{I} & \mathbf{0} & \mathbf{0} \end{array} \right] \begin{bmatrix} \Delta \vec{b}_1 \\ \Delta \vec{p}_1 \\ \vec{\alpha} \end{bmatrix} \leq \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \\ \beta \\ \beta \\ \beta \end{bmatrix}$$



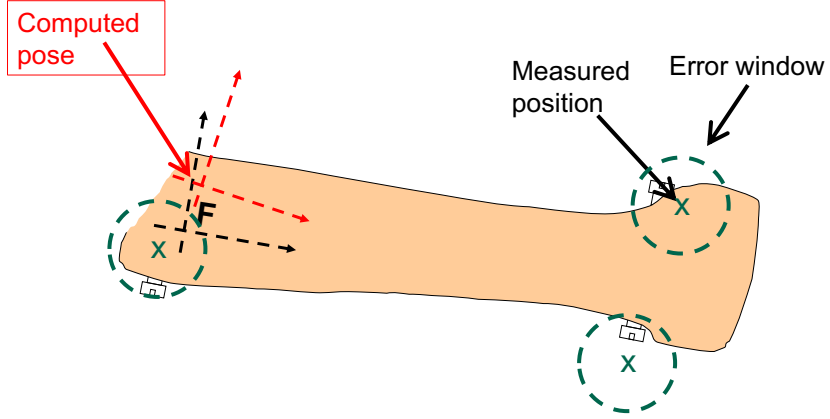


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Errors & Sensitivity



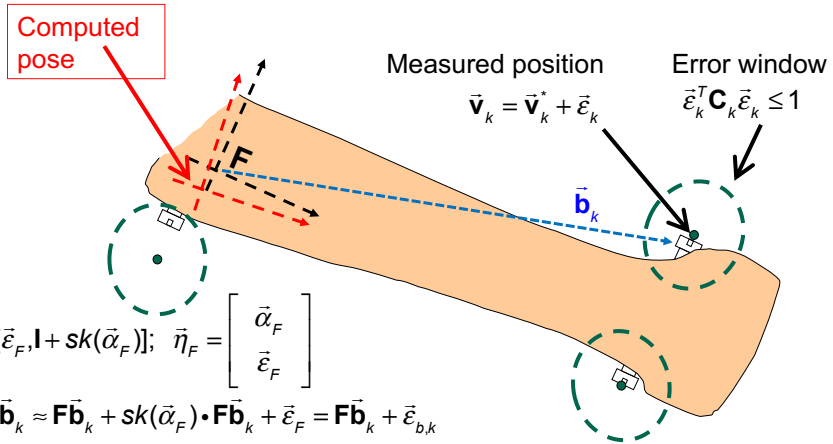
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Errors & Sensitivity



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Errors & Sensitivity

Computed pose

Measured position

Error window

$$\bar{\mathbf{v}}_k = \bar{\mathbf{v}}_k^* + \bar{\epsilon}_k$$

$$\bar{\epsilon}_k^T \mathbf{C}_k \bar{\epsilon}_k \leq 1$$

$$\left. \begin{array}{l} \bar{\eta}_F^T \mathbf{M}_1^T \mathbf{C}_1 \mathbf{M}_1 \bar{\eta}_F \leq 1 \\ \vdots \\ \bar{\eta}_F^T \mathbf{M}_k^T \mathbf{C}_k \mathbf{M}_k \bar{\eta}_F \leq 1 \\ \vdots \\ \bar{\eta}_F^T \mathbf{M}_N^T \mathbf{C}_N \mathbf{M}_N \bar{\eta}_F \leq 1 \end{array} \right\} \text{define feasible region}$$

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Error from frame composition

Consider $\mathbf{R}_1^* \mathbf{R}_2^* = \mathbf{R}_3^*$ where $\mathbf{R}_1^* = \mathbf{R}_1 \Delta \mathbf{R}_1$, $\mathbf{R}_2^* = \mathbf{R}_2 \Delta \mathbf{R}_2$, $\mathbf{R}_3^* = \mathbf{R}_3 \Delta \mathbf{R}_3$
and $\Delta \mathbf{R}_1 \approx \mathbf{I} + sk(\bar{\alpha}_1)$, $\Delta \mathbf{R}_2 \approx \mathbf{I} + sk(\bar{\alpha}_2)$, estimate $\Delta \mathbf{R}_3 \approx \mathbf{I} + sk(\bar{\alpha}_3)$

$$\mathbf{R}_1 \Delta \mathbf{R}_1 \mathbf{R}_2 \Delta \mathbf{R}_2 = \mathbf{R}_1 \mathbf{R}_2 \Delta \mathbf{R}_3$$

$$\mathbf{R}_1 (\mathbf{I} + sk(\bar{\alpha}_1)) \mathbf{R}_2 (\mathbf{I} + sk(\bar{\alpha}_2)) \approx \mathbf{R}_1 \mathbf{R}_2 (\mathbf{I} + sk(\bar{\alpha}_3))$$

$$(\mathbf{R}_1 \mathbf{R}_2)^{-1} \mathbf{R}_1 (\mathbf{I} + sk(\bar{\alpha}_1)) \mathbf{R}_2 (\mathbf{I} + sk(\bar{\alpha}_2)) \approx \mathbf{I} + sk(\bar{\alpha}_3)$$

$$\mathbf{R}_2^{-1} \cancel{\mathbf{R}_1} \mathbf{R}_1 (\mathbf{I} + sk(\bar{\alpha}_1)) \mathbf{R}_2 (\mathbf{I} + sk(\bar{\alpha}_2)) \approx \mathbf{I} + sk(\bar{\alpha}_3)$$

$$\mathbf{I} + \mathbf{R}_2^{-1} sk(\bar{\alpha}_1) \mathbf{R}_2 + sk(\bar{\alpha}_2) + \mathbf{R}_2^{-1} sk(\bar{\alpha}_1) \mathbf{R}_2 sk(\bar{\alpha}_2) \approx \mathbf{I} + sk(\bar{\alpha}_3)$$

$$\mathbf{R}_2^{-1} sk(\bar{\alpha}_1) \mathbf{R}_2 + sk(\bar{\alpha}_2) \approx sk(\bar{\alpha}_3)$$

