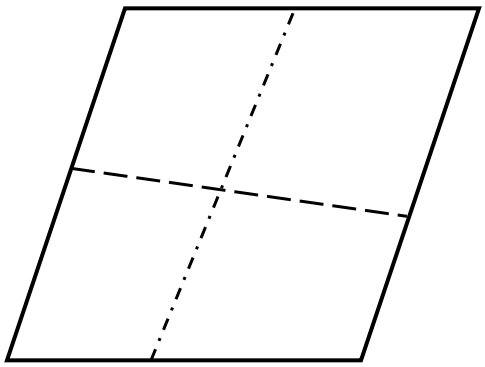
Interpolation and Deformations
A short cookbook



600.445 Fall 2000; Updated: 5 October 2021

Linear Interpolation

$\vec{\mathbf{p}}_2 = \begin{bmatrix} 40 & 30 & 20 \end{bmatrix}^T$

$\rho_2 = 20$

$\vec{\mathbf{p}}_3 = \begin{bmatrix} 20 & 20 & 20 \end{bmatrix}^T$

$\rho_3 = \mathbf{10}$

$\vec{\mathbf{p}}_1 = \begin{bmatrix} 10 & 15 & 20 \end{bmatrix}^T$

$\rho_1 = 5$

600.445 Fall 2000; Updated: 5 October 2021

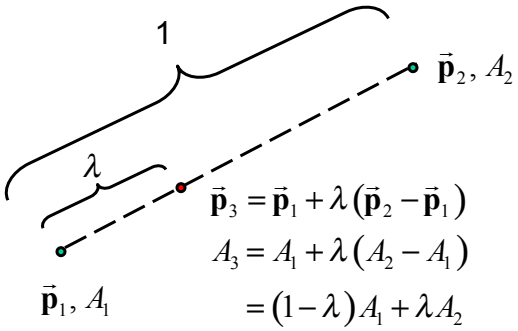
Linear Interpolation

$$\vec{\mathbf{p}}_2 = \begin{bmatrix} 40 & 30 & 20 \end{bmatrix}^T$$
$$\vec{\mathbf{q}}_2 = \vec{\mathbf{b}}$$

$$\vec{\mathbf{p}}_3 = \begin{bmatrix} 20 & 20 & 20 \end{bmatrix}^T$$
$$\vec{\mathbf{q}}_3 = \vec{\mathbf{a}} + \tfrac{1}{3}\left(\vec{\mathbf{b}} - \vec{\mathbf{a}}\right)$$

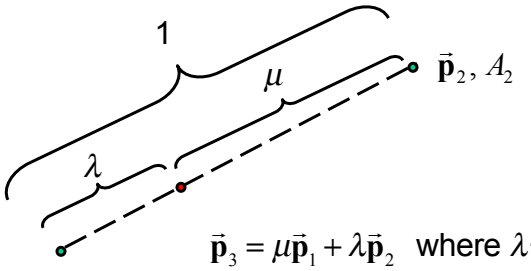
$$\vec{\mathbf{p}}_1 = \begin{bmatrix} 10 & 15 & 20 \end{bmatrix}^T$$
$$\vec{\mathbf{q}}_1 = \vec{\mathbf{a}}$$

Linear Interpolation



$$\vec{\mathbf{p}}_3 = \vec{\mathbf{p}}_1 + \lambda\left(\vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1\right)$$
$$A_3 = A_1 + \lambda\left(A_2 - A_1\right)$$
$$= \left(1-\lambda\right)A_1 + \lambda A_2$$
$$\lambda = \frac{\left(\vec{\mathbf{p}}_3 - \vec{\mathbf{p}}_1\right)\bullet\left(\vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1\right)}{\left(\vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1\right)\bullet\left(\vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1\right)}$$

Linear Interpolation (Barycentric Form)

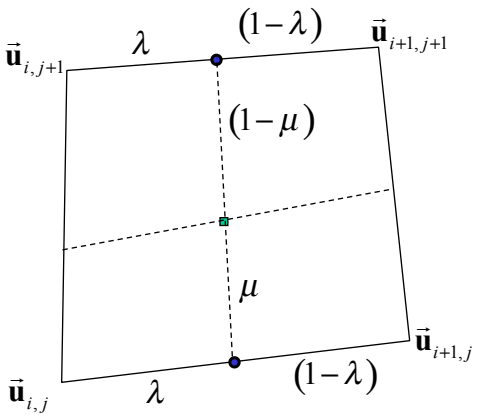


$$\vec{\mathbf{p}}_3 = \mu\vec{\mathbf{p}}_1 + \lambda\vec{\mathbf{p}}_2 \quad \text{where } \lambda+\mu=1$$

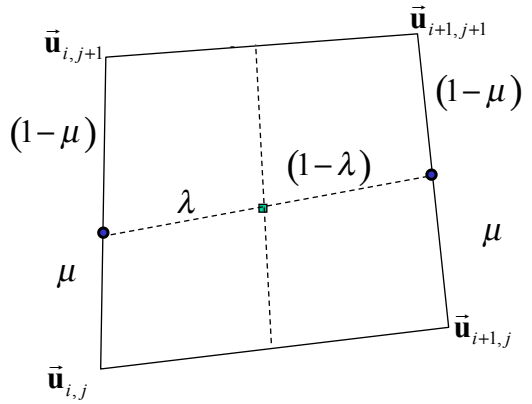
$$A_3 = \mu A_1 + \lambda A_2$$

$$\lambda = \frac{(\vec{\mathbf{p}}_3 - \vec{\mathbf{p}}_1)\bullet(\vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1)}{(\vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1)\bullet(\vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1)}$$

