

Point cloud to point cloud rigid transformations

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Minimizing Rigid Registration Errors

Typically, given a set of points $\{\mathbf{a}_i\}$ in one coordinate system and another set of points $\{\mathbf{b}_i\}$ in a second coordinate system
Goal is to find $[\mathbf{R}, \mathbf{p}]$ that minimizes

$$\eta = \sum_i \mathbf{e}_i \bullet \mathbf{e}_i$$

where

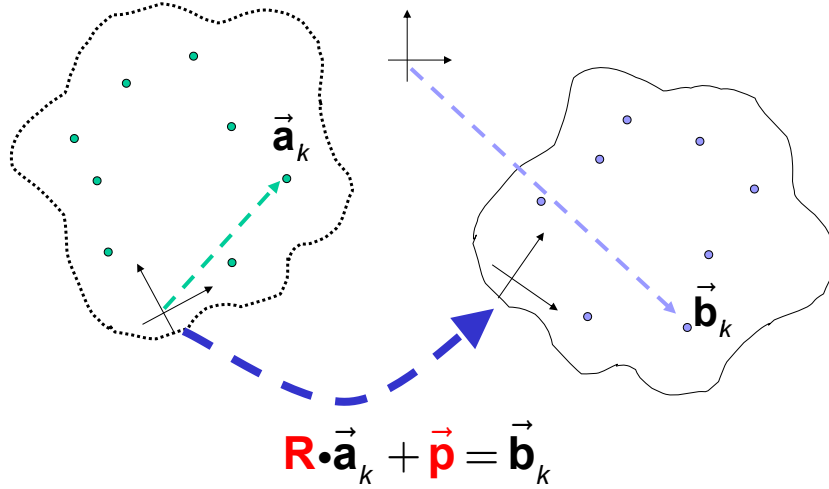
$$\mathbf{e}_i = (\mathbf{R} \bullet \mathbf{a}_i + \mathbf{p}) - \mathbf{b}_i$$

This is tricky, because of \mathbf{R} .



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Point cloud to point cloud registration



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Minimizing Rigid Registration Errors

Step 1: Compute

$$\bar{\mathbf{a}} = \frac{1}{N} \sum_{i=1}^N \vec{\mathbf{a}}_i \quad \bar{\mathbf{b}} = \frac{1}{N} \sum_{i=1}^N \vec{\mathbf{b}}_i$$

$$\tilde{\mathbf{a}}_i = \vec{\mathbf{a}}_i - \bar{\mathbf{a}} \quad \tilde{\mathbf{b}}_i = \vec{\mathbf{b}}_i - \bar{\mathbf{b}}$$

Step 2: Find \mathbf{R} that minimizes

$$\sum_i (\mathbf{R} \cdot \tilde{\mathbf{a}}_i - \tilde{\mathbf{b}}_i)^2$$

Step 3: Find $\bar{\mathbf{p}}$

$$\bar{\mathbf{p}} = \bar{\mathbf{b}} - \mathbf{R} \cdot \bar{\mathbf{a}}$$

Step 4: Desired transformation is

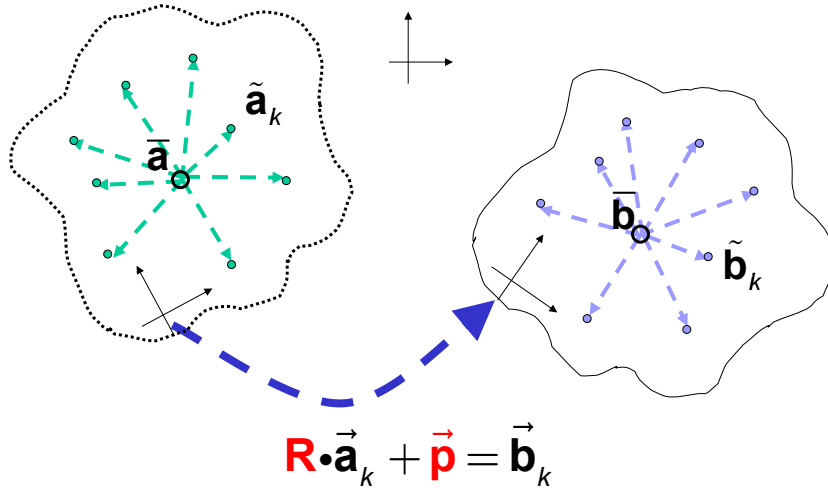
$$\mathbf{F} = \text{Frame}(\mathbf{R}, \bar{\mathbf{p}})$$