Since $\mathbf{R}^{-1} \cdot (\bar{\mathbf{a}} \times \mathbf{R} \bar{\mathbf{b}}) = (\mathbf{R}^{-1} \bar{\mathbf{a}}) \times \bar{\mathbf{b}}$ for all \mathbf{R} , $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ we get $\mathbf{R}_2^{-1} sk(\bar{\alpha}_1) \mathbf{R}_2 = sk(\mathbf{R}_2^{-1} \bar{\alpha}_1)$

$$sk(\bar{\alpha}_3) \approx sk(\mathbf{R}_2^{-1} \bar{\alpha}_1) + sk(\bar{\alpha}_2) = sk(\mathbf{R}_2^{-1} \bar{\alpha}_1 + \bar{\alpha}_2)$$

$$\bar{\alpha}_3 \approx \mathbf{R}_2^{-1} \bar{\alpha}_1 + \bar{\alpha}_2$$

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Error from frame composition

Consider $\mathbf{F}_1^* \mathbf{F}_2^* = \mathbf{F}_3^*$ where $\mathbf{F}_1^* = \mathbf{F}_1 \Delta \mathbf{F}_1$, $\mathbf{F}_2^* = \mathbf{F}_2 \Delta \mathbf{F}_2$, $\mathbf{F}_3^* = \mathbf{F}_3 \Delta \mathbf{F}_3$
 and $\Delta \mathbf{F}_1 \approx [\mathbf{I} + \text{sk}(\vec{\alpha}_1), \vec{\varepsilon}_1]$, $\Delta \mathbf{F}_2 \approx [\mathbf{I} + \text{sk}(\vec{\alpha}_2), \vec{\varepsilon}_2]$,
 estimate $\Delta \mathbf{F}_3 \approx [\mathbf{I} + \text{sk}(\vec{\alpha}_3), \vec{\varepsilon}_3]$

From before, we have $\vec{\alpha}_3 \approx \mathbf{R}_2^{-1} \vec{\alpha}_1 + \vec{\alpha}_2$. So now we just need $\vec{\varepsilon}_3$.

$$\begin{aligned} \mathbf{F}_3 \Delta \mathbf{F}_3 &= [\mathbf{R}_3 \Delta \mathbf{R}_3, \mathbf{R}_3 \Delta \vec{p}_3 + \vec{p}_3] \\ \vec{p}_3 + \mathbf{R}_3 \Delta \vec{p}_3 &= \mathbf{R}_1 (\Delta \mathbf{R}_1 (\vec{p}_2 + \mathbf{R}_2 \vec{\varepsilon}_2) + \vec{\varepsilon}_1) + \vec{p}_1 \\ \vec{p}_3 + \mathbf{R}_3 \vec{\varepsilon}_3 &\approx \mathbf{R}_1 (\mathbf{I} + \text{sk}(\vec{\alpha}_1)) (\vec{p}_2 + \mathbf{R}_2 \vec{\varepsilon}_2) + \mathbf{R}_1 \vec{\varepsilon}_1 + \vec{p}_1 \\ &= \mathbf{R}_1 \vec{p}_2 + \mathbf{R}_1 \mathbf{R}_2 \vec{\varepsilon}_2 + \mathbf{R}_1 \cdot (\vec{\alpha}_1 \times \vec{p}_2 + \vec{\alpha}_1 \times \mathbf{R}_2 \vec{\varepsilon}_2) + \vec{p}_1 + \mathbf{R}_1 \vec{\varepsilon}_1 \\ &= \vec{p}_3 + \mathbf{R}_1 \mathbf{R}_2 \vec{\varepsilon}_2 + \mathbf{R}_1 \cdot (\vec{\alpha}_1 \times \vec{p}_2 + \vec{\alpha}_1 \times \mathbf{R}_2 \vec{\varepsilon}_2) + \mathbf{R}_1 \vec{\varepsilon}_1 \\ \mathbf{R}_3 \vec{\varepsilon}_3 &\approx \mathbf{R}_1 \mathbf{R}_2 \vec{\varepsilon}_2 + \mathbf{R}_1 \cdot \vec{\alpha}_1 \times \vec{p}_2 + \mathbf{R}_1 \vec{\varepsilon}_1 \\ \vec{\varepsilon}_3 &\approx \mathbf{R}_2^{-1} \mathbf{R}_1^{-1} \mathbf{R}_1 \mathbf{R}_2 \vec{\varepsilon}_2 + \mathbf{R}_2^{-1} \mathbf{R}_1^{-1} \mathbf{R}_1 \cdot \vec{\alpha}_1 \times \vec{p}_2 + \mathbf{R}_2^{-1} \mathbf{R}_1^{-1} \mathbf{R}_1 \vec{\varepsilon}_1 \\ &= \vec{\varepsilon}_2 + \mathbf{R}_2^{-1} \cdot \vec{\alpha}_1 \times \vec{p}_2 + \mathbf{R}_2^{-1} \vec{\varepsilon}_1 \\ &= \vec{\varepsilon}_2 - \mathbf{R}_2^{-1} \text{sk}(\vec{p}_2) \vec{\alpha}_1 + \mathbf{R}_2^{-1} \vec{\varepsilon}_1 \end{aligned}$$