Bilinear Interpolation



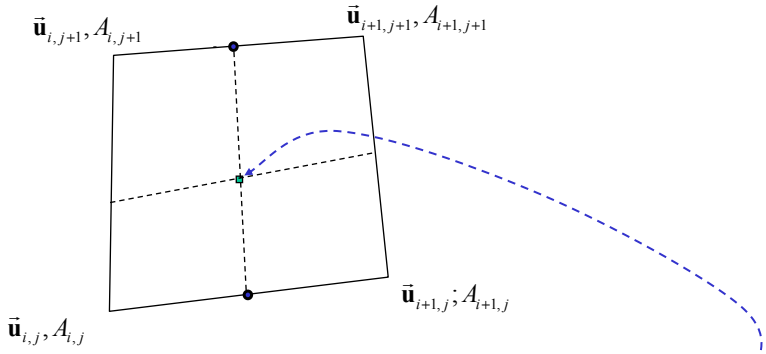
Bilinear Interpolation



$$\vec{\mathbf{u}}(\lambda,\mu) = \lambda\left(\mu\vec{\mathbf{u}}_{i+1,j+1} + (1-\mu)\vec{\mathbf{u}}_{i+1,j}\right) + (1-\lambda)\left(\mu\vec{\mathbf{u}}_{i,j+1} + (1-\mu)\vec{\mathbf{u}}_{i,j}\right)$$

$$= \vec{\mathbf{u}}_{i,j} + \lambda\left(\vec{\mathbf{u}}_{i+1,j} - \vec{\mathbf{u}}_{i,j}\right) + \mu\left(\vec{\mathbf{u}}_{i,j+1} - \vec{\mathbf{u}}_{i,j}\right) + \lambda\mu\left(\vec{\mathbf{u}}_{i+1,j+1} - \vec{\mathbf{u}}_{i,j}\right)$$

Bilinear Interpolation



$$\vec{\mathbf{u}}(\lambda,\mu) = \text{interpolate}(\{\lambda,\mu\},\{\vec{\mathbf{u}}_{i,j},\vec{\mathbf{u}}_{i+1,j},\vec{\mathbf{u}}_{i+1,j+1},\vec{\mathbf{u}}_{i,j+1}\})$$

$$A(\lambda,\mu) = \text{interpolate}(\{\lambda,\mu\},\{A_{i,j},A_{i+1,j},A_{i+1,j+1},A_{i,j+1}\})$$

N-linear Interpolation

Let

$$\bar{\Lambda}_N = \{\lambda_1, \ldots, \lambda_N\}, \text{ with } 0 \le \lambda_k \le 1$$

be a set of interpolation parameters, and let

$$\bar{\mathbf{A}} = \{A_1, \ldots, A_{2^N}\}$$

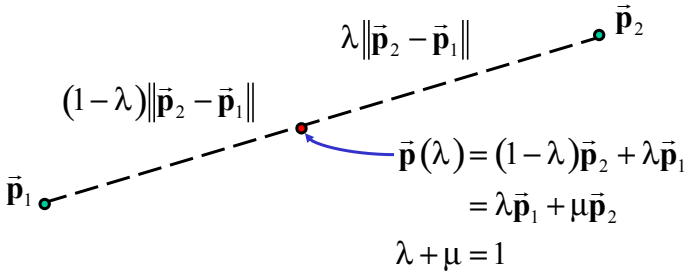
be a set of constants. Then we define:

$$\begin{aligned}\text{NlinearInterpolate}(\Lambda_N, \mathbf{A}) = \\ (1-\lambda_N)\,\text{NlinearInterpolate}(\Lambda_{N-1}, \{A_1, \ldots, A_{2^{N-1}}\}) \\ +\lambda_N\ \text{NlinearInterpolate}(\Lambda_{N-1}, \{A_{2^{N-1}+1}, \ldots, A_{2^N}\})\end{aligned}$$

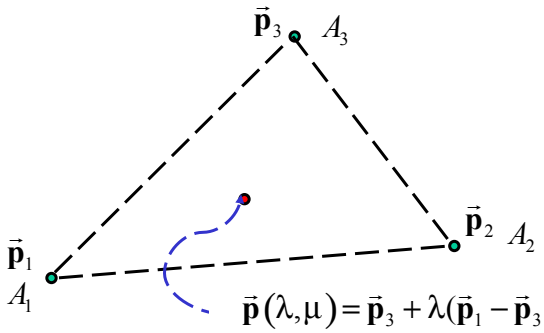
NOTE: Sometimes in this situation we will use notation

$$\begin{aligned}A(\bar{\Lambda}_N) &= A(\lambda_1, \ldots, \lambda_N) \\ &= \text{NlinearInterpolate}(\bar{\Lambda}_N, \bar{\mathbf{A}})\end{aligned}$$

