Feature-Based 2D-3D Registration

**Given**
- 3D surface model of an anatomic structure
- Multiple 2D x-ray projection images taken at known poses relative to some coordinate system C
- Initial estimate of the pose $F$ of the anatomic object relative to the x-ray imaging coordinate system C

**Goal**
- Compute an accurate value for $F$
Feature-Based 2D-3D Registration
Feature-Based 2D-3D Registration
A contour-based 2D-3D method ...

Gueziec et al., 1998

Step 0: Extract contours from x-ray images and compute corresponding lines between source and detector
A contour-based 2D-3D method ...

Gueziec et al., 1998

Step 1: Given the current estimate for $F = [R, t]$, compute the apparent projection contours of the model for each viewing direction.

Step 2: For each x-ray path line $L_i$, identify the closest point $p_i$ on an apparent projection contour. This will give a set of points on the body surface to be moved toward the corresponding x-ray lines.

Distance $d = \| (\hat{p} - \hat{c}) \times \hat{v} \|

Line direction $\hat{v}$  \( (\|\hat{v}\| = 1) \)

Note: It is convenient to use the x-ray source position (i.e., the center of convergence for a bundle of x-ray projection lines) as the value for $\hat{c}$.

A contour-based 2D-3D method ...
Gueziec et al., 1998

Step 3: Solve an optimization problem to compute a value of $F$ that minimizes the distance between the $p_i$ and the $L_i$.

$$\min_{R,t} \sum_i \alpha_i^2 = \min_{R,t} \sum_i \left\| \mathbf{v}_i \times \left( \mathbf{c}_i - \left( R \mathbf{p}_i + t \right) \right) \right\|^2$$

$$= \min_{R,t} \sum_i \left\| \text{skew} \mathbf{(v}_i \mathbf{)} \cdot \left( \mathbf{c}_i - \left( R \mathbf{p}_i + t \right) \right) \right\|^2$$

Step 4: Iterate steps 1-3 until reach convergence.

11

Computational Note

Gueziec uses the Cayley parameterization for rotations:

$$R(\mathbf{v}) = (I - \text{skew}(\mathbf{u}))(I + \text{skew}(\mathbf{u}))^{-1}$$

This leads to the approximation

$$R(\mathbf{u}) \approx I + \text{skew}(2\mathbf{u})$$

which is similar to our familiar $R(\alpha) \approx I + \text{skew}(\alpha)$. He also uses the notation $U = \text{skew}(\mathbf{u})$. So $R(\mathbf{u}) = (I - U)(I + U)^{-1}$

Similarly, we will see $V = \text{skew}(\mathbf{v})$, etc.
Guezic compared three different methods for performing the minimization in Step 3:

- Levenberg Marquardt (LM) nonlinear minimization.
- Linearization and constrained minimization
- Use of a Robust M-Estimator

Levenberg-Marquardt ...

(Following development in Guezic et al., 1998)

Define \( f_i(\tilde{x}) = \left\| V_i (\tilde{c}_i - R(\tilde{u})\tilde{p}_i - \tilde{t}) \right\| \) where \( \tilde{x}' = [\tilde{u}', \tilde{t}']^T, V_j = \text{skew}(\tilde{v}_j) \)

Our goal is to minimize

\[ \varepsilon(\tilde{x}) = \sum_i f_i(\tilde{x})^2 = \sum_i \left\| V_i (\tilde{c}_i - R(\tilde{u})\tilde{p}_i - \tilde{t}) \right\|^2 \]

We note that \( \varepsilon(\tilde{x}) \) is nonlinear. Levenberg-Marquardt is a widely used optimization method for problems of this type. However, it requires us to evaluate the partial derivatives \( \partial f_i / \partial x_j \). Guezic worked these out symbolically for his problem
Levenberg-Marquardt ...
(Following development in Gueziec et al., 1998)

Define \( f(\tilde{x}) = \| \mathbf{V}_j (\tilde{c} - \mathbf{R}(\tilde{u})\tilde{p}_j - \tilde{t}) \| \) where \( \tilde{x}' = [\tilde{u}', \tilde{t}'] \), \( \mathbf{V}_j = \text{skew}(\mathbf{v}_j) \)

\[
\mathbf{J} = \left[ \begin{array}{ccc}
\cdots & \frac{\partial f}{\partial \tilde{x}} & \cdots \\
\end{array} \right] = \left[ \begin{array}{c}
\frac{\partial f}{\partial \tilde{u}} \\
\frac{\partial f}{\partial \tilde{t}} \\
\end{array} \right]
\]

\[
\frac{\partial f}{\partial \tilde{t}} = \frac{\mathbf{V}_j^{\top} \mathbf{V}_j (\tilde{R}\tilde{p}_j - \mathbf{c} + \tilde{t})}{f_j}
\]

\[
\frac{\partial f}{\partial \tilde{u}} = \left( \frac{\partial \tilde{R}\tilde{p}_j}{\partial \tilde{u}} \right)^{\top} \mathbf{V}_j^{\top} \mathbf{V}_j (\tilde{R}\tilde{p}_j - \mathbf{c} + \tilde{t}) \frac{1}{f_j}
\]

Details on this may be found in reference [45] of Gueziec’s paper.