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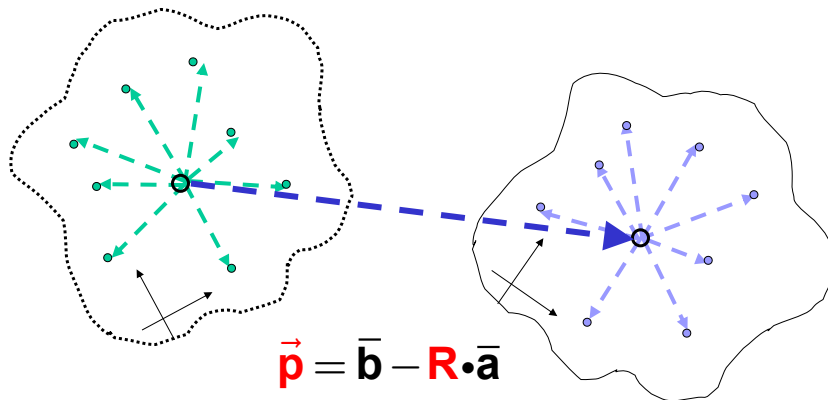
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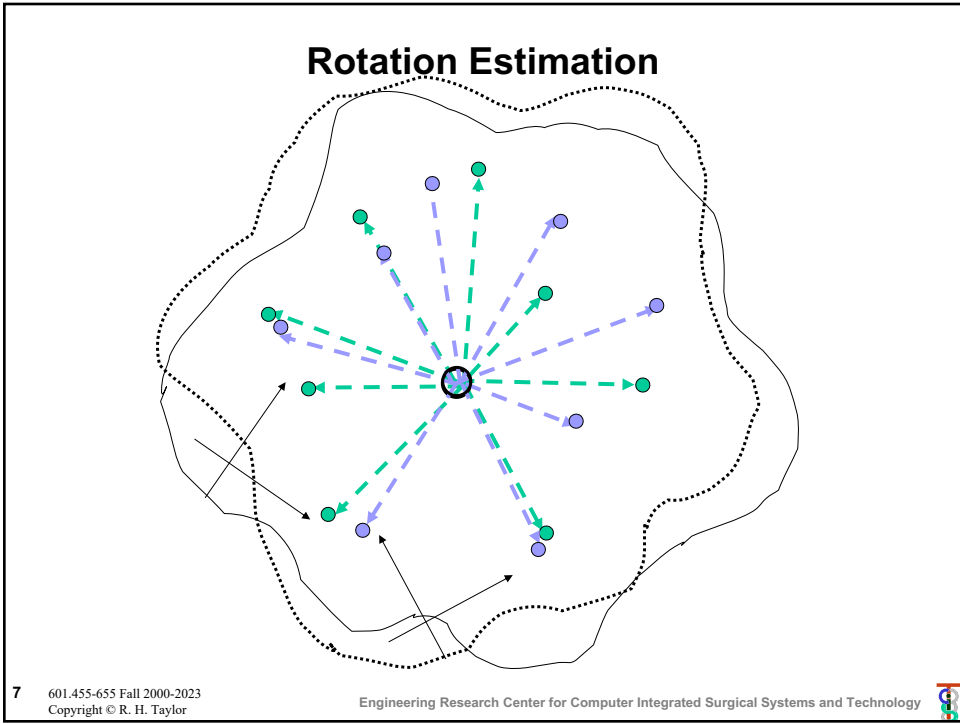