Barycentric Interpolation

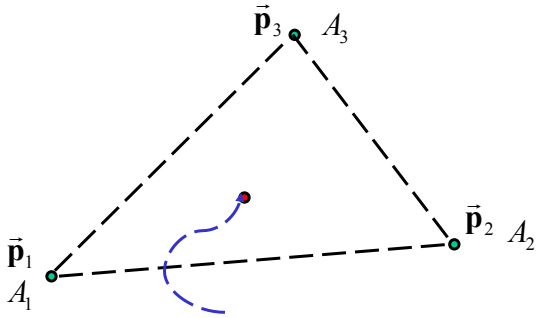


Barycentric Interpolation



$$\vec{\mathbf{p}}(\lambda,\mu) = \vec{\mathbf{p}}_3 + \lambda(\vec{\mathbf{p}}_1 - \vec{\mathbf{p}}_3) + \mu(\vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_3)$$
$$= \lambda\vec{\mathbf{p}}_1 + \mu\vec{\mathbf{p}}_2 + (1-\lambda-\mu)\vec{\mathbf{p}}_3$$
$$\vec{\mathbf{p}}(\lambda,\mu,\nu) = \lambda\vec{\mathbf{p}}_1 + \mu\vec{\mathbf{p}}_2 + \nu\vec{\mathbf{p}}_3 \quad \text{where } \lambda+\mu+\nu = 1$$
$$A(\lambda,\mu,\nu) = \lambda A_1 + \mu A_2 + \nu A_3$$

Barycentric Interpolation



$$\begin{bmatrix}\vec{\mathbf{p}}\\ 1\end{bmatrix} = \begin{bmatrix}\vec{\mathbf{p}}_1 & \vec{\mathbf{p}}_2 & \vec{\mathbf{p}}_3\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}\lambda\\ \mu\\ \nu\end{bmatrix}$$

Barycentric Interpolation

Let

$$\vec{\Lambda} = \{\lambda_1, \ldots, \lambda_N\}, \text{ with } 0 \le \lambda_k \le 1 \text{ and } \sum_{k=1}^{N} \lambda_k = 1$$

be a set of interpolation parameters, and let

$$\vec{\mathbf{A}} = \{A_1, \ldots, A_{2^N}\}$$

be a set of constants. Then we define:

$$\text{BarycentricInterpolate}(\vec{\Lambda}, \vec{\mathbf{A}}) = \vec{\Lambda} \bullet \vec{\mathbf{A}} = \sum_{k=1}^{N} \lambda_k A_k$$

NOTE: Sometimes in this situation we will use notation

$$\mathbf{A}(\Lambda_N) = \mathbf{A}(\lambda_1, \ldots, \lambda_N) = \text{BarycentricInterpolate}(\Lambda_N, \mathbf{A})$$

NOTE: This is a special case of barycentric Bezier polynomial interpolations (here, 1st degree)

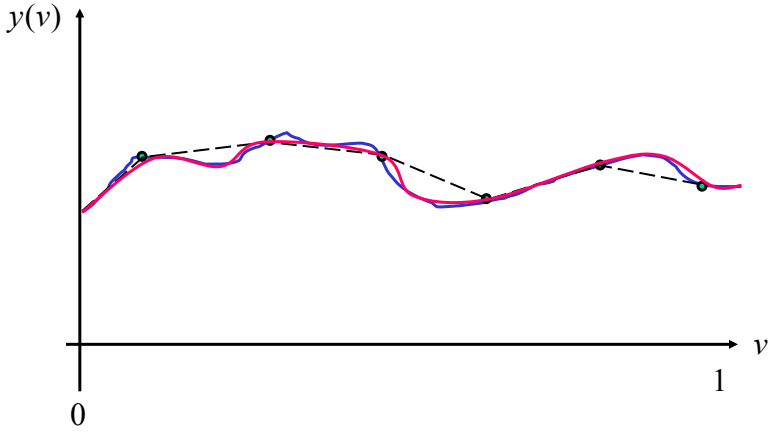
Barycentric Interpolation

Given n+1 n-dimensional points $\vec{\mathbf{a}}_0 \cdots \vec{\mathbf{a}}_n$ and a test point $\vec{\mathbf{a}}_{test}$ find barycentric coordinates $\vec{\lambda}$=[λ_0,...λ_n] such that $\vec{\mathbf{a}}_{test} = \sum_k \lambda_k \vec{\mathbf{a}}_k$ and $\sum_k \lambda_k = 1$.

Solve

$$\begin{bmatrix} \vec{\mathbf{a}}_0 & \cdots & \vec{\mathbf{a}}_n \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{a}}_{test} \\ 1 \end{bmatrix}$$

Interpolation of functions



Fitting of interpolation curves

- The discussion below follows (in part)

  G. Farin, Curves and surfaces for computer-aided geometric design, a practical guide, Academic Press, Boston, 1990, chapter 10 and pp 281-284.

1-D Interpolation

Given set of known values $\{y_0(v_0),...,y_m(v_m)\}$,

find an approximating polynomial $y \approx P(c_0,...,c_N;v)$

$$P(c_0,...,c_N;v) = \sum_{k=0}^{N} c_k P_{N,k}(v)$$

Note that many forms of polynomial may be used for the $P_{N,k}(v)$. One common (not very good) choice is the power basis:

$$P_{N,k}(v) = v^k$$

Better choices are the Bernstein polynomials and the B-spline basis functions, which we will discuss in a moment