Inverse of frame transformation with errors

For $\mathbf{F}^* = \mathbf{F} \Delta \mathbf{F}$, if we want $\Delta \mathbf{F}_i$ such that $\mathbf{F}_i^* = \mathbf{F}_i \Delta \mathbf{F}_i$, we have

$$\begin{aligned} \mathbf{F}_i &= \mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1} \vec{p}] \\ \mathbf{F}_i^* &= \mathbf{F}_i \Delta \mathbf{F}_i = (\mathbf{F} \Delta \mathbf{F})^{-1} = \Delta \mathbf{F}^{-1} \mathbf{F}^{-1} = [\Delta \mathbf{R}^{-1}, -\Delta \mathbf{R}^{-1} \Delta \vec{p}] \cdot [\mathbf{R}^{-1}, -\mathbf{R}^{-1} \vec{p}] \\ \mathbf{F}_i \Delta \mathbf{F}_i &= [\Delta \mathbf{R}^{-1} \mathbf{R}^{-1}, -\Delta \mathbf{R}^{-1} \mathbf{R}^{-1} \vec{p} - \Delta \mathbf{R}^{-1} \Delta \vec{p}] \\ \Delta \mathbf{F}_i &= \mathbf{F}_i^{-1} [\Delta \mathbf{R}^{-1} \mathbf{R}^{-1}, -\Delta \mathbf{R}^{-1} \mathbf{R}^{-1} \vec{p} - \Delta \mathbf{R}^{-1} \Delta \vec{p}] \\ &= (\mathbf{F}^{-1})^{-1} [\Delta \mathbf{R}^{-1} \mathbf{R}^{-1}, -\Delta \mathbf{R}^{-1} \mathbf{R}^{-1} \vec{p} - \Delta \mathbf{R}^{-1} \Delta \vec{p}] \\ &= [\mathbf{R}, \vec{p}] \cdot [\Delta \mathbf{R}^{-1} \mathbf{R}^{-1}, -\Delta \mathbf{R}^{-1} \mathbf{R}^{-1} \vec{p} - \Delta \mathbf{R}^{-1} \Delta \vec{p}] \\ &= [\mathbf{R} \Delta \mathbf{R}^{-1} \mathbf{R}^{-1}, -\mathbf{R} \Delta \mathbf{R}^{-1} \mathbf{R}^{-1} \vec{p} - \mathbf{R} \Delta \mathbf{R}^{-1} \Delta \vec{p} + \vec{p}] \end{aligned}$$



Inverse of frame transformation with errors

For $\mathbf{F}^* = \Delta\mathbf{F}\mathbf{F}$, if we want $\mathbf{F}_i^* = \Delta\mathbf{F}_i\mathbf{F}_i$, we have

$$\begin{aligned}\mathbf{F}_i &= \mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1}\vec{\mathbf{p}}] \\ \mathbf{F}_i^* &= \Delta\mathbf{F}_i\mathbf{F}_i = (\Delta\mathbf{F}\mathbf{F})^{-1} = \mathbf{F}^{-1}\Delta\mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1}\vec{\mathbf{p}}] \cdot [\Delta\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}}] \\ \Delta\mathbf{F}_i\mathbf{F}_i &= [\mathbf{R}^{-1}\Delta\mathbf{R}^{-1}, -\mathbf{R}^{-1}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \mathbf{R}^{-1}\vec{\mathbf{p}}] \\ \Delta\mathbf{F}_i &= [\mathbf{R}^{-1}\Delta\mathbf{R}^{-1}, -\mathbf{R}^{-1}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \mathbf{R}^{-1}\vec{\mathbf{p}}] \cdot [\mathbf{R}, \vec{\mathbf{p}}] \\ &= [\mathbf{R}^{-1}\Delta\mathbf{R}^{-1}\mathbf{R}, \mathbf{R}^{-1}\Delta\mathbf{R}^{-1}\vec{\mathbf{p}} - \mathbf{R}^{-1}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \mathbf{R}^{-1}\vec{\mathbf{p}}]\end{aligned}$$



Inverse of frame transformation with errors

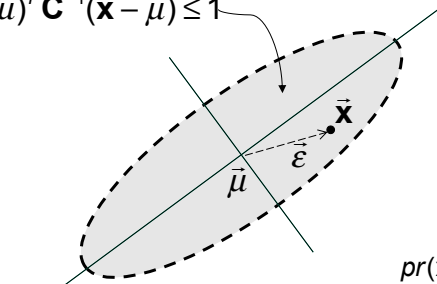
Suppose we know that $\Delta\mathbf{R}$ is "small", i.e., $\Delta\mathbf{R} \approx \mathbf{I} + sk(\vec{\alpha})$, and for notational convenience we write $\Delta\vec{\mathbf{p}} = \vec{\epsilon}$, we get

$$\begin{aligned}\Delta\mathbf{R}_i &= \mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1} \approx \mathbf{R}(\mathbf{I} + sk(\vec{\alpha}))^{-1}\mathbf{R}^{-1} \\ &\approx \mathbf{R}(\mathbf{I} - sk(\vec{\alpha}))\mathbf{R}^{-1} \\ &= \mathbf{R}\mathbf{R}^{-1} - \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1} \\ &= \mathbf{I} - \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1} \\ &= \mathbf{I} - sk\left((\mathbf{R}^{-1})^{-1}\vec{\alpha}\right) = \mathbf{I} - sk(\mathbf{R}\vec{\alpha}) \\ \Delta\vec{\mathbf{p}}_i &= -\mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}} - \mathbf{R}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} + \vec{\mathbf{p}} \\ &\approx -\mathbf{R}(\mathbf{I} - sk(\vec{\alpha}))\mathbf{R}^{-1}\vec{\mathbf{p}} - \mathbf{R}(\mathbf{I} - sk(\vec{\alpha}))\vec{\epsilon} + \vec{\mathbf{p}} \\ &= -\vec{\mathbf{p}} + \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1}\vec{\mathbf{p}} - \mathbf{R}\vec{\epsilon} + \mathbf{R}sk(\vec{\alpha})\vec{\epsilon} + \vec{\mathbf{p}} \\ &\approx \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1}\vec{\mathbf{p}} - \mathbf{R}\vec{\epsilon} = \mathbf{R}(\vec{\alpha} \times \mathbf{R}^{-1}\vec{\mathbf{p}}) - \mathbf{R}\vec{\epsilon} \\ \Delta\vec{\mathbf{p}} &\approx -\mathbf{R}\vec{\epsilon} - \mathbf{R}sk(\mathbf{R}^{-1}\vec{\mathbf{p}})\vec{\alpha}\end{aligned}$$