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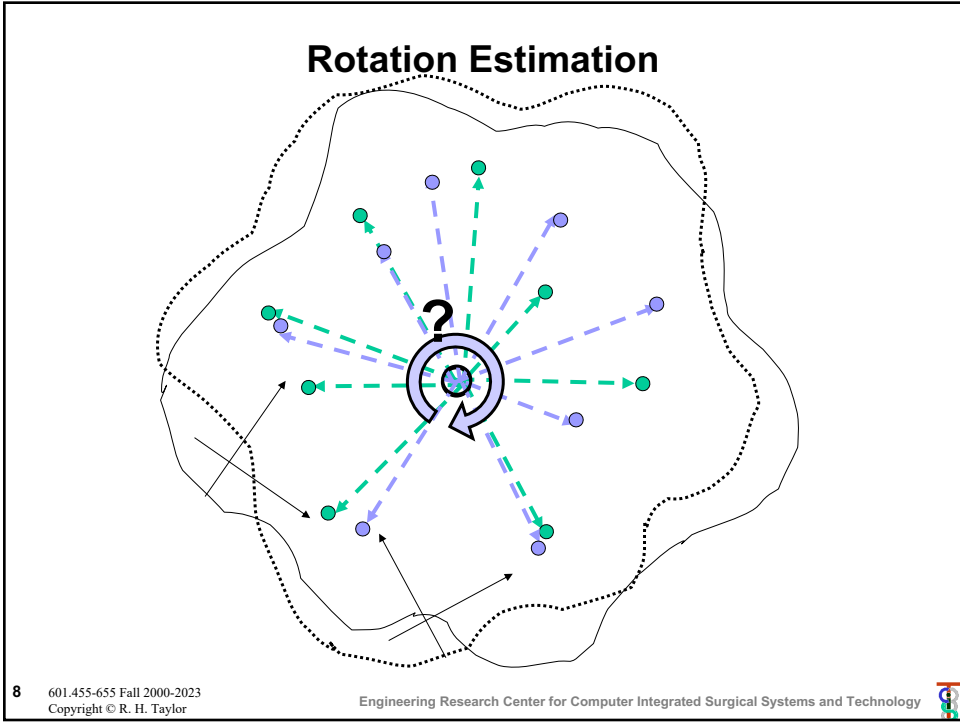
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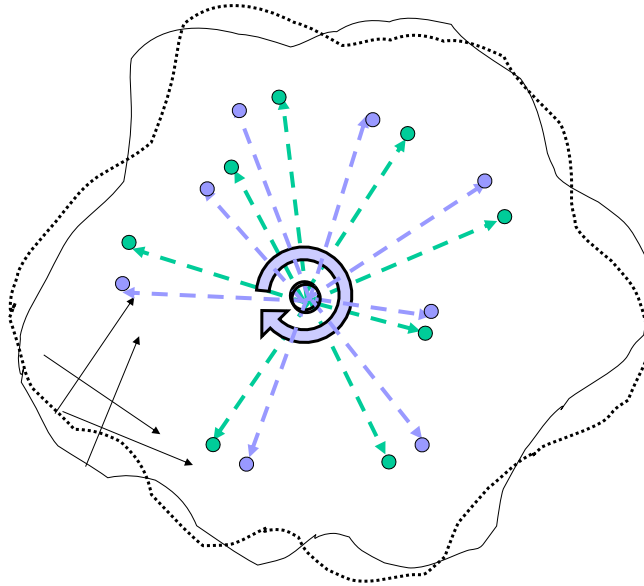


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Rotation Estimation



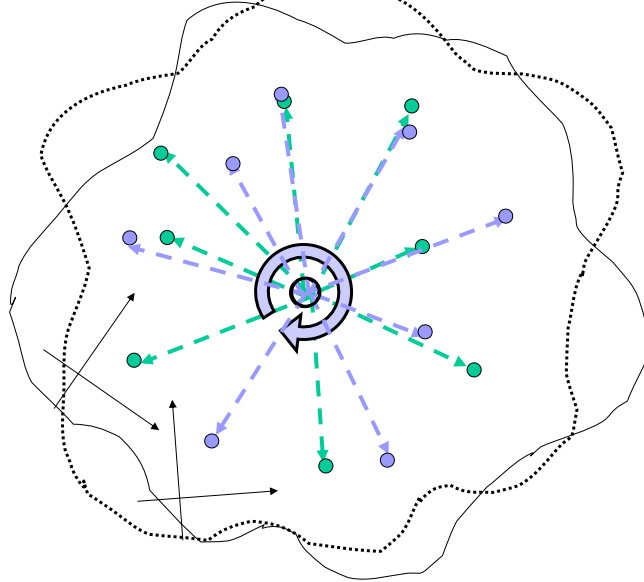
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Rotation Estimation