Levenberg-Marquardt ...
(Following development in Gueziec et al., 1998)

Step 1: Pick \( \lambda = a \) small number; pick initial guess for \( \tilde{x} \)
Step 2: Evaluate \( f(\tilde{x}) \) and \( \mathbf{J} \) and solve the least squares problem

\[
\left( \mathbf{J}^{\top} \mathbf{J} + \lambda \mathbf{I} \right) \Delta \tilde{x} = \mathbf{J}^{\top} f_j
\]

for \( \Delta \tilde{x} \).

Step 3: \( \tilde{x} \leftarrow \tilde{x} + \Delta \tilde{x} \); update \( \lambda \).

Step 4: Evaluate termination condition. If not done, go back to step 2.

Note: Usually \( \lambda \) starts small and grows larger. Consult standard references (e.g., Numerical Recipes) for more information.

Constrained Linearized Least Squares ...

(Following development in Gueziec et al., 1998)

Step 0: Make an initial guess for $\mathbf{R}$ and $\mathbf{t}$

Step 1: Compute $\mathbf{p}_i \leftarrow \mathbf{Rp}_i + \mathbf{t}$

Step 2: Define $\mathbf{P}_i = \text{skew}(\mathbf{p}_i)$, $\mathbf{V}_i = \text{skew}(\mathbf{v}_i)$

Step 3: Solve the least squares problem:

$$
\varepsilon^2 = \min \begin{bmatrix}
2\mathbf{V}_i & \mathbf{V}_i & \mathbf{u} \\
\vdots & \vdots & \vdots \\
\mathbf{V}_i & \mathbf{V}_i & \mathbf{v}_i \\
\vdots & \vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
\Delta \mathbf{t} \\
\Delta \mathbf{R} \\
\end{bmatrix}
- \begin{bmatrix}
\mathbf{v}_i \mathbf{c}_i - \mathbf{p}_i \\
\vdots \\
\end{bmatrix}
$$

subject to $\|\mathbf{u}\| \leq \rho$

where $\rho$ is sufficiently small so that $I + 2\mathbf{U}$ approximates a rotation

Step 4: Compute $\Delta \mathbf{R} = (I - \mathbf{U})(I + \mathbf{U})^{-1}$

Update $\mathbf{p}_i \leftarrow \Delta \mathbf{Rp}_i + \Delta \mathbf{t}$; $\mathbf{R} \leftarrow \Delta \mathbf{RR}$; $\mathbf{t} \leftarrow \Delta \mathbf{Rt} + \Delta \mathbf{t}$

Step 5: If $\varepsilon$ is small enough or some other termination condition is met, then stop. Otherwise go back to Step 2.

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Robust Pose Estimation ...

- Basic idea is to identify outliers and give them little or no weight.

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Robust Pose Estimation ...

• Basic idea is to identify outliers and give them little or no weight.

Robust M-Estimator ...
(Following development in Gueziec et al., 1998)

Step 0: Make an initial guess for \( R \) and \( \hat{t} \)
Step 1: Compute \( \hat{p}_j \leftarrow R \hat{p}_j + \hat{t} \)
Step 2: Define \( \mathbf{P}_j = \text{skew}(\hat{p}_j), \ V_j = \text{skew}(\hat{v}_j), \)
Step 3: Solve a robust linearized problem

\[
\hat{e} = \arg\min_{\hat{e}, \hat{t}} \sum_j \rho \left( \frac{0.6745}{\text{med} \{(e_j)\}} \right) \text{ where } e_j = \| V_j (\hat{p}_j - c_j + 2\hat{P}_j + \Delta \hat{t}) \|
\]

(See next slide)
Step 4: Compute \( \Delta R = (I - U)(I + U)^{-1} \)
Update \( p_j \leftarrow \Delta R \hat{p}_j + \Delta \hat{t}; \ R \leftarrow \Delta RR; \ \hat{t} \leftarrow \Delta R \hat{t} + \Delta \hat{t} \)
Step 5: If \( \hat{e} \) is small enough or some other termination condition is met, then stop. Otherwise go back to Step 2.


Robust M-Estimator ...
(Following development in Gueziec et al., 1998)

Step 3.0: Set \( \hat{u} = 0, \hat{x} = 0 \)
Step 3.1: Compute \( e_i = \| V_i (\hat{p}_i - \bar{c}_i + 2\hat{P}_i + \Delta \hat{t}) \|, \ s = \text{med} \{(\cdots, e_i, \cdots)\} / 0.6745, \)
Step 3.2: Solve \( Cx = d, \) where \( x^t = [\hat{u}, \hat{x}] \)

\[
C = \sum_i \psi\left( \frac{e_i}{s} \right) \left[ \begin{array}{cc} 2P_i W_i & P_i W_i \\ 2P_i W_i & W_i \end{array} \right] \text{ and } d = \sum_i \psi\left( \frac{e_i}{s} \right) \left[ \begin{array}{c} P_i W_i (\bar{c}_i - \hat{p}_i) \\ W_i (\bar{c}_i - \hat{p}_i) \end{array} \right]
\]

where \( W_i = V_i V_i = I - \bar{v}_i v_i^t \) \( \psi(\mu) = \left\{ \begin{array}{ll} \mu(1 - \mu^2 / \alpha^2)^2 & \text{if } \|\mu\| \leq \alpha \\ 0 & \text{otherwise} \end{array} \right. \)

(Note: We use \( \alpha = 2 \))
Step 3.3: Iterate steps 3.1 and 3.2 until a suitable termination condition is reached.