1-D Interpolation

Given set of known values $\{y_0(v_0),...,y_m(v_m)\}$,

find an approximating polynomial $y \approx P(c_0,...,c_N;v)$

$$P(c_0,...,c_N;v) = \sum_{k=0}^{N} c_k P_{N,k}(v)$$

To do this, solve:

$$\begin{bmatrix} P_{N,0}(v_0) & \cdots & P_{N,N}(v_0) \\ \vdots & \ddots & \vdots \\ P_{N,0}(v_m) & \cdots & P_{N,N}(v_m) \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_N \end{bmatrix} \approx \begin{bmatrix} y_0 \\ \vdots \\ y_m \end{bmatrix}$$

Bezier and Bernstein Polynomials

$$P(c_0,\ldots,c_N;v) = \sum_{k=0}^{N} c_k \binom{N}{k} (1-v)^{N-k} v^k$$

$$= \sum_{k=0}^{N} c_k B_{N,k}(v)$$

$$\text{where} \quad B_{N,k}(v) = \binom{N}{k} (1-v)^{N-k} v^k$$

- Excellent numerical stability for 0<v<1
- There exist good ways to convert to more conventional power basis

Barycentric Bezier Polynomials

$$P(c_0,\ldots,c_N;u,v) = \sum_{k=0}^{N} c_k \binom{N}{k} u^{N-k} v^k$$

$$= \sum_{k=0}^{N} c_k B_{N,k}(u,v)$$

$$\text{where} \quad B_{N,k}(u,v) = \binom{N}{k} u^{N-k} v^k \qquad u+v=1$$

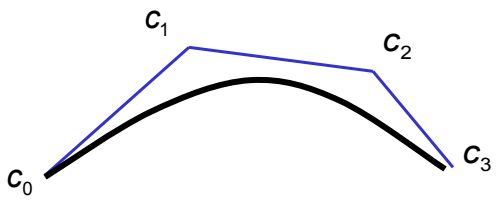
- Excellent numerical stability for 0<u,v<1
- There exist good ways to convert to more conventional power basis

Bezier Curves

Suppose that the coefficients $\vec{c}_j$ are multi-dimensional vectors (e.g., 2D or 3D points). Then the polynomial

$$P\left(\vec{c}_0,\ldots,\vec{c}_N;v\right)=\sum_{k=0}^{N}\vec{c}_k B_{N,k}(v)$$

computed over the range $0 \le v \le 1$ generates a Bezier curve with control vertices $\vec{c}_j$.



Bezier Curves: de Casteljau Algorithm

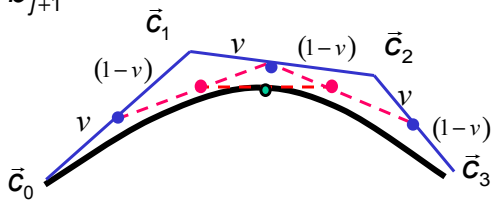
Given coefficients $\vec{c}_j$, Bezier curves can be generated recursively by repeated linear interpolation:

$$P\left(\vec{c}_0,\ldots,\vec{c}_N;v\right)=b_0^N$$

where

$$b_j^0=\vec{c}_j$$

$$b_j^k=(1-v)b_j^{k-1}+vb_{j+1}^{k-1}$$



Iterative Form of deCasteljau Algorithm

Step 1: $b_j \leftarrow c_j$ for $0 \le j \le N$

Step 2: for $k \leftarrow 1$ step 1 until $k = N$ do

for $j \leftarrow 0$ step 1 until $j = N - k$ do

$b_j \leftarrow (1-v)b_j + vb_{j+1}$

Step 3: return b_0

Advantages of Bezier Curves

- Numerically very robust
- Many nice mathematical properties
- Smooth

- "Global" (may be viewed as a disadvantage)

B-splines

Given

coefficient values $\bar{\mathbf{C}} = \{\vec{c}_0, \cdots, \vec{c}_{L+D-1}\}$

"knot points" $\bar{\mathbf{u}} = \{u_0, \cdots, u_{L+2D-2}\}$ with $u_i \le u_{i+1}$

D = "degree" of desired B-spline

Can define an interpolated curve $P(\bar{\mathbf{C}}, \bar{\mathbf{u}};\ u)$ on $u_{D-1} \le u < u_{L+D-1}$

Then

$$P(\bar{\mathbf{C}};u) = \sum_{j=0}^{L+D-1} \vec{c}_j \ \ N_j^D(u)$$

where $N_j^D(u)$ are B-spline basis polynomials (discussed later)

B-Spline Polynomials

Some useful references include

- http://en.wikipedia.org/wiki/B-spline
- http://vision.ucsd.edu/~kbranson/research/bsplines/bsplines.pdf
- http://scholar.lib.vt.edu/theses/available/etd-100699-171723/
- https://www.cs.drexel.edu/~david/Classes/CS430/Lectures/L-09_BSplines_NURBS.pdf
- http://www.stat.columbia.edu/~ruf/ruf_bspline.pdf

B-spline polynomials & B-spline basis functions

Given $\bar{\mathbf{C}}, \bar{\mathbf{u}}, \mathrm{D}$ as before

$$P(\bar{\mathbf{C}}, \bar{\mathbf{u}}; u) = \sum_{j=0}^{L+D-1} \vec{c}_j \quad N_j^D(u)$$

where

$$N_j^0(u) = \begin{cases} 1 & u_{j-1} \le u \le u_j \\ 0 & \text{Otherwise} \end{cases}$$

$$N_j^k(u) = \frac{u - u_{j-1}}{u_{j+k-1} - u_{j-1}} N_j^{k-1}(u) + \frac{u_{j+k} - u}{u_{j+k} - u_j} N_{j+1}^{k-1}(u) \quad \text{for } k > 0$$