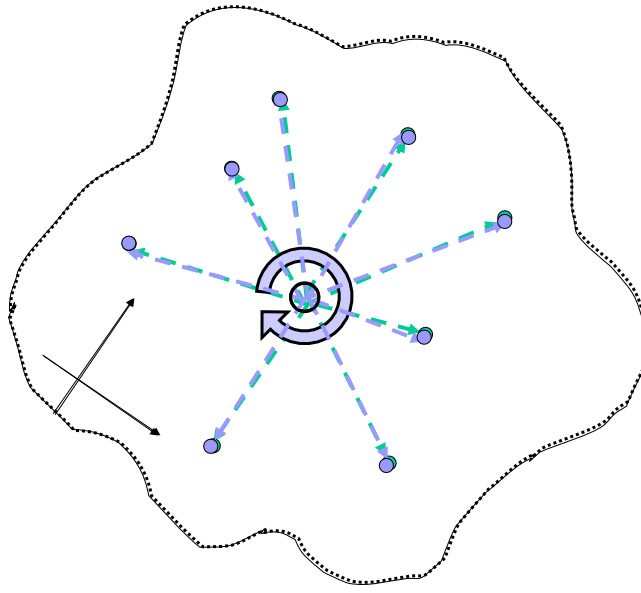
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Solving for R: iteration method

Given $\{\dots, (\tilde{\mathbf{a}}_i, \tilde{\mathbf{b}}_i), \dots\}$, want to find $\mathbf{R} = \arg \min \sum_i \|\mathbf{R}\tilde{\mathbf{a}}_i - \tilde{\mathbf{b}}_i\|^2$

Step 0: Make an initial guess \mathbf{R}_0

Step 1: Given \mathbf{R}_k , compute $\tilde{\tilde{\mathbf{b}}}_i = \mathbf{R}_k^{-1}\tilde{\mathbf{b}}_i$

Step 2: Compute $\Delta\mathbf{R}$ that minimizes

$$\sum_i (\Delta\mathbf{R} \tilde{\mathbf{a}}_i - \tilde{\tilde{\mathbf{b}}}_i)^2$$

Step 3: Set $\mathbf{R}_{k+1} = \mathbf{R}_k \Delta\mathbf{R}$

Step 4: Iterate Steps 1-3 until residual error is sufficiently small
(or other termination condition)

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Iterative method: Getting Initial Guess

We want to find an approximate solution \mathbf{R}_0 to

$$\mathbf{R}_0 \cdot [\dots \tilde{\mathbf{a}}_i \dots] \approx [\dots \tilde{\mathbf{b}}_i \dots]$$

One way to do this is as follows. Form matrices

$$\mathbf{A} = [\dots \tilde{\mathbf{a}}_i \dots] \quad \mathbf{B} = [\dots \tilde{\mathbf{b}}_i \dots]$$

Solve least-squares problem $\mathbf{M}_{3 \times 3} \mathbf{A}_{3 \times N} \approx \mathbf{B}_{3 \times N}$

Note: You may find it easier to solve $\mathbf{A}_{3 \times N}^T \mathbf{M}_{3 \times 3}^T \approx \mathbf{B}_{3 \times N}^T$

Set $\mathbf{R}_0 = \text{orthogonalize}(\mathbf{M}_{3 \times 3})$. Verify that \mathbf{R} is a rotation

Our problem is now to solve $\mathbf{R}_0 \Delta \mathbf{R} \mathbf{A} \approx \mathbf{B}$. I.e., $\Delta \mathbf{R} \mathbf{A} \approx \mathbf{R}_0^{-1} \mathbf{B}$



Iterative method: Solving for $\Delta \mathbf{R}$

Approximate $\Delta \mathbf{R}$ as $(\mathbf{I} + \text{skew}(\bar{\alpha}))$. I.e.,

$$\Delta \mathbf{R} \cdot \mathbf{v} \approx \mathbf{v} + \bar{\alpha} \times \mathbf{v}$$

for any vector \mathbf{v} . Then, our least squares problem becomes

$$\min_{\Delta \mathbf{R}} \sum_i (\Delta \mathbf{R} \cdot \tilde{\mathbf{a}}_i - \tilde{\mathbf{b}}_i)^2 \approx \min_{\bar{\alpha}} \sum_i (\tilde{\mathbf{a}}_i - \tilde{\mathbf{b}}_i + \bar{\alpha} \times \tilde{\mathbf{a}}_i)^2$$

This is linear least squares problem in $\bar{\alpha}$.

Then compute $\Delta \mathbf{R}(\bar{\alpha})$.