A contour-based 2D-3D method ... results
Gueziec et al., 1998

Before

After

A contour-based 2D-3D method ... times

**Gueziec et al., 1998**

<table>
<thead>
<tr>
<th>TABLE I</th>
<th>AVERAGE EXECUTION TIMES IN MS FOR THE THREE REGISTRATION METHODS APPLIED TO DATA SETS THAT COMPRIS 100 POINTS (TOP) AND 20 POINTS (BOTTOM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number Points/Method</td>
<td>LM</td>
</tr>
<tr>
<td>100 points (CPU time)</td>
<td>790</td>
</tr>
<tr>
<td>20 points (CPU time)</td>
<td>200</td>
</tr>
</tbody>
</table>


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**Sample Set Analysis**

- **Question:** How good is a particular set of 3D sample points for the purpose of registration to a 3D surface?

- Long line of authors have looked at this question

Sample Set Analysis: Distance Estimates

Let

$$F(x) = 0$$

be the implicit equation of a surface, then one good estimate of the distance of a point $x$ to the surface is

$$D(x) = \frac{F(x)}{||\nabla F(x)||}$$

Sample set analysis: sensitivity

Let $x_s$ be a point on the surface, and let $T(\eta)$ represent a small perturbation with parameters $\eta$ with respect to the surface of point $x_s$:

$$x'_s = T(\eta)x_s$$

Then we define $V(x_s)$ to be

$$V(x_s) = \frac{\partial D(T(\eta)x_s)}{\partial \eta} = \begin{bmatrix} n_s \\ x_s \times n_s \end{bmatrix}$$

where $n_s$ is the unit normal to the surface at $x_s$. So,

$$D(T(\eta)x_s) \approx V(x_s)\eta$$

Squaring this gives

$$D^2(T(\eta)x_s) \approx \eta^T V(x_s)V^T(x_s)\eta$$

$$= \eta^T M(x_s)\eta$$

Note that $M$ is $6 \times 6$ positive, semi-definite, symmetric matrix.
Sample set analysis: sensitivity

For a region $\mathcal{R}$, define

$$E_R(\eta) = \eta^T \left[ \sum_{\mathbf{x}_i \in \mathcal{R}} \mathbf{M}(\mathbf{x}_i) \right] \eta$$

$$= \eta^T \Psi_R \eta$$

$$= \eta^T \mathbf{Q} \Lambda \mathbf{Q}^T \eta$$

$$= \sum_{1 \leq i \leq 6} \lambda_i (\eta^T \cdot \mathbf{q}_i)^2$$

- Note that the eigenvectors $\mathbf{q}_i$ correspond to small differential transformations $\mathbf{T}(\mathbf{q}_i)$, and can sort eigenvalues so that

  $$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_6$$

- Note that eigenvector $\mathbf{q}_1$ corresponds to direction of greatest constraint.

- Similarly, can also think of $\mathbf{q}_6$ as the least constrained direction.

Sample Set Analysis: Goodness Measures

- Magnitude of smallest eigenvalue (Simon)

- (Kim and Khosla)

$$\frac{\sqrt[6]{\lambda_1 \cdot \ldots \cdot \lambda_6}}{\lambda_1 + \ldots + \lambda_6}$$

- Nahvi

$$\frac{\lambda_6^2}{\lambda_1}$$
Sample Set Selection

• One blind search method (similar to Simon, 1995) is:
  
  – Randomly select sample points on surface
  – (prune for reachability)
  – evaluate goodness of sample set using some criterion
  – repeat many times and choose the best one found

Sample Set Selection

• Refinement of blind search (hill climbing):
  
  – Randomly select sample points on surface
  – (prune for reachability)
  – evaluate goodness of sample set using some criterion
  – replace a point from sample set with a randomly selected point
  – evaluate goodness
  – if better, keep it
  – else revert to original point and try again

• Variations include simulated annealing, “genetic” algorithms
Sample Set Selection: Another Alternative

- Select large number of random points $\mathbf{x}_i$
- Prune for reachability
- For each point, compute constraint direction $\mathbf{V}_i = \mathbf{V}(\mathbf{x}_i)$. To a first approximation, a measurement at $\mathbf{x}_i$ with accuracy $\epsilon_i$ constrains $\eta$ by
  $$ |\mathbf{V}_i \eta| \leq \epsilon_i $$
- Now select subset of the $\mathbf{x}_i$ that minimizes, e.g.,
  $$ \min_{\delta} \max \eta^T \Sigma \eta $$
  subject to
  $$ \delta_i \in \{0, 1\} $$
  $$ |\delta_i \mathbf{V}_i \eta| \leq \epsilon_i $$
  $$ \sum \delta_i \leq \text{subset size} $$

There are various ways to do this.