B-Spline Polynomials

For a B-spline polynomial

$$P(\bar{\mathbf{C}}, \bar{\mathbf{u}}; t) = \sum_{j=0}^{L+D-1} \vec{c}_j \quad N_j^D(\bar{\mathbf{u}}, t)$$

the basis functions $N_j^D(\bar{\mathbf{u}}, t)$ are a function of the degree of the polynomial and the vector $\bar{\mathbf{u}} = \left[u_0, \cdots, u_n\right]$ of "knot points". The polynomial is "uniform" if the distance between knot points is evenly spaced and "non-uniform" otherwise.

deBoor Algorithm (Farin)

Given **u**, **c**, D as before, can evaluate $P(\mathbf{c},\mathbf{u};u)$
recursively as follows:

Step 1: Determine index i such that $u_i \le u < u_{i+1}$

Step 2: Determine multplicity r such that

$$u_{i-r} = u_{i-r+1} = \cdots = u_i$$

Step 3: Set $\vec{\mathbf{d}}_j^0 = c_j$ for $i - D + 1 \le j \le i + 1$

Step 4: Compute $P(\mathbf{c},\mathbf{u};u) = d_{i+1}^{D-r}$ recursively, where

$$\vec{\mathbf{d}}_j^k = \frac{u_{j+D-k} - u}{u_{j+D-k} - u_{j-1}} \vec{\mathbf{d}}_{j-1}^{k-1} + \frac{u - u_{j-1}}{u_{j+D-k} - u_{j-1}} \vec{\mathbf{d}}_j^{k-1} = \frac{\alpha_j{}^k \vec{\mathbf{d}}_{j-1}^{k-1}}{\gamma_j{}^k} + \frac{\beta_j{}^k \vec{\mathbf{d}}_j^{k-1}}{\gamma_j{}^k}$$

deBoor Algorithm: Example D=3, r=0

$$\vec{\mathbf{d}}_j^k = \frac{u_{j+D-k} - u}{u_{j+D-k} - u_{j-1}} \vec{\mathbf{d}}_{j-1}^{k-1} + \frac{u - u_{j-1}}{u_{j+D-k} - u_{j-1}} \vec{\mathbf{d}}_j^{k-1} = \frac{\alpha_j^k \vec{\mathbf{d}}_{j-1}^{k-1}}{\gamma_j^k} + \frac{\beta_j^k \vec{\mathbf{d}}_j^{k-1}}{\gamma_j^k}$$

$$\vec{\mathbf{d}}_{i+1}^{3-0} = \vec{\mathbf{p}}(\{\cdots, \vec{\mathbf{c}}_i, \cdots\}, 3; u)$$

$$= \left(\alpha_{i+1}^3 \vec{\mathbf{d}}_i^2 + \beta_{i+1}^3 \vec{\mathbf{d}}_{i+1}^2\right) \Big/ \gamma_{i+1}^3 = \left((u_{i+1} - u)\vec{\mathbf{d}}_i^2 + (u - u_i)\vec{\mathbf{d}}_{i+1}^2\right) \Big/ (u_{i+1} - u_i)$$

$$\vec{\mathbf{d}}_{i+1}^2 = \left(\alpha_{i+1}^2 \vec{\mathbf{d}}_i^1 + \beta_{i+1}^2 \vec{\mathbf{d}}_{i+1}^1\right) \Big/ \gamma_{i+1}^2 = \left((u_{i+2} - u)\vec{\mathbf{d}}_i^1 + (u - u_i)\vec{\mathbf{d}}_{i+1}^1\right) \Big/ (u_{i+2} - u_i)$$

$$\vec{\mathbf{d}}_i^2 = \left(\alpha_i^2 \vec{\mathbf{d}}_{i-1}^1 + \beta_i^2 \vec{\mathbf{d}}_i^1\right) \Big/ \gamma_i^2 = \left((u_{i+1} - u)\vec{\mathbf{d}}_{i-1}^1 + (u - u_{i-1})\vec{\mathbf{d}}_i^1\right) \Big/ (u_{i+1} - u_{i-1})$$

$$\vec{\mathbf{d}}_{i+1}^1 = \left(\alpha_{i+1}^1 \vec{\mathbf{d}}_i^0 + \beta_{i+1}^1 \vec{\mathbf{d}}_{i+1}^0\right) \Big/ \gamma_{i+1}^1 = \left((u_{i+3} - u)\vec{\mathbf{d}}_i^0 + (u - u_i)\vec{\mathbf{d}}_{i+1}^0\right) \Big/ (u_{i+3} - u_i)$$

$$\vec{\mathbf{d}}_i^1 = \left(\alpha_i^1 \vec{\mathbf{d}}_{i-1}^0 + \beta_i^1 \vec{\mathbf{d}}_i^0\right) \Big/ \gamma_i^1 = \left((u_{i+2} - u)\vec{\mathbf{d}}_{i-1}^0 + (u - u_{i-1})\vec{\mathbf{d}}_i^0\right) \Big/ (u_{i+2} - u_{i-1})$$

$$\vec{\mathbf{d}}_{i-1}^1 = \left(\alpha_{i-1}^1 \vec{\mathbf{d}}_{i-2}^0 + \beta_{i-1}^1 \vec{\mathbf{d}}_{i-1}^0\right) \Big/ \gamma_{i-1}^1 = \left((u_{i+1} - u)\vec{\mathbf{d}}_{i-2}^0 + (u - u_{i-2})\vec{\mathbf{d}}_{i-1}^0\right) \Big/ (u_{i+1} - u_{i-2})$$

deBoor Algorithm (alternative formula)