Probabilistic Error Modeling: Multivariable Gaussian

$$(\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}})^T \mathbf{C}^{-1} (\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}}) \leq 1$$



$$\bar{\mathbf{x}} \sim \mathcal{N}(\bar{\boldsymbol{\mu}}, \mathbf{C})$$

$$\bar{\boldsymbol{\mu}} = E[\bar{\mathbf{x}}]$$

$$\begin{aligned} \mathbf{C} &= E\left[(\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}})(\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}})^T\right] \\ &= E[\bar{\mathbf{x}}\bar{\mathbf{x}}^T] - \bar{\boldsymbol{\mu}}\bar{\boldsymbol{\mu}}^T \end{aligned}$$

$$pr(\bar{\mathbf{x}}) = \frac{\exp(-(\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}})^T \mathbf{C}^{-1} (\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}}) / 2)}{\sqrt{(2\pi)^n |\mathbf{C}|}}$$

If $\bar{\mathbf{x}} \sim \mathcal{N}(\bar{\boldsymbol{\mu}}_x, \mathbf{C}_{xx})$, then

$$\bar{\boldsymbol{\epsilon}} = \bar{\mathbf{x}} - \bar{\boldsymbol{\mu}} \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$$

$$\mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{c}} \sim \mathcal{N}(\mathbf{A}\bar{\boldsymbol{\mu}}_x + \bar{\mathbf{c}}, \mathbf{A}\mathbf{C}_{xx}\mathbf{A}^T) \text{ for constants } \mathbf{A}, \bar{\mathbf{c}}$$

Also there will be a random vector $\bar{\boldsymbol{\theta}}$ with independent elements

$$\bar{\theta}_i \sim \mathcal{N}(0, 1) \text{ and matrix } \mathbf{M} \text{ such that } \bar{\mathbf{x}} = \bar{\boldsymbol{\mu}} + \mathbf{M}\bar{\boldsymbol{\theta}}, \text{ and}$$

$$\mathbf{C}_{xx} = \mathbf{M}\mathbf{M}^T = \mathbf{Q}\boldsymbol{\Lambda}^2\mathbf{Q}^T \text{ where } \mathbf{Q}\mathbf{Q}^T = \mathbf{I}, \boldsymbol{\Lambda} = \text{diag}(\bar{\boldsymbol{\lambda}}), \mathbf{M} = \mathbf{Q}\text{diag}(\bar{\boldsymbol{\lambda}})$$

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Mahalanobis Distance

Given an n -dimensional random variable $\bar{\mathbf{x}}$ with mean $\bar{\boldsymbol{\mu}}$ and positive-definite covariance \mathbf{C} , the Mahalanobis distance

$$d_M(\bar{\mathbf{x}}; \bar{\boldsymbol{\mu}}, \mathbf{C}) = (\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}})^T \mathbf{C}^{-1} (\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}})$$

is a random variable follows a chi-squared distribution with n degrees of freedom. I.e.

$$d_M(\bar{\mathbf{x}}; \bar{\boldsymbol{\mu}}, \mathbf{C}) \sim \chi^2(n)$$

Further, the probability that $d_M^2 \leq \rho^2$ is given by

$$\Pr[d_M^2 \leq \rho^2] = \frac{1}{\Gamma(n/2)} \gamma\left(\frac{n}{2}, \frac{d_M^2}{2}\right)$$

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Probabilistic Error Modeling: Multivariable Gaussian

Suppose $\bar{\mathbf{x}} \sim N(\bar{\boldsymbol{\mu}}_x, \mathbf{C}_x)$ and $\bar{\mathbf{y}} \sim N(\bar{\boldsymbol{\mu}}_y, \mathbf{C}_y)$

$$\bar{\boldsymbol{\mu}}_{x-y} = E[\bar{\mathbf{x}} - \bar{\mathbf{y}}] = \bar{\boldsymbol{\mu}}_x - \bar{\boldsymbol{\mu}}_y$$

$$\begin{aligned} \mathbf{C}_{x-y} &= E[(\bar{\mathbf{x}} - \bar{\mathbf{y}})(\bar{\mathbf{x}} - \bar{\mathbf{y}})^T] - (\bar{\boldsymbol{\mu}}_x - \bar{\boldsymbol{\mu}}_y)(\bar{\boldsymbol{\mu}}_x - \bar{\boldsymbol{\mu}}_y)^T \\ &= \mathbf{C}_{xx} - 2\mathbf{C}_{xy} + \mathbf{C}_{yy} \\ &= \mathbf{C}_{xx} + \mathbf{C}_{yy} \text{ if } \bar{\mathbf{x}} \text{ and } \bar{\mathbf{y}} \text{ are independent} \end{aligned}$$

Similarly, $\bar{\boldsymbol{\mu}}_{x+y} = E[\bar{\mathbf{x}} + \bar{\mathbf{y}}] = \bar{\boldsymbol{\mu}}_x + \bar{\boldsymbol{\mu}}_y$

$$\begin{aligned} \mathbf{C}_{x+y} &= E[(\bar{\mathbf{x}} + \bar{\mathbf{y}})(\bar{\mathbf{x}} + \bar{\mathbf{y}})^T] - (\bar{\boldsymbol{\mu}}_x + \bar{\boldsymbol{\mu}}_y)(\bar{\boldsymbol{\mu}}_x + \bar{\boldsymbol{\mu}}_y)^T \\ &= \mathbf{C}_{xx} + 2\mathbf{C}_{xy} + \mathbf{C}_{yy} \\ &= \mathbf{C}_{xx} + \mathbf{C}_{yy} \text{ if } \bar{\mathbf{x}} \text{ and } \bar{\mathbf{y}} \text{ are independent} \end{aligned}$$



Probabilistic Error Modeling: Multivariable Gaussian

Suppose $\bar{\mathbf{x}} \sim N(\bar{\boldsymbol{\mu}}_x, \mathbf{C}_{xx})$, $\bar{\mathbf{y}} \sim N(\bar{\boldsymbol{\mu}}_y, \mathbf{C}_{yy})$, and $\bar{\mathbf{z}} = \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{y}} \end{bmatrix}$

$$\text{Then } \mathbf{C}_{zz} = \begin{bmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xy} \\ \mathbf{C}_{yx} & \mathbf{C}_{yy} \end{bmatrix}$$

If we know that $\bar{\mathbf{y}} = \bar{\mathbf{a}}$ then $[\bar{\mathbf{x}}|\bar{\mathbf{y}} = \bar{\mathbf{a}}] \sim N(\bar{\boldsymbol{\mu}}_{[x|y=a]}, \mathbf{C}_{[x|y=a]})$, where

$$\bar{\boldsymbol{\mu}}_{[x|y=a]} = \bar{\boldsymbol{\mu}}_x + \mathbf{C}_{xy} \mathbf{C}_{yy}^{-1} (\bar{\mathbf{a}} - \bar{\boldsymbol{\mu}}_y)$$

$$\mathbf{C}_{[x|y=a]} = \mathbf{C}_{xx} - \mathbf{C}_{xy} \mathbf{C}_{yy}^{-1} \mathbf{C}_{yx}$$

Also $\bar{\mathbf{y}}$ and $\bar{\mathbf{x}} - \mathbf{C}_{xy} \mathbf{C}_{yy}^{-1} \bar{\mathbf{y}}$ are independent.