Sample Set Selection: Another Alternative (con’d)

- One can also minimize other forms, e.g.,
  $$ \min_i \max |\sigma_i \eta_i| $$
  subject to similar constraints
- An alternative is to minimize the number of sample points required to ensure that some constraints on $\eta$ are guaranteed to be met. E.g.,
  $$ \min_{\delta} \sum \delta_i $$
  such that
  $$ \delta_i \in \{0, 1\} $$
  $$ \xi \leq \xi_{\text{limit}} $$
  where
  $$ \xi = \max \eta^T \Sigma \eta $$
  or some other form subject to
  $$ |\delta_i \mathbf{V}_i \eta| \leq \epsilon_i $$
Probabilistic Registration

- Registration methods typically use some optimization algorithm to find a “best” transformation between one data set and the other.
- It makes sense to try to find the “most likely” registration transformation.
- ICP minimizes sum-of-squares distances.
- This is equivalent to assuming that point-pair match probabilities are independent and symmetric Gaussian distributions based on distances.
- But there are a number of other methods that explicitly consider probabilities ...

Coherent Point Drift

- Alignment of point clouds
  - Fast method follows “EM” paradigm
  - Tolerates outliers and noise
  - Transformations: Rigid, affine, general deformable

Click here for slides
**Question:** How to combine multiple data sources, in order to improve the accuracy and robustness of registration outcomes?

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Multi-Modal Feature-Based Registration

**Question:** How to combine multiple data sources, in order to improve the accuracy and robustness of registration outcomes?

Iterative Closest Point (ICP) Revisited

- Widely popular and useful method for point cloud to surface registration introduced by Besl & McKay in 1992
- Many variants proposed since its inception affecting all aspects of the algorithm (robustness, matching criteria, match alignment, etc.)

- **Matching Phase:**
  - for each point in the source shape, find the closest point on the target shape
  \[ y_i = C_{CP}(T(x_i), \Psi) = \arg\min_{y \in \Psi} \| y - T(x_i) \|_2 \]

- **Registration Phase:**
  - compute transformation to minimize sum of square distances between matches
  \[ T = \arg\min_T \sum_{i=1}^{n} \| y_i - T(x_i) \|_2^2 \]

Most-Likely Point Paradigm Illustrated with ICP

1. Probability Model: isotropic Gaussian
   \[ f_{\text{match}}(\mathbf{x} \mid \mathbf{y}, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-\frac{1}{2\sigma^2} \|\mathbf{y} - \mathbf{x}\|^2} \]

2. Match Phase:
   \[ y_i = \arg \max_{y_i \in \phi} f_{\text{match}}(T(x_i) \mid y_i, \sigma^2) \]
   \[ = \arg \max_{y_i \in \phi} \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-\frac{1}{2\sigma^2} \|y_i - T(x_i)\|^2} \]
   \[ \rightarrow \arg \min_{y_i \in \phi} \|y_i - T(x_i)\| \]

3. Registration Phase:
   \[ T = \arg \max_T \prod_i f_{\text{match}}(T(x_i) \mid y_i, \sigma^2) \]
   \[ = \arg \max_T \prod_i \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-\frac{1}{2\sigma^2} \|y_i - T(x_i)\|^2} \]
   \[ \rightarrow \arg \max_T \left[ -n \log \left( (2\pi\sigma^2)^{3/2} \right) - \frac{1}{2\sigma^2} \sum_i \|y_i - T(x_i)\|^2 \right] \]
   \[ \rightarrow \arg \min_T \sum_i \|y_i - T(x_i)\|^2 \]

Outline of Registration Algorithms

- ICP - Iterative Closest Point
  - isotropic position data

- IMLP - Iterative Most Likely Point
  - anisotropic position data
  - robust to outliers

- IMLOP - Iterative Most Likely Oriented Point
  - isotropic position & orientation data

- G-IMLOP - Generalized IMLOP
  - anisotropic position & orientation data

- P-IMLOP - Projected IMLOP
  - anisotropic position & projected orientation data
Sources of Anisotropic Uncertainty

Prior Work: Anisotropic Registration

- Generalized Total Least Squares ICP (GTLS-ICP)
  - Registration Phase
    - anisotropic noise model
    - ad-hoc implementation less accurate / efficient; can be unstable
  - Match Phase
    - isotropic (i.e. closest-point matching)

- Generalized ICP (G-ICP)
  - Registration Phase
    - anisotropic noise model limited to model locally-linear surface regions surrounding each feature point of a point cloud shape
    - uses off-the-shelf conjugate gradient solver
  - Match Phase
    - isotropic (i.e. closest-point matching)

Prior Work: Anisotropic Registration

- Anisotropic ICP (A-ICP)


- Registration Phase
  - anisotropic noise model
    - ad-hoc implementation does not fully account for noise in both shapes (i.e., lacks ability to reorient the data-shape covariances during optimization)