An alternative formulation from Wikipedia is given as:

Given **u**, **c**, D as before, can evaluate $P(\mathbf{c},\mathbf{u};u)$ recursively as follows:

Step 1: Determine index i such that $u_i \le u < u_{i+1}$

Step 2: Determine multplicity r such that

$$u_{i-r} = u_{i-r+1} = \cdots = u_i$$

Step 3: Set $\vec{\mathbf{d}}_j^0 = c_j$ for $i - D + 1 \le j \le i + 1$

Step 4: Compute $P(\mathbf{c},\mathbf{u};u) = d_{i+1}^{D-r}$ recursively, where

$$\vec{\mathbf{d}}_j^k = (1-\alpha_{k,j})\vec{\mathbf{d}}_{j-1}^{k-1} + \alpha_{k,j}\vec{\mathbf{d}}_j^{k-1} \text{ where } \alpha_{k,i} = \frac{u-u_j}{u_{j+D+1-k}-u_j}$$

Source: https://en.wikipedia.org/wiki/De_Boor%27s_algorithm

Uniform B-Spline Polynomials

Third degree uniform B-spline $P(\bar{\mathbf{C}},\bar{\mathbf{u}};t) = \sum_j \vec{c}_j N_j^2(\bar{\mathbf{u}},t)$ with $t_j = j$

$$N_j^3(\bar{\mathbf{u}},t) = \begin{cases} \frac{1}{6}(t-j)^2 & \text{if } j \le t < j+1 \\ \frac{1}{6}\left[-3(t-j-1)^3 + 3(t-j-1)^2 + 3(t-j-1) + 1\right] & \text{if } j+1 \le t < j+2 \\ \frac{1}{6}\left[3(t-j-1)^3 - 6(t-j-1)^2 + 4\right] & \text{if } j+2 \le t < j+3 \\ \frac{1}{6}\left[1-(t-j-1)\right]^3 & \text{if } j+3 \le t < j+4 \\ 0 & \text{otherwise} \end{cases}$$

http://vision.ucsd.edu/~kbranson/research/bsplines/bsplines.pdf

Some advantages of B-splines

- Efficient
- Numerically stable
- Smooth
- Local

2D Interpolation (tensor form)

Consider the 2D polynomial

$$P(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij} A_i(u) B_j(v)$$

$$= [A_0(u), \cdots, A_m(u)] \begin{bmatrix} c_{00} & \cdots & c_{0n} \\ \vdots & \ddots & \vdots \\ c_{m0} & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} B_0(v) \\ \vdots \\ B_n(v) \end{bmatrix}$$

where $A_i(u)$ and $B_j(v)$ can be arbitrary functions (good choices Bernstein polynomials or B-Spline basis functions. Suppose that we have samples

$\mathbf{y_s} = \mathbf{y}(u_s, v_s)$ for $s = 0,...,N_s$

We want to find an approximating polynomial P.

2D Interpolation: Finding the best fit

Given a set of sample values $\mathbf{y}_s(u_s,v_s)$ corresponding to 2D coordinates (u_s,v_s), left hand side basis functions $\left[A_0(u),\cdots,A_m(u)\right]$ and right hand side basis functions $\left[B_0(v),\cdots,B_n(v)\right]$, the goal is to find the matrix $\mathbf{C}$ of coefficients $\mathbf{c}_{ij}$.

To do this, solve the least squares problem

$$\begin{bmatrix} \vdots \\ \mathbf{y}_s(u_s,v_s) \\ \vdots \end{bmatrix} \approx \begin{bmatrix} \vdots & \vdots & & \vdots & & \vdots \\ A_0(u_s)B_0(v_s) & A_0(u_s)B_1(v_s) & \cdots & A_i(u_s)B_j(v_s) & \cdots & A_m(u_s)B_n(v_s) \\ \vdots & \vdots & & \vdots & & \vdots \end{bmatrix} \bullet \begin{bmatrix} \mathbf{c}_{00} \\ \mathbf{c}_{01} \\ \vdots \\ \mathbf{c}_{ij} \\ \vdots \\ \mathbf{c}_{mn} \end{bmatrix}$$

2D Interpolation: Sampling on a regular grid

A common special case arises when the (u_s,v_s) form a regular grid. In this case we have $u_s \in \{u_0,\cdots,u_{N_u}\}$ and $v_s \in \{v_0,\cdots,v_{N_v}\}$. For each value $v_j \in \{v_0,\cdots,v_{N_v}\}$ solve the N_s row least squares problem

$$\begin{bmatrix} \vdots \\ \mathbf{y}_s(u_s,v_j) \\ \vdots \end{bmatrix} \approx \begin{bmatrix} \vdots & \cdots & \vdots \\ A_0(u_s) & \cdots & A_m(u_s) \\ \vdots & \cdots & \vdots \end{bmatrix} \bullet \begin{bmatrix} \mathbf{X}_{j0} \\ \vdots \\ \mathbf{X}_{jm} \end{bmatrix}$$

for the unknown m-vector $\mathbf{X}_j$. Then solve m n-variable least squares problems