Note: Use trigonometric formulas to compute this



Direct Iterative approach for Rigid Frame

Given $\{\dots, (\vec{\mathbf{a}}_i, \vec{\mathbf{b}}_i), \dots\}$, want to find $\mathbf{F} = \arg \min \sum_i \|\mathbf{F}\vec{\mathbf{a}}_i - \vec{\mathbf{b}}_i\|^2$

Step 0: Make an initial guess \mathbf{F}_0

Step 1: Given \mathbf{F}_k , compute $\vec{\mathbf{a}}_i^k = \mathbf{F}_k \vec{\mathbf{a}}_i$

Step 2: Compute $\Delta\mathbf{F}$ that minimizes

$$\sum_i \|\Delta\mathbf{F}\vec{\mathbf{a}}_i^k - \vec{\mathbf{b}}_i\|^2$$

Step 3: Set $\mathbf{F}_{k+1} = \Delta\mathbf{F}\mathbf{F}_k$

Step 4: Iterate Steps 1-3 until residual error is sufficiently small
(or other termination condition)



Direct Iterative approach for Rigid Frame

To solve for $\Delta\mathbf{F} = \arg \min \sum_i \|\Delta\mathbf{F}\vec{\mathbf{a}}_i^k - \vec{\mathbf{b}}_i\|^2$

$$\Delta\mathbf{F}\vec{\mathbf{a}}_i^k - \vec{\mathbf{b}}_i \approx \vec{\alpha} \times \vec{\mathbf{a}}_i^k + \vec{\varepsilon} + \vec{\mathbf{a}}_i^k - \vec{\mathbf{b}}_i$$

$$\vec{\alpha} \times \vec{\mathbf{a}}_i^k + \vec{\varepsilon} \approx \vec{\mathbf{b}}_i - \vec{\mathbf{a}}_i^k$$

$$sk(-\vec{\mathbf{a}}_i^k)\vec{\alpha} + \vec{\varepsilon} \approx \vec{\mathbf{b}}_i - \vec{\mathbf{a}}_i^k$$

Solve the least-squares problem

$$\begin{bmatrix} \vdots & \vdots \\ sk(-\vec{\mathbf{a}}_i^k) & \mathbf{I} \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vec{\alpha} \\ \vec{\varepsilon} \end{bmatrix} \approx \begin{bmatrix} \vdots \\ \vec{\mathbf{b}}_i - \vec{\mathbf{a}}_i^k \\ \vdots \end{bmatrix}$$

Now set $\Delta\mathbf{F} = [\Delta\mathbf{R}(\vec{\alpha}), \vec{\varepsilon}]$



Direct Techniques to solve for R

- Method due to K. Arun, et. al., IEEE PAMI, Vol 9, no 5, pp 698-700, Sept 1987

Step 1: Compute

$$\mathbf{H} = \sum_i \begin{bmatrix} \tilde{a}_{i,x} \tilde{b}_{i,x} & \tilde{a}_{i,x} \tilde{b}_{i,y} & \tilde{a}_{i,x} \tilde{b}_{i,z} \\ \tilde{a}_{i,y} \tilde{b}_{i,x} & \tilde{a}_{i,y} \tilde{b}_{i,y} & \tilde{a}_{i,y} \tilde{b}_{i,z} \\ \tilde{a}_{i,z} \tilde{b}_{i,x} & \tilde{a}_{i,z} \tilde{b}_{i,y} & \tilde{a}_{i,z} \tilde{b}_{i,z} \end{bmatrix}$$

NOTE well

Step 2: Compute the SVD of $\mathbf{H} = \mathbf{USV}^t$

Step 3: $\mathbf{R} = \mathbf{VU}^t$

Step 4: Verify $Det(\mathbf{R}) = 1$. If not, then algorithm may fail.

- Failure is rare, and mostly fixable. The paper has details.

Quaternions Technique to solve for R

- B.K.P. Horn, “Closed form solution of absolute orientation using unit quaternions”, J. Opt. Soc. America, A vol. 4, no. 4, pp 629-642, Apr. 1987.
- Method described as reported in Besl and McKay, “A method for registration of 3D shapes”, IEEE Trans. on Pattern Analysis and Machine Intelligence, vol. 14, no. 2, February 1992.
- Solves a 4x4 eigenvalue problem to find a unit quaternion corresponding to the rotation
- This quaternion may be converted in closed form to get a more conventional rotation matrix

Digression: quaternions

Invented by Hamilton in 1843. Can be thought of as

$$\begin{aligned} \text{4 elements:} & \quad \mathbf{q} = [q_0, q_1, q_2, q_3] \\ \text{scalar \& vector:} & \quad \mathbf{q} = s + \vec{v} = [s, \vec{v}] \\ \text{Complex number:} & \quad \mathbf{q} = q_0 + q_1i + q_2j + q_3k \\ & \quad \text{where } i^2 = j^2 = k^2 = ijk = -1 \end{aligned}$$