Probabilistic Error Modeling: Multivariable Gaussian

Suppose $\bar{\mathbf{x}} \sim N(\bar{\mu}_x, \mathbf{C}_{xx})$, $\bar{\mathbf{y}} \sim N(\bar{\mu}_y, \mathbf{C}_{yy})$ are two different ways to estimate the same quantity, then we can define $\bar{\mathbf{z}} = \bar{\mathbf{x}} - \bar{\mathbf{y}}$

$$\bar{\mu}_z = E[\bar{\mathbf{x}} - \bar{\mathbf{y}}] = \bar{\mu}_x - \bar{\mu}_y$$

$$\mathbf{C}_{zz} = \mathbf{C}_{xx} - 2\mathbf{C}_{xy} + \mathbf{C}_{yy}; \mathbf{C}_{xz} = \mathbf{C}_{xx} - \mathbf{C}_{xy}; \mathbf{C}_{zx} = \mathbf{C}_{xx} - \mathbf{C}_{yx}$$

$$\text{Define } \bar{\mathbf{w}} = \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{z}} \end{bmatrix}$$

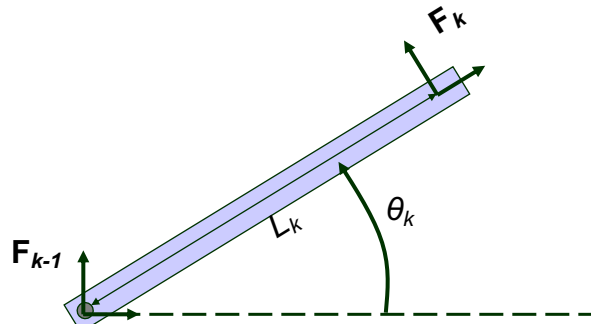
Then

$$E[\bar{\mathbf{x}} | \bar{\mathbf{z}} = \bar{\mathbf{0}}] = \bar{\mu}_x + \mathbf{C}_{xz} \mathbf{C}_{zz}^{-1} (\bar{\mathbf{0}} - \bar{\mu}_z) = \bar{\mu}_x + \mathbf{C}_{xz} \mathbf{C}_{zz}^{-1} (\bar{\mu}_y - \bar{\mu}_x)$$

$$\text{Cov}[\bar{\mathbf{x}} | \bar{\mathbf{z}} = \bar{\mathbf{0}}] = \mathbf{C}_{xx} - \mathbf{C}_{xz} \mathbf{C}_{zz}^{-1} \mathbf{C}_{zx}$$



Error Propagation in Chains



$$\mathbf{F}_k^* = \mathbf{F}_{k-1}^* \bullet \mathbf{F}_{k-1,k}^*$$

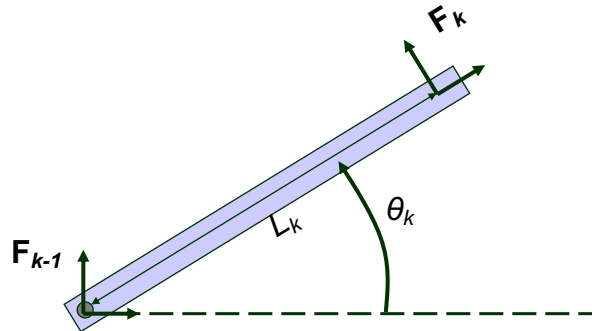
$$\mathbf{F}_k \Delta \mathbf{F}_k = \mathbf{F}_{k-1} \Delta \mathbf{F}_{k-1} \mathbf{F}_{k-1,k} \Delta \mathbf{F}_{k-1,k}$$

$$\Delta \mathbf{F}_k = (\mathbf{F}_k^{-1} \mathbf{F}_{k-1}) \Delta \mathbf{F}_{k-1} \mathbf{F}_{k-1,k} \Delta \mathbf{F}_{k-1,k}$$

$$= (\mathbf{F}_{k-1,k}^{-1} \Delta \mathbf{F}_{k-1} \mathbf{F}_{k-1,k}) \Delta \mathbf{F}_{k-1,k}$$



Error Propagation in Chains



$$\begin{aligned}\Delta \mathbf{F}_k &= (\mathbf{F}_k^{-1} \mathbf{F}_{k-1}) \Delta \mathbf{F}_{k-1} \mathbf{F}_{k-1,k} \Delta \mathbf{F}_{k-1,k} \\ &= (\mathbf{F}_{k-1,k}^{-1} \Delta \mathbf{F}_{k-1} \mathbf{F}_{k-1,k}) \Delta \mathbf{F}_{k-1,k} \\ \Delta \mathbf{R}_k &= (\mathbf{R}_{k-1,k}^{-1} \Delta \mathbf{R}_{k-1} \mathbf{R}_{k-1,k}) \Delta \mathbf{R}_{k-1,k} \\ &\approx (\mathbf{R}_{k-1,k}^{-1} (\mathbf{I} + \text{skew}(\vec{\alpha}_{k-1})) \mathbf{R}_{k-1,k}) (\mathbf{I} + \text{skew}(\vec{\alpha}_{k-1,k})) \\ &\approx \mathbf{I} + (\mathbf{R}_{k-1,k}^{-1} \text{skew}(\vec{\alpha}_{k-1}) \mathbf{R}_{k-1,k}) + \text{skew}(\vec{\alpha}_{k-1,k}) = \mathbf{I} + \text{skew}(\mathbf{R}_{k-1,k}^{-1} \vec{\alpha}_{k-1} + \vec{\alpha}_{k-1,k})\end{aligned}$$

Note: This is same as what we could have obtained by substituting in formulas from the "error from frame composition" slides given earlier.