- Match Phase
  - anisotropic noise model with non-optimal matching (finds minimal Mahalanobis distance match rather than most-likely match)
    - inefficient implementation; also cannot guarantee that the “best” match is found
  - Initializes registration by ICP (due to inefficient match phase)

Iterative Most Likely Point (IMLP)

Probability Model: anisotropic Gaussian

\[
 f_{\text{match}}(x | y, \Sigma_x, \Sigma_y) = \frac{1}{(2\pi)^{3/2} |\Sigma_x + \Sigma_y|^{1/2}} \cdot e^{-\frac{1}{2} (y-x)^T (\Sigma_x + \Sigma_y)^{-1} (y-x)}
\]

Match Phase:

\[
 [y_i, \Sigma_{yi}] = \arg\min_{[y_i, \Sigma_{yi}] \in \Psi} \left[ \log \left(\mathbf{R}\Sigma_{xi}\mathbf{R}^T + \Sigma_{yi} \right) + (y_i - T(x_i))^T (\mathbf{R}\Sigma_{xi}\mathbf{R}^T + \Sigma_{yi})^{-1} (y_i - T(x_i)) \right]
\]

Registration Phase:

\[
 T = \arg\min_{T \in \mathbb{R}^d} \sum_{i=1}^{n} (y_i - T(x_i))^T (\mathbf{R}\Sigma_{xi}\mathbf{R}^T + \Sigma_{yi})^{-1} (y_i - T(x_i))
\]

IMLP: Match Phase

- Due to anisotropic distance metric, standard KD-tree search techniques do not apply.
- **Approach**: PD-tree search with modified node test

**Constructing the PD tree:**
1. Add all datums to a root node
2. Compute covariance of datum positions within the node
3. Create minimally-sized bounding box aligned to the covariance eigenvectors
4. Partition node along the direction of greatest extent
5. Form left and right child nodes from the datums in each partition
6. Repeat from Step 2 for left and right child nodes until # datums in node < threshold or node size < threshold

**Searching the PD tree:**

*Assume the current match candidate has a match error equal to $E_{best}$*

**Question:** can any feature in this node possibly provide a match error less than $E_{best}$?

$$\{y_i, \Sigma_{y_i}\} = \arg\min_{\{y, \Sigma_y\} \in \mathcal{Y}} \left[ \log \left( \frac{1}{(2\pi)^{d/2} |\Sigma_y|^{1/2}} \right) \right.$$  
$$\left. + \frac{1}{2} (y_i - T(x_i))^T \left( \Sigma_y + \Sigma_{node} \right)^{-1} (y_i - T(x_i)) \right]$$  

**True if:**  
$$\left( y_i - T(x_i) \right)^T \left( \Sigma_{node} \right)^{-1} (y_i - T(x_i)) < E_{best} - \log_{min}$$

**Node Test:** if the ellipsoid 

$$\mathcal{E} = \{y \mid (y - T(x_i))^T \left( \Sigma_{node} \right)^{-1} (y - T(x_i)) \leq E_{best} - \log_{min}\}$$

intersects the bounding box of the node, then search the node.
**IMLP: Match Phase**

Searching the PD tree:

Assume the current match candidate has a match error equal to $E_{\text{best}}$.

**Question:** can any feature in this node possibly provide a match error less than $E_{\text{best}}$?

$$\mathbf{y}_i, \Sigma_{\mathbf{y}_i} = \arg\min_{\{\mathbf{y}_i, \Sigma_{\mathbf{y}_i}\} \in \Psi} \left[ \log(\mathbf{R} \Sigma_{\mathbf{x}_i} \mathbf{R}^T + \Sigma_{\mathbf{y}_i}) + (\mathbf{y}_i - \mathbf{T}(\mathbf{x}_i))^T (\mathbf{R} \Sigma_{\mathbf{x}_i} \mathbf{R}^T + \Sigma_{\mathbf{y}_i})^{-1} (\mathbf{y}_i - \mathbf{T}(\mathbf{x}_i)) \right]$$

True if: $$(\mathbf{y}_i - \mathbf{T}(\mathbf{x}_i))^T (\mathbf{R} \Sigma_{\mathbf{x}_i} \mathbf{R}^T + \Sigma_{\text{node}})^{-1} (\mathbf{y}_i - \mathbf{T}(\mathbf{x}_i)) < E_{\text{best}} - \log_{\text{min}}$$

**Node Test:** if the ellipsoid

$$\mathcal{E} = \{ \mathbf{y} \mid (\mathbf{y} - \mathbf{T}(\mathbf{x}_i))^T (\mathbf{R} \Sigma_{\mathbf{x}_i} \mathbf{R}^T + \Sigma_{\text{node}})^{-1} (\mathbf{y} - \mathbf{T}(\mathbf{x}_i)) \leq E_{\text{best}} - \log_{\text{min}} \}$$

intersects the bounding box of the node, then search the node.