Properties:

$$\begin{aligned} \text{Linearity:} & \quad \lambda \mathbf{q}_1 + \mu \mathbf{q}_2 = [\lambda s_1 + \mu s_2, \lambda \vec{v}_1 + \mu \vec{v}_2] \\ \text{Conjugate:} & \quad \mathbf{q}^* = s - \vec{v} = [s, -\vec{v}] \\ \text{Product:} & \quad \mathbf{q}_1 \circ \mathbf{q}_2 = [s_1 s_2 - \vec{v}_1 \cdot \vec{v}_2, s_1 \vec{v}_2 + s_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2] \\ \text{Transform vector:} & \quad \mathbf{q} \circ \vec{p} = \mathbf{q} \circ [0, \vec{p}] \circ \mathbf{q}^* \\ \text{Norm:} & \quad \|\mathbf{q}\| = \sqrt{s^2 + \vec{v} \cdot \vec{v}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \end{aligned}$$



Digression continued: unit quaternions

We can associate a rotation by angle θ about an axis \vec{n} with the unit quaternion:

$$\text{Rot}(\vec{n}, \theta) \Leftrightarrow \left[\cos \frac{\theta}{2}, \sin \frac{\theta}{2} \vec{n} \right]$$

Exercise: Demonstrate this relationship. I.e., show

$$\text{Rot}((\vec{n}, \theta) \cdot \vec{p} = \left[\cos \frac{\theta}{2}, \sin \frac{\theta}{2} \vec{n} \right] \circ [0, \vec{p}] \circ \left[\cos \frac{\theta}{2}, -\sin \frac{\theta}{2} \vec{n} \right]$$

Hint: Substitute and reduce to see if you get Rodrigues' formula.



A bit more on quaternions

Exercise: show by substitution that the various formulations for quaternions are equivalent

A few web references:

<http://mathworld.wolfram.com/Quaternion.html>

<http://en.wikipedia.org/wiki/Quaternion>

http://en.wikipedia.org/wiki/Quaternions_and_spatial_rotation

<http://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/index.htm>

CAUTION: Different software packages are not always consistent in the order of elements if a quaternion is represented as a 4 element vector. Some put the scalar part first, others (including cisst libraries) put it last.

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Rotation matrix from unit quaternion

$$\mathbf{q} = [q_0, q_1, q_2, q_3]; \quad \|\mathbf{q}\| = 1$$

$$\mathbf{R}(\mathbf{q}) = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

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Unit quaternion from rotation matrix

$$\mathbf{R}(\mathbf{q}) = \begin{bmatrix} r_{xx} & r_{yx} & r_{zx} \\ r_{xy} & r_{yy} & r_{zy} \\ r_{xz} & r_{yz} & r_{zz} \end{bmatrix}; \quad \begin{aligned} \mathbf{a}_0 &= 1 + r_{xx} + r_{yy} + r_{zz}; & \mathbf{a}_1 &= 1 + r_{xx} - r_{yy} - r_{zz} \\ \mathbf{a}_2 &= 1 - r_{xx} + r_{yy} - r_{zz}; & \mathbf{a}_3 &= 1 - r_{xx} - r_{yy} + r_{zz} \end{aligned}$$

$\mathbf{a}_0 = \max\{\mathbf{a}_k\}$	$\mathbf{a}_1 = \max\{\mathbf{a}_k\}$	$\mathbf{a}_2 = \max\{\mathbf{a}_k\}$	$\mathbf{a}_3 = \max\{\mathbf{a}_k\}$
$q_0 = \frac{\sqrt{\mathbf{a}_0}}{2}$	$q_0 = \frac{r_{yz} - r_{zy}}{4q_1}$	$q_0 = \frac{r_{zx} - r_{xz}}{4q_2}$	$q_0 = \frac{r_{xy} - r_{yx}}{4q_3}$
$q_1 = \frac{r_{xy} - r_{yx}}{4q_0}$	$q_1 = \frac{\sqrt{\mathbf{a}_1}}{2}$	$q_1 = \frac{r_{xy} + r_{yx}}{4q_2}$	$q_1 = \frac{r_{xz} + r_{zx}}{4q_3}$
$q_2 = \frac{r_{zx} - r_{xz}}{4q_0}$	$q_2 = \frac{r_{xy} + r_{yx}}{4q_1}$	$q_2 = \frac{\sqrt{\mathbf{a}_2}}{2}$	$q_2 = \frac{r_{yz} + r_{zy}}{4q_3}$
$q_3 = \frac{r_{yz} - r_{zy}}{4q_0}$	$q_3 = \frac{r_{xz} + r_{zx}}{4q_1}$	$q_3 = \frac{r_{yz} + r_{zy}}{4q_2}$	$q_3 = \frac{\sqrt{\mathbf{a}_3}}{2}$