$$\left[\begin{array}{c|c|c|c} \mathbf{X}_{00} & \mathbf{X}_{01} & \cdots & \mathbf{X}_{0m} \\ \mathbf{X}_{10} & \mathbf{X}_{11} & \cdots & \mathbf{X}_{1m} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ \mathbf{X}_{N_v 0} & \mathbf{X}_{N_v 1} & \cdots & \mathbf{X}_{N_v m} \end{array}\right] \approx \begin{bmatrix} B_0(v_0) & B_1(v_0) & \cdots & B_n(v_0) \\ B_0(v_1) & B_1(v_1) & \cdots & B_n(v_1) \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ B_0(v_{N_v}) & B_1(v_{N_v}) & \cdots & B_n(v_{N_v}) \end{bmatrix} \bullet \left[\begin{array}{c|c|c|c} \mathbf{c}_{00} & \mathbf{c}_{10} & \cdots & \mathbf{c}_{m0} \\ \mathbf{c}_{01} & \mathbf{c}_{11} & \cdots & \mathbf{c}_{m1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{c}_{0n} & \mathbf{c}_{1n} & \cdots & \mathbf{c}_{mn} \end{array}\right]$$

for the vectors $\left[\mathbf{c}_{j0},\cdots,\mathbf{c}_{jn}\right]$. Note that this latter step requires only 1 SVD or similar matrix computation.

2D Interpolation: Sampling on a regular grid

- There are a number of caveats to the "grid" method on the previous slide. (E.g., you need enough data for each of the least squares problems). But where applicable the method can save computation time since it replaces a number of m and n variable least squares problems for one big m x n problem
- Note that there is a similar trick that you can play by grouping all the common u_i elements together.
- Note that the **y**'s and the **c**'s do not have to be scalar numbers. They can be Vectors, Matrices, or other objects that have appropriate algebraic properties

N-dimensional interpolation

Define

$$F_{i_1 \cdots i_N}(\vec{\mathbf{u}}) = A^1{}_{i_1}(u_1) \cdots A^N{}_{i_N}(u_N)$$

Then solve the least squares problem

$$\begin{bmatrix} \vdots & & \\ F_{00\cdots0}(\vec{\mathbf{u}}_s) & F_{10\cdots0}(\vec{\mathbf{u}}_s) & \cdots \quad F_{m_1\cdots m_n}(\vec{\mathbf{u}}_s) \\ \vdots & & \end{bmatrix} \begin{bmatrix} c_{00\cdots0} \\ c_{10\cdots0} \\ \vdots \\ c_{m_1\cdots m_n} \end{bmatrix} \cong \begin{bmatrix} \vdots \\ \vec{\mathbf{y}}_s \\ \vdots \end{bmatrix}$$

N-dimensional interpolation

- The methods described earlier generalize naturally to N dimensions.

$$P(\vec{\mathbf{u}}) = P(u_1, \cdots, u_N) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_N=0}^{m_N} c_{i_1 \cdots i_n} A^1{}_{i_1}(u_1) \cdots A^N{}_{i_N}(u_N)$$

where $A_i^K(u)$ can be arbitrary functions

(good choices are Bernstein polynomials or

B-Spline basis functions). Suppose that we have samples

$\mathbf{y_s} = \mathbf{y}(\vec{\mathbf{u}}_s)$ for $s = 0, ..., N_s$

We want to find coefficients of $c_{i_1 \cdots i_n}$ approximating polynomial P.

Example: 3D Calibration of Distortion

Suppose we want to compute a distortion correction function for a distorted 3D navigational sensor. Let

$\vec{\mathbf{p}}_i$ = known 3D "ground truth"

$\vec{\mathbf{q}}_i$ = Values returned by navigational sensor

Here we will construct a "tensor form" interpolation polynomial using 5th degree Bernstein polynomials

$$F_{ijk}(u_x, u_y, u_z) = B_{5,i}(u_x) B_{5,j}(u_y) B_{5,k}(u_z)$$

We need to do the following:

1. Bernstein polynomials are really designed to work well in the range $0 \le u \le 1$, so we need to determine a "bounding box" to scale our $\vec{\mathbf{q}}_i$ values. I.e., we pick upper and lower limits $\vec{\mathbf{q}}^{\min}$ and $\vec{\mathbf{q}}^{\max}$ and compute $\vec{\mathbf{u}}_s = ScaleToBox(\vec{\mathbf{q}}_s, \vec{\mathbf{q}}^{\min}, \vec{\mathbf{q}}^{\max})$ where

$$ScaleToBox(x, x^{\min}, x^{\max}) = \frac{x - x^{\min}}{x^{\max} - x^{\min}}$$

2. Now, we set up and solve the least squares problem:

Example: 3D Calibration of Distortion

$$\begin{bmatrix} & \vdots & \\ F_{000}(\vec{\mathbf{u}}_s) & \cdots & F_{555}(\vec{\mathbf{u}}_s) \\ & \vdots & \end{bmatrix} \begin{bmatrix} c^x_{000} & c^y_{000} & c^z_{000} \\ \vdots & \vdots & \vdots \\ c^x_{555} & c^y_{555} & c^z_{555} \end{bmatrix} \approx \begin{bmatrix} & \vdots & \\ p^x_s & p^y_s & p^z_s \\ & \vdots & \end{bmatrix}$$