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Exercise

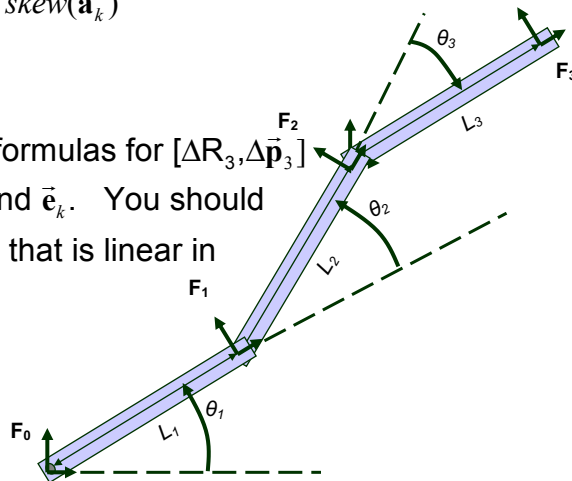
Suppose that you have

$$\Delta \mathbf{R}_{k-1,k} = \Delta \mathbf{R}(\vec{\mathbf{a}}_k) \cong \mathbf{I} + \text{skew}(\vec{\mathbf{a}}_k)$$

$$\Delta \vec{\mathbf{p}}_{k-1,k} = \vec{\mathbf{e}}_k$$

Work out approximate formulas for $[\Delta \mathbf{R}_3, \Delta \vec{\mathbf{p}}_3]$

in terms of $L_k, \vec{\mathbf{r}}_k, \theta_k, \vec{\mathbf{a}}_k$ and $\vec{\mathbf{e}}_k$. You should come up with a formula that is linear in $L_k, \vec{\mathbf{a}}_k$, and $\vec{\mathbf{e}}_k$.



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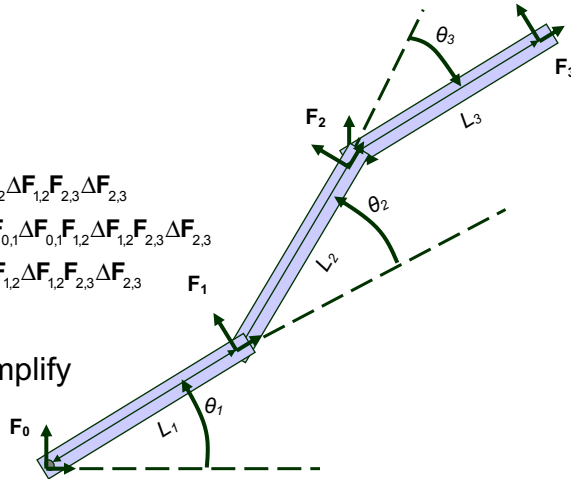
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Exercise

Suppose we want to know error in $F_{0,3} = F_0^{-1}F_3$

$$\begin{aligned}
 F_{0,3} &= F_0^{-1}F_1F_2F_3 \\
 F_{0,3}^* &= F_{0,1}^*F_{1,2}^*F_{2,3}^* \\
 F_{0,3}\Delta F_{0,3} &= F_{0,1}^*F_{1,2}^*F_{2,3}^* \\
 \Delta F_{0,3} &= F_{0,3}^{-1}F_{0,1}\Delta F_{0,1}F_{0,1}^{-1}F_{1,2}\Delta F_{1,2}F_{1,2}^{-1}F_{2,3}\Delta F_{2,3} \\
 &= F_{2,3}^{-1}F_{1,2}^{-1}F_{0,1}^{-1}F_{0,1}\Delta F_{0,1}F_{0,1}^{-1}F_{1,2}\Delta F_{1,2}F_{1,2}^{-1}F_{2,3}\Delta F_{2,3} \\
 &= F_{2,3}^{-1}F_{1,2}^{-1}\Delta F_{0,1}F_{0,1}\Delta F_{1,2}F_{1,2}\Delta F_{2,3}
 \end{aligned}$$

Now substitute and simplify



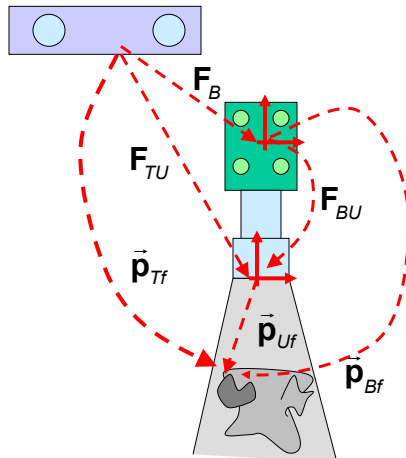
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Another Example



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Another Example

$$\vec{p}_{Tf} = F_{TU} \cdot \vec{p}_{Uf}$$

$$\begin{aligned} F_{TU} &= F_B \cdot F_{BU} \\ &= [R_B \cdot R_{BU}, R_B \cdot \vec{p}_{BU} + \vec{p}_B] \end{aligned}$$

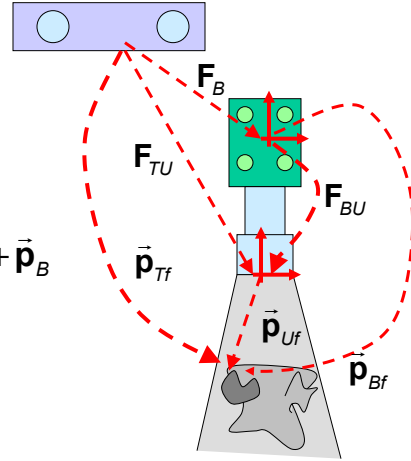
$$\vec{p}_{Tf} = R_B \cdot R_{BU} \cdot \vec{p}_{Uf} + R_B \cdot \vec{p}_{BU} + \vec{p}_B$$