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Rotation axis and angle from rotation matrix

Many options, including direct trigonometric solution.
But this works:

```
[n̄, θ] ← ExtractAxisAngle(R)
{
  [s, v̄] ← ConvertToQuaternion(R)
  return([v̄ / ||v̄||, 2atan(s / ||v̄||)])
}
```

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Quaternion method for R

Step 1: Compute

$$\mathbf{H} = \sum_i \begin{bmatrix} \tilde{a}_{i,x} \tilde{b}_{i,x} & \tilde{a}_{i,x} \tilde{b}_{i,y} & \tilde{a}_{i,x} \tilde{b}_{i,z} \\ \tilde{a}_{i,y} \tilde{b}_{i,x} & \tilde{a}_{i,y} \tilde{b}_{i,y} & \tilde{a}_{i,y} \tilde{b}_{i,z} \\ \tilde{a}_{i,z} \tilde{b}_{i,x} & \tilde{a}_{i,z} \tilde{b}_{i,y} & \tilde{a}_{i,z} \tilde{b}_{i,z} \end{bmatrix}$$

Step 2: Compute

$$\mathbf{G} = \begin{bmatrix} \text{trace}(\mathbf{H}) & \Delta^T \\ \Delta & \mathbf{H} + \mathbf{H}^T - \text{trace}(\mathbf{H})\mathbf{I} \end{bmatrix}$$

$$\text{where } \Delta^T = \begin{bmatrix} \mathbf{H}_{2,3} - \mathbf{H}_{3,2} & \mathbf{H}_{3,1} - \mathbf{H}_{1,3} & \mathbf{H}_{1,2} - \mathbf{H}_{2,1} \end{bmatrix}$$

Step 3: Compute eigen value decomposition of G

$$\text{diag}(\bar{\lambda}) = \mathbf{Q}^T \mathbf{G} \mathbf{Q}$$

Step 4: The eigenvector $\mathbf{Q}_k = [q_0, q_1, q_2, q_3]$ corresponding to the largest eigenvalue λ_k is a unit quaternion corresponding to the rotation.



Another Quaternion Method for R

Let $\mathbf{q} = s + \bar{\mathbf{v}}$ be the unit quaternion corresponding to \mathbf{R} . Let $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ be vectors with $\tilde{\mathbf{b}} = \mathbf{R} \cdot \tilde{\mathbf{a}}$ then we have the quaternion equation

$$(s + \bar{\mathbf{v}}) \cdot (0 + \tilde{\mathbf{a}})(s - \bar{\mathbf{v}}) = 0 + \tilde{\mathbf{b}}$$

$$(s + \bar{\mathbf{v}}) \cdot (0 + \tilde{\mathbf{a}}) = (0 + \tilde{\mathbf{b}}) \cdot (s + \bar{\mathbf{v}}) \quad \text{since } (s - \bar{\mathbf{v}})(s + \bar{\mathbf{v}}) = 1 + \bar{\mathbf{0}}$$

Expanding the scalar and vector parts gives

$$-\bar{\mathbf{v}} \cdot \tilde{\mathbf{a}} = -\bar{\mathbf{v}} \cdot \tilde{\mathbf{b}}$$

$$s\tilde{\mathbf{a}} + \bar{\mathbf{v}} \times \tilde{\mathbf{a}} = s\tilde{\mathbf{b}} + \tilde{\mathbf{b}} \times \bar{\mathbf{v}}$$

Rearranging ...

$$(\tilde{\mathbf{b}} - \tilde{\mathbf{a}}) \cdot \bar{\mathbf{v}} = 0$$

$$s(\tilde{\mathbf{b}} - \tilde{\mathbf{a}}) + (\tilde{\mathbf{b}} + \tilde{\mathbf{a}}) \times \bar{\mathbf{v}} = \bar{\mathbf{0}}_3$$

NOTE: This method works for any set of vectors $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$. We are using the symbols $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ to maintain consistency with the discussion of the previous method.