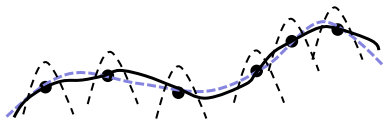
Example: 3D Calibration of Distortion

The correction function will then look like this:

$$\vec{\mathbf{p}} = CorrectDistortion(\vec{\mathbf{q}})$$
$$\{ \quad \vec{\mathbf{u}} = ScaleToBox(\vec{\mathbf{q}}, \vec{\mathbf{q}}^{\min}, \vec{\mathbf{q}}^{\max})$$
$$\text{return} \sum_{i=0}^{5}\sum_{j=0}^{5}\sum_{k=0}^{5} \vec{\mathbf{c}}_{i,j,k} B_{5,i}(u_x) B_{5,j}(u_y) B_{5,k}(u_z)$$
$$\}$$

Radial Basis Function Interpolation

$$\vec{\mathbf{f}}(\vec{\mathbf{x}}) = \sum_{k=1}^{n} \vec{\mathbf{f}}_k \varphi\left(\left\|\vec{\mathbf{x}} - \vec{\mathbf{x}}_k\right\|\right) \quad \text{where } \vec{\mathbf{x}} \in \mathbb{R}^d$$



Given a set of points $\left\{\cdots \vec{\mathbf{x}}_k \cdots\right\}$, solve this system to find the $\vec{\mathbf{f}}_k$:

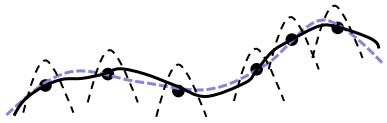
$$\begin{bmatrix} 0 & \varphi\left(\left\|\vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1\right\|\right) & \cdots & \varphi\left(\left\|\vec{\mathbf{x}}_n - \vec{\mathbf{x}}_1\right\|\right) \\ \varphi\left(\left\|\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2\right\|\right) & 0 & \cdots & \varphi\left(\left\|\vec{\mathbf{x}}_n - \vec{\mathbf{x}}_2\right\|\right) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi\left(\left\|\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_n\right\|\right) & \varphi\left(\left\|\vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_n\right\|\right) & \cdots & 0 \end{bmatrix} \begin{bmatrix} \vec{\mathbf{f}}_1^{\,T} \\ \vec{\mathbf{f}}_2^{\,T} \\ \vdots \\ \vec{\mathbf{f}}_n^{\,T} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{x}}_1^{\,T} \\ \vec{\mathbf{x}}_2^{\,T} \\ \vdots \\ \vec{\mathbf{x}}_n^{\,T} \end{bmatrix}$$

Sometimes add linear combination of polynomials $\vec{\gamma}_j(\vec{\mathbf{x}})$ to the method

$$\vec{\mathbf{f}}(\vec{\mathbf{x}}) = \sum_{k=1}^{n} \vec{\mathbf{f}}_k \varphi\left(\left\|\vec{\mathbf{x}} - \vec{\mathbf{x}}_k\right\|\right) + \sum_{j=1}^{m} \vec{\mathbf{g}}_j \gamma_j(\vec{\mathbf{x}}) \quad \text{where } \sum_{k=1}^{n} \vec{\mathbf{g}}_k \gamma_k(\vec{\mathbf{x}}_k) = 0$$

Radial Basis Function Interpolation

$$\vec{\mathbf{f}}(\vec{\mathbf{x}}) = \sum_{k=1}^{n} \vec{\mathbf{f}}_k \varphi\left(\left\|\vec{\mathbf{x}} - \vec{\mathbf{x}}_k\right\|\right) \quad \text{where } \vec{\mathbf{x}} \in \mathbb{R}^d$$



Typical radial basis functions with global support[1]

Radial Basis Function	$\phi(r)$	parameters	order
Gaussians	$e^{-(cr)^2}$	$c > 0$	0
Polyharmonic Splines	r^{2k-1}	$k \in \mathbb{N}$	$m = k$
	$r^{2k}\log(r)$	$k \in \mathbb{N}$	$m = k+1$
Multiquadrics	$\sqrt{r^2 + c^2}$	$c > 0$	1
Inverse Multiquadrics	$\frac{1}{\sqrt{r^2+c^2}}$	$c > 0$	0
Inverse Quadratics	$\frac{1}{r^2+c^2}$	$c > 0$	0

[1] W. du Toit, *Radial Basis Function Interpolation,*, MS Thesis, University of Stellenbosch, 2008

Example radial basis function with compact support

$$\varphi(r) = \begin{cases} e^{-\left(\frac{1}{1-(\varepsilon r)^2}\right)} & \text{for } 0 \le r < \frac{1}{\varepsilon} \\ 0 & \text{otherwise} \end{cases}$$

Note: The choice of constants in design of RBFs is important

Radial Basis Function Interpolation

- As mentioned in the previous slide, the choice of specific RBF basis and associated parameters is important and is generally problem-specific. The design process typically involves an optimization problem of some sort, trading off accuracy, coverage, and computational efficiency.
- Here are some useful web links to start further exploration:
  - https://en.wikipedia.org/wiki/Radial_basis_function_interpolation
  - https://core.ac.uk/download/pdf/37320748.pdf
  - http://www.scholarpedia.org/article/Radial_basis_function#Compactly_supported_radial_basis_functions
  - https://link.springer.com/content/pdf/10.1007/s00500-020-05211-0.pdf
  - https://www.researchgate.net/publication/340082978_Compactly_supported_radial_basis_functions_how_and_why