---

**IMLP: Registration Phase**

1. Re-formulate the cost function from an unconstrained optimization

$$\mathbf{T} = \arg\min_{[\mathbf{R}, \mathbf{t}]} \sum_{i=1}^{n} (\mathbf{y}_i - \mathbf{Rx}_i - \mathbf{t})^T (\mathbf{R} \Sigma_{\mathbf{x}_i} \mathbf{R}^T + \Sigma_{\mathbf{y}_i})^{-1} (\mathbf{y}_i - \mathbf{Rx}_i - \mathbf{t})$$

2. Linearize the constraints with a Taylor series centered at the measured (known) data

$$F_i(\mathbf{x}_i^*, \mathbf{y}_i^*, \mathbf{R}, \mathbf{t}) \approx F_i(\mathbf{x}_i, \mathbf{y}_i, \mathbf{R}, \mathbf{t}) + (\mathbf{x}_i^* - \mathbf{x}_i) \frac{\partial F_i}{\partial \mathbf{x}} + (\mathbf{y}_i^* - \mathbf{y}_i) \frac{\partial F_i}{\partial \mathbf{y}} + \mathbf{R} \delta \mathbf{R} + \text{skew}(\delta \mathbf{R})$$

$$\mathbf{R} \approx \mathbf{I} + \text{skew}(\delta \mathbf{R})$$

$$\delta \mathbf{R} = \Delta \mathbf{R}$$
IMLP: Registration Phase

3. Apply the method of Lagrange multipliers to solve constrained optimization.
   3a. Form the Lagrange function using the linearized constraints
   \[ \mathcal{L}(d_0, d_t, \lambda) = \sum_{i=1}^{N} r_i^T \Sigma_{x_i}^{-1} r_i + \sum_{j=1}^{M} r_j^T \Sigma_{y_j}^{-1} r_j + \sum_{i=1}^{N} \lambda_i^T f_i(x_i, y_j, d_0, d_t) \]

3b. Solve zero gradient w.r.t. the optimization parameters and the Lagrange multipliers

\[
\begin{align*}
J^T \Sigma^{-1} \delta p &= -J^T \Sigma^{-1} f^0 \\
\delta p &= 
\begin{bmatrix}
d_0 \\
d_t \\
f^0 \\
f^1 \\
f^2 \\
f^3 \\
\Sigma_{x_1} \\
\Sigma_{x_2} \\
\Sigma_{x_3} \\
\Sigma_{y_1} \\
\Sigma_{y_2} \\
\Sigma_{y_3}
\end{bmatrix} \\
J &= 
\begin{bmatrix}
s(\lambda_1 x_1) & -1 \\
\vdots & \vdots \\
s(\lambda_n x_n) & -1
\end{bmatrix} \\
\Sigma &= 
\begin{bmatrix}
f_1 f_1^T & \Sigma_{x_1} \\
f_2 f_2^T & \Sigma_{x_2} \\
f_3 f_3^T & \Sigma_{x_3} \\
f_4 f_4^T & \Sigma_{y_1} \\
f_5 f_5^T & \Sigma_{y_2} \\
f_6 f_6^T & \Sigma_{y_3}
\end{bmatrix}
\end{align*}
\]

4. Iteratively solve 3b by linear least squares until convergence.
\[ R_{k+1} = R(d_k) R_k , \quad t_{k+1} = t_k + dt \]

IMLP: Experiments

- **Data Shape**: 100 noisy points + outliers simulated from a mesh model of a human hip
- **Model Shape**: point-cloud formed from the center points of the mesh triangles
- **Random initial misalignments** [30,60] mm and [30,60] degrees
- **Target registration error (TRE)** averaged over 300 randomized trials for each test case

<table>
<thead>
<tr>
<th>Alg</th>
<th>1</th>
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<td>0.009</td>
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<td>0.019</td>
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<td>0.099</td>
<td>0.067</td>
<td>0.084</td>
<td>0.073</td>
</tr>
</tbody>
</table>
IMLP: Experiments

- **Data Shape**: 100 noisy points simulated from a mesh model of a human femur
- **Model Shape**: point-cloud formed from the center points of the mesh triangles
- **Random initial misalignments**: [10, 20] mm and [10, 20] degrees
- **Target registration error (TRE)** averaged over 300 randomized trials for each test case

![Image of femur and histogram]

**Fig. 5. Registration of shapes having partial overlap.** (Experiment 7). (A): The statue Laumana sub-divided into (B): front and (C): right half-section, such that (D): 50% overlap exists between the two sub-shapes. The sub-shapes were (E): misaligned by 10 mm and 10 degrees in a random direction and then registered using (F): CPD, (G): IMLP, (H): IMLP-CP, (I): IMLP-MD, and (J): the proposed IMLP algorithm. Sub-figures (E-H) show the initial misalignment and the final registered alignments of the two shapes for the 6th randomized trial of Experiment 7, which involved 10 randomized trials in total.
**Iterative Most Likely Oriented Point (IMLOP)**

- **Matching Phase:**
  
  for each oriented point in the source shape, find the most likely oriented point on the target shape

  \[ y_i = C_{MLP}(T(x_i), \Psi) = \arg \max_{y \in \Phi} f_{match}(T(x_i), y) \]

- **Registration Phase:**
  
  compute transformation to maximize the likelihood (i.e. minimize negative log-likelihood) of oriented point matches

  \[ T = \arg \min_T \left( \frac{1}{2 \sigma^2} \sum_{i=1}^{n} \| y_{\psi i} - T(x_{\psi i}) \|^2_2 - k \sum_{i=1}^{n} y_{\psi i}^T R x_{\psi i} \right) \]


**Sources of Orientation Data**

- **Video**
- **X-Ray**
- **Shape Models**
- **Tracked Pointer**
- **Oriented Fiducials**
- **Ultrasound**
Experiments

Performance comparison of IMLOP vs. ICP was made through a simulation study using a human femur surface mesh segmented from CT imaging.