Also

$$\vec{p}_{Tf} = F_B \cdot \vec{p}_{Bf}$$

$$\begin{aligned} \vec{p}_{Bf} &= F_{BU} \cdot \vec{p}_{Uf} \\ &= R_{BU} \cdot \vec{p}_{Uf} + \vec{p}_{BU} \end{aligned}$$

$$\vec{p}_{Tf} = R_B \cdot R_{BU} \cdot \vec{p}_{Uf} + R_B \cdot \vec{p}_{BU} + \vec{p}_B$$



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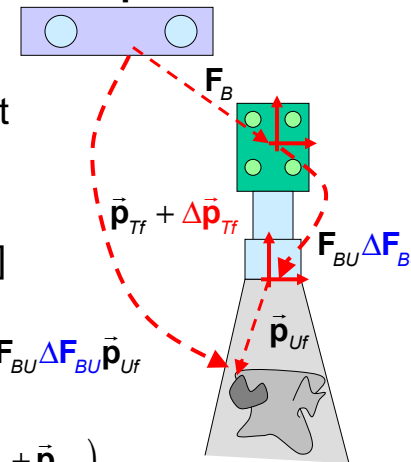
Another Example

Suppose that the track body to US calibration is not perfect

$$\begin{aligned} F_{BU}^* &= F_{BU} \Delta F_{BU} \\ &= [R_{BU} \Delta R_{BU}, R_{BU} \Delta \vec{p}_{BU} + \vec{p}_{BU}] \end{aligned}$$

$$\vec{p}_{Bf}^* = F_{BU}^* \cdot \vec{p}_{Uf} \quad \text{i.e.,} \quad \vec{p}_{Bf} + \Delta \vec{p}_{Bf} = F_{BU} \Delta F_{BU} \vec{p}_{Uf}$$

$$\begin{aligned} \Delta \vec{p}_{Bf} &= F_{BU} \Delta F_{BU} \vec{p}_{Uf} - \vec{p}_{Bf} \\ &= F_{BU} (\Delta R_{BU} \vec{p}_{Uf} + \Delta \vec{p}_{BU}) - (R_{BU} \vec{p}_{Uf} + \vec{p}_{BU}) \\ &= R_{BU} \Delta R_{BU} \vec{p}_{Uf} + R_{BU} \Delta \vec{p}_{BU} + \vec{p}_{BU} - R_{BU} \vec{p}_{Uf} - \vec{p}_{BU} \\ &= R_{BU} \Delta R_{BU} \vec{p}_{Uf} + R_{BU} \Delta \vec{p}_{BU} - R_{BU} \vec{p}_{Uf} \end{aligned}$$



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Another Example

Continuing ...

$$\begin{aligned}
 \Delta \vec{p}_{Bf} &= \mathbf{R}_{BU} \Delta \mathbf{R}_{BU} \vec{p}_{Uf} + \mathbf{R}_{BU} \Delta \vec{p}_{BU} - \mathbf{R}_{BU} \vec{p}_{Uf} \\
 &\approx \mathbf{R}_{BU} (\mathbf{I} + \text{skew}(\vec{\alpha}_{BU})) \vec{p}_{Uf} + \mathbf{R}_{BU} \Delta \vec{p}_{BU} - \mathbf{R}_{BU} \vec{p}_{Uf} \\
 &= \cancel{\mathbf{R}_{BU} \vec{p}_{Uf}} + \mathbf{R}_{BU} \bullet \vec{\alpha}_{BU} \times \vec{p}_{Uf} + \mathbf{R}_{BU} \Delta \vec{p}_{BU} - \cancel{\mathbf{R}_{BU} \vec{p}_{Uf}} \\
 &= \mathbf{R}_{BU} \bullet \vec{\alpha}_{BU} \times \vec{p}_{Uf} + \mathbf{R}_{BU} \Delta \vec{p}_{BU} \\
 &= -\mathbf{R}_{BU} \bullet \vec{p}_{Uf} \times \vec{\alpha}_{BU} + \mathbf{R}_{BU} \Delta \vec{p}_{BU} \\
 &= \mathbf{R}_{BU} \text{skew}(-\vec{p}_{Uf}) \vec{\alpha}_{BU} + \mathbf{R}_{BU} \Delta \vec{p}_{BU}
 \end{aligned}$$

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Another Example

$$\begin{aligned}
 \vec{p}_{Tf} + \Delta \vec{p}_{Tf} &= \mathbf{F}_B \Delta \mathbf{F}_B (\vec{p}_{Bf} + \Delta \vec{p}_{Bf}) \\
 \Delta \vec{p}_{Tf} &= \mathbf{F}_B \Delta \mathbf{F}_B (\vec{p}_{Bf} + \Delta \vec{p}_{Bf}) - \mathbf{F}_B \vec{p}_{Bf} \\
 \Delta \mathbf{F}_B (\vec{p}_{Bf} + \Delta \vec{p}_{Bf}) &= \Delta \mathbf{R}_B (\vec{p}_{Bf} + \Delta \vec{p}_{Bf}) + \Delta \vec{p}_B \\
 &\approx (\mathbf{I} + \text{skew}(\vec{\alpha}_B)) (\vec{p}_{Bf} + \Delta \vec{p}_{Bf}) + \Delta \vec{p}_B \\
 &= (\vec{p}_{Bf} + \Delta \vec{p}_{Bf}) + \vec{\alpha}_B \times \vec{p}_{Bf} + \cancel{\vec{\alpha}_B \times \Delta \vec{p}_{Bf}} + \Delta \vec{p}_B \\
 &\approx \vec{p}_{Bf} + \Delta \vec{p}_{Bf} + \vec{\alpha}_B \times \vec{p}_{Bf} + \Delta \vec{p}_B \\
 \Delta \vec{p}_{Tf} &\approx \mathbf{F}_B (\vec{p}_{Bf} + \Delta \vec{p}_{Bf} + \vec{\alpha}_B \times \vec{p}_{Bf} + \Delta \vec{p}_B) - \mathbf{F}_B \vec{p}_{Bf} \\
 &= \mathbf{R}_B (\vec{p}_{Bf} + \Delta \vec{p}_{Bf} + \vec{\alpha}_B \times \vec{p}_{Bf} + \Delta \vec{p}_B) + \vec{p}_B - (\mathbf{R}_B \vec{p}_{Bf} + \vec{p}_B) \\
 &= \mathbf{R}_B (\Delta \vec{p}_{Bf} + \vec{\alpha}_B \times \vec{p}_{Bf} + \Delta \vec{p}_B)
 \end{aligned}$$

$$\Delta \vec{p}_{Bf} \approx \mathbf{R}_{BU} \text{skew}(-\vec{p}_{BU}) \vec{\alpha}_{BU} + \mathbf{R}_{BU} \Delta \vec{p}_{BU}$$

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Another Example

$$\Delta \vec{p}_{Tf} \approx \mathbf{R}_B (\Delta \vec{p}_{Bf} + \vec{\alpha}_B \times \vec{p}_{Bf} + \Delta \vec{p}_B)$$

$$\Delta \vec{p}_{Bf} \approx \mathbf{R}_{BU} \text{skew}(-\vec{p}_{BU}) \vec{\alpha}_{BU} + \mathbf{R}_{BU} \Delta \vec{p}_{BU}$$

$$\Delta \vec{p}_{Tf} \approx \mathbf{R}_B (\mathbf{R}_{BU} \text{skew}(-\vec{p}_{BU}) \vec{\alpha}_{BU} + \mathbf{R}_{BU} \Delta \vec{p}_{BU} + \vec{\alpha}_B \times \vec{p}_{Bf} + \Delta \vec{p}_B)$$

$$= \begin{pmatrix} \mathbf{R}_B \mathbf{R}_{BU} \text{skew}(-\vec{p}_{BU}) \vec{\alpha}_{BU} + \mathbf{R}_B \mathbf{R}_{BU} \Delta \vec{p}_{BU} \\ \mathbf{R}_B \text{skew}(-\vec{p}_{Bf}) \vec{\alpha}_B + \mathbf{R}_B \Delta \vec{p}_B \end{pmatrix}$$

$$= \begin{bmatrix} \mathbf{R}_B \mathbf{R}_{BU} \text{skew}(-\vec{p}_{BU}) & | & \mathbf{R}_B \mathbf{R}_{BU} & | & \mathbf{R}_B \text{skew}(-\vec{p}_{Bf}) & | & \mathbf{R}_B \end{bmatrix} \begin{bmatrix} \vec{\alpha}_{BU} \\ \Delta \vec{p}_{BU} \\ \vec{\alpha}_B \\ \Delta \vec{p}_B \end{bmatrix}$$



Parametric Sensitivity

Suppose you have an explicit formula like

$$\vec{p}_3 = \begin{bmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ 0 \end{bmatrix}$$

and know that the only variation is in parameters like L_k and θ_k . Then you can estimate the variation in \vec{p}_3 as a function of variation in L_k and θ_k by remembering your calculus.

$$\Delta \vec{p}_3 \cong \begin{bmatrix} \frac{\partial \vec{p}_3}{\partial L} & \frac{\partial \vec{p}_3}{\partial \theta} \end{bmatrix} \begin{bmatrix} \Delta \vec{L} \\ \Delta \vec{\theta} \end{bmatrix}$$



Parametric Sensitivity

Grinding this out gives:

$$\Delta \bar{\mathbf{p}}_3 \cong \begin{bmatrix} \frac{\partial \bar{\mathbf{p}}_3}{\partial \bar{L}} & \frac{\partial \bar{\mathbf{p}}_3}{\partial \bar{\theta}} \end{bmatrix} \begin{bmatrix} \Delta \bar{L} \\ \Delta \bar{\theta} \end{bmatrix}$$

where

$$\bar{L} = [L_1, L_2, L_3]^T$$

$$\bar{\theta} = [\theta_1, \theta_2, \theta_3]^T$$

$$\frac{\partial \bar{\mathbf{p}}_3}{\partial \bar{L}} = \begin{bmatrix} \cos(\theta_1) & \cos(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2 + \theta_3) \\ \sin(\theta_1) & \sin(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2 + \theta_3) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \bar{\mathbf{p}}_3}{\partial \bar{\theta}} = \begin{bmatrix} -L_1 \sin(\theta_1) - L_2 \sin(\theta_1 + \theta_2) - L_3 \sin(\theta_1 + \theta_2 + \theta_3) & -L_2 \sin(\theta_1 + \theta_2) - L_3 \sin(\theta_1 + \theta_2 + \theta_3) & -L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) & L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) & L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ 0 & 0 & 0 \end{bmatrix}$$



More generally ...

Suppose that we have a vector function

$$\bar{\mathbf{v}} = \bar{\mathbf{g}}(\bar{\mathbf{q}}) = [g_1(\bar{\mathbf{q}}), \dots, g_m(\bar{\mathbf{q}})]^T$$

of parameters $\bar{\mathbf{q}} = [q_1, \dots, q_n]$. Then we can estimate the value of

$$\bar{\mathbf{v}} + \Delta \bar{\mathbf{v}} = \bar{\mathbf{g}}(\bar{\mathbf{q}} + \Delta \bar{\mathbf{q}})$$

by

$$\bar{\mathbf{v}} + \Delta \bar{\mathbf{v}} \approx \bar{\mathbf{g}}(\bar{\mathbf{q}}) + \mathbf{J}_{\bar{\mathbf{g}}}(\bar{\mathbf{q}}) \bullet \Delta \bar{\mathbf{q}}$$

where

$$\mathbf{J}_{\bar{\mathbf{g}}}(\bar{\mathbf{q}}) = \begin{bmatrix} \frac{\partial g_1}{\partial q_1} & \frac{\partial g_1}{\partial q_j} & \frac{\partial g_1}{\partial q_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial g_j}{\partial q_1} & \dots & \frac{\partial g_j}{\partial q_j} & \dots & \frac{\partial g_j}{\partial q_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_m}{\partial q_1} & \frac{\partial g_m}{\partial q_j} & \frac{\partial g_m}{\partial q_n} \end{bmatrix}$$