- Source shape created by randomly sampling points from the mesh surface (10, 20, 35, 50, 75, and 100 points tested)
- Gaussian [wrapped Gaussian] noise added to the source points (0, 0.5, 1.0, and 2.0 mm [degrees] tested)
- Applied random misalignment of [10,20] mm / degrees
- 300 trials performed for each sample size / noise level
- Registration accuracy (TRE) evaluated using 100 validation points randomly sampled from the mesh
- Registration failures automatically detected using threshold on final residual match errors

ICP: threshold on position residuals only
IMLOP: threshold on position & orientation residuals


Average TRE of successful registrations and registration failure rates across all sample sizes for noise levels of 1 (A) and 2 (B) mm [degrees].

Registration failure threshold set to twice the noise level for both position and orientation.

Experiments

Results from 300 trials within a single sample size (75 points) and noise level (1.0 mm [degree]). NOTE: improved accuracy and failure detection capability for IMLOP.

ICP

![Graph showing widely distributed TRE follows drop in residual error](Registration Trials Sorted by TRE (Positions))

IMLOP

![Graph showing sharp drop in TRE accompanies drop in residual error](Registration Trials Sorted by TRE (Positions))


Generalized IMLOP Results

- Extends IMLOP to account for anisotropic measurement error distributions
  - Model orientations with Kent distributions
  - Model positions with Gaussian distributions
- Simulation results for 50 samples shown below

Experiments: TRE for Rejected and Non-Rejected Registrations

![Graphs showing ICP, IMLOP, and G-IMLOP performance](image)


Ultrasound-assisted Registration

1. Generate surface model from CT
2. Digitize proximal bone using tracked pointer
3. Collect tracked US images of distal bone
4. Register points/contours to surface model


http://dx.doi.org/10.1007/s11548-015-1188-z   DOI 10.1007/s11548-015-1188-z]
Intensity-based methods

• Typically performed between images
• The “features” in this case are the intensities associated with pixels (2D) or voxels (3D) in the images.
• General framework:

\[ \rho^* = \min_\rho E(Image_1, \Theta(\rho, Image_2)) \]

• Methods differ mostly in choice of transformation function \( \Theta(\cdot) \) and Energy function \( E(\cdot, \cdot) \).
**Typical energy functions**  
(not an exhaustive list)

**Normalized image subtraction**

\[
E(IM_1, IM_2) = \sum_k \frac{\|IM_1[k] - IM_2[k]\|}{\max_j \|IM_1[j] - IM_2[j]\|}
\]

**Normalized cross correlation (NCC)**

\[
E(IM_1, IM_2) = \frac{\sum_k (IM_1[k] - \text{avg}(IM_1))(IM_2[k] - \text{avg}(IM_2))}{\sqrt{\sum_k (IM_1[k] - \text{avg}(IM_1))^2} \sqrt{\sum_k (IM_2[k] - \text{avg}(IM_2))^2}}
\]

**Mutual information**

\[
E(IM_1, IM_2) = \sum_{p: IM_1, q: IM_2} Pr(p, q)\log Pr(p, q) - Pr_{IM_1}(p)\log Pr_{IM_1}(p) - Pr_{IM_2}(q)\log Pr_{IM_2}(q)
\]

---

**Mutual Information**

- First proposed independently in 1995 by Collignon and Viola & Wells.
- Very widely practiced
- Is able to co-register images with very different sensor modalities so long as there is a stable relationship between intensities in one modality with those in another
- Many “flavors” and variations
Mutual Information

Entropy

\[ H(a) = \text{Pr}(a) \log \text{Pr}(a) \]
\[ H(a,b) = \text{Pr}(a,b) \log \text{Pr}(a,b) \]

Mutual Information (Viola & Wells '95, Colligen '95)

\[ \text{Similarity}(A,B) = H(A) + H(B) - H(A,B) \]

Normalized mutual information (Maes et al. '97)

\[ \text{Similarity}(A,B) = \frac{H(A) + H(B)}{H(A,B)} \]

Objective function

\[ E(\text{Im}_1, \text{Im}_2) = -\text{Similarity}(\text{Im}_1, \text{Im}_2) \]

Basic Idea of Intensity-Based 2D/3D Registration

- Assumes a pre-op CT is available
- Simulate many C-Arm images and choose the most similar to the intraoperative image
- Solves the following optimization problem:

\[
\arg\min_{\theta \in \mathbb{SE}(3)} S(I_{\text{Intra-Op}}, P(\theta, I_{\text{CT}}))
\]
Rigid 3D/2D Registration

Ofri Sadowsky

Optimizer: Downhill Simplex

Prior CT

Estimated position and orientation

Predict images

Patient under fluoroscopy

Simulated images

Patient images

Similarity measure (MI)

Examples: LaRose, Zollei, ...

A clinical example (periacetublar osteotomy)

Problem: Acetabular Dysplasia

Image Source: ouh.nhs.uk

Image Source: James Heilman, MD

Distraction Caused by Equinus

Femur

Hip socket deformed

Shallow hip socket

Pelvis

Normal hip bones

Hip dysplasia

Femoral Head

Hip socket

Image Source: James Heilman, MD

Slide credit: Robert Grupp
A clinical example (periacetabular osteotomy)

**One Solution: Periacetabular Osteotomy (PAO)**

**FIGS. 3A-3C.**

(A) This AP view of a 13-year-old girl with hip pain shows mild dysplasia; Shenton's line is interrupted. Computed tomography (CT) evaluation showed a predominantly anterior deficiency with a total cartilage coverage of 58% (normal, 70%)

(B) Intraoperative verification of the correction held by threaded Kirschner wires. The teardrop has migrated cranially and the femoral head is slightly medialized. Shenton's line has been restored, the ilioischial line is in continuity, and the dorsal pillar is intact. The K-wires are replaced by cortical screws and trimming of the acetabular fragment.

(C) The acetabulum has healed and shows normal bone structure nine months postoperation. Femoral head coverage is 73% on CT.

**FIGS. 4A-4C.**

(A) This 22-year-old woman had marked dysplasia with acetabular cyst and corresponding femoral head defect: VCE angle is 6° (left), VCA angle in the false profile (right) is 5°.

(B) In the postoperative roentgenograms the acetabulum covers the femoral head defect (left). In the alar view (right) the dorsal pillar with the rotated acetabular fragment is intact.

(C) Two years after surgery the VCE angle is 32° (left) and the VCA, 40°. Flexion is limited to 90° due to ectopic ossification near the origin of the rectus femoris (right).

Image Source: Ganz 1988

**Goal: Automatic visualization and guidance**

Total Rotation: 20.5°
Anterior/Posterior Rotation: 3.7°
Left/Right Rotation: 16.3°
Inferior/Superior Rotation: 12.5°
Movement of the Osteotomy Fragment is Challenging

One Approach for Computer-Assistance: Optical Tracking Devices
Intraoperative Fluoroscopy is Available

Slide credit: Robert Grupp
Engineering Research Center for Computer Integrated Surgical Systems and Technology

Intraoperative X-Ray Imaging with Mobile C-Arm

Slide credit: Robert Grupp
Engineering Research Center for Computer Integrated Surgical Systems and Technology
3D-2D Registration of Osteotomy Fragments

\[
\arg\min_{\theta_1, \ldots, \theta_N \in \mathbb{R}^3} \sum_{n=1}^{M} S \left( I_m, \sum_{n=1}^{N} P_m \left( I_{CT}; \theta_n \right) \right)
\]

- Compute the Sobel derivatives in the X and Y directions of the two input images:
  \[ \nabla_X I_1, \nabla_X I_2, \nabla_Y I_1, \nabla_Y I_2 \]
- Compute NCC between the corresponding gradient images:
  \[ S(I_1, I_2) = NCC(\nabla_X I_1, \nabla_X I_2) + NCC(\nabla_Y I_1, \nabla_Y I_2) \]
Initialize Using a Nominal AP View?

Use a Single Landmark to Initialize Registration

- Assume the pelvis is in an AP orientation – this may be computed preoperatively
- Manually annotate a single landmark to recover translation
Example of a Single Landmark Initialization

Automatically Initialize Second and Third Views

- Constrain C-arm motion to orbital rotation
- Perform an exhaustive search over \(\pm 90^\circ\) in 1° increments
Example Initializations From Orbital Search

View #2

View #3

Automatic Landmark-Based Initialization

- Train a CNN to recognize approximate landmark positions in x-ray images

- Use landmark-based 2D-3D registration to initialize registration

- Combine landmark and intensity objective functions
Why Not Simultaneously Use Intensities and Features?

- Registration objective function:

\[
\min_{\phi_r, \theta_r, \theta_{RF}, \theta_{FP}} \lambda S(\phi_r; \phi_r, \theta_{RF}) + (1 - \lambda) R(\theta_r; \theta_r, \theta_{RF})
\]

- Usually, regularization penalizes the amount of rotation and translation away from initialization
- Why not directly include the landmark re-projection as regularization?

\[
R(\theta_r) = \frac{1}{2N} \sum_{i=1}^{N} \left\| \phi_r(p_i^D; \theta_r) - p_i^G \right\|_2^2
\]

- Can also think of this as running landmark registration and regularizing on image appearance

---

Objective Function When Combining Landmark Re-Projection

[Diagram showing the objective function with initialization, ground truth, and screw components]