Another Quaternion Method for R

Expressing this as a matrix equation

$$\left[\begin{array}{c|c} 0 & (\tilde{\mathbf{b}} - \tilde{\mathbf{a}})^T \\ \hline (\tilde{\mathbf{b}} - \tilde{\mathbf{a}}) & sk(\tilde{\mathbf{b}} + \tilde{\mathbf{a}}) \end{array} \right] \begin{bmatrix} s \\ \tilde{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{\mathbf{0}}_3 \end{bmatrix}$$

If we now express the quaternion \mathbf{q} as a 4-vector $\vec{\mathbf{q}} = [s, \tilde{\mathbf{v}}]^T$, we can express the rotation problem as the constrained linear system

$$\begin{aligned} \mathbf{M}(\vec{\mathbf{a}}, \vec{\mathbf{b}})\vec{\mathbf{q}} &= \vec{\mathbf{0}}_4 \\ \|\vec{\mathbf{q}}\| &= 1 \end{aligned}$$

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Another Quaternion Method for R

In general, we have many observations, and we want to solve the problem in a least squares sense:

$$\min \|\mathbf{M}\vec{\mathbf{q}}\| \text{ subject to } \|\vec{\mathbf{q}}\| = 1$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}(\vec{\mathbf{a}}_1, \vec{\mathbf{b}}_1) \\ \vdots \\ \mathbf{M}(\vec{\mathbf{a}}_n, \vec{\mathbf{b}}_n) \end{bmatrix} \text{ and } n \text{ is the number of observations}$$

Taking the singular value decomposition of $\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^T$ reduces this to the easier problem

$$\min \|\mathbf{U}\Sigma\mathbf{V}^T\vec{\mathbf{q}}_x\| = \|\mathbf{U}(\Sigma\vec{\mathbf{y}})\| = \|\Sigma\vec{\mathbf{y}}\| \text{ subject to } \|\vec{\mathbf{y}}\| = \|\mathbf{V}^T\vec{\mathbf{q}}\| = \|\vec{\mathbf{q}}\| = 1$$

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Another Quaternion Method for R

This problem is just

$$\min \|\Sigma \bar{\mathbf{y}}\| = \left\| \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_4 \end{bmatrix} \bar{\mathbf{y}} \right\| \quad \text{subject to } \|\bar{\mathbf{y}}\| = 1$$

where σ_i are the singular values. Recall that SVD routines typically return the $\sigma_i \geq 0$ and sorted in decreasing magnitude. So σ_4 is the smallest singular value and the value of $\bar{\mathbf{y}}$ with $\|\bar{\mathbf{y}}\| = 1$ that minimizes $\|\Sigma \bar{\mathbf{y}}\|$ is $\bar{\mathbf{y}} = [0, 0, 0, 1]^T$. The corresponding value of $\bar{\mathbf{q}}$ is given by $\bar{\mathbf{q}} = \mathbf{V}\bar{\mathbf{y}} = \mathbf{V}_4$. Where \mathbf{V}_4 is the 4th column of \mathbf{V} .

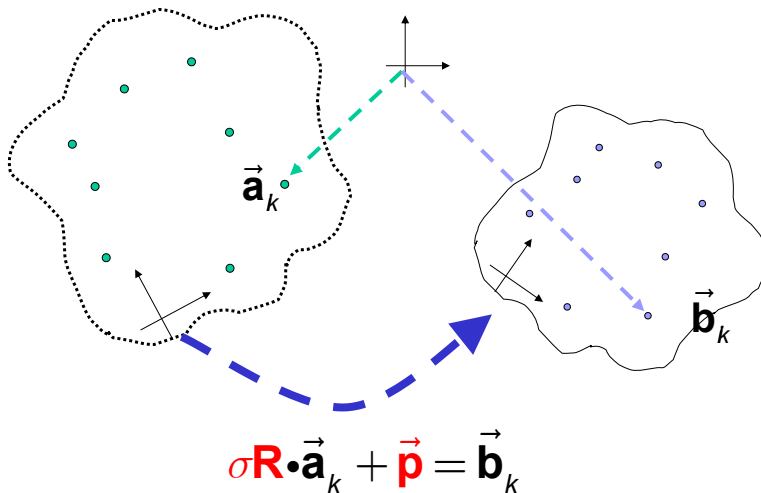
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Non-reflective spatial similarity (rigid+scale)



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Non-reflective spatial similarity

Step 1: Compute

$$\bar{\mathbf{a}} = \frac{1}{N} \sum_{i=1}^N \bar{\mathbf{a}}_i \quad \bar{\mathbf{b}} = \frac{1}{N} \sum_{i=1}^N \bar{\mathbf{b}}_i$$

$$\tilde{\mathbf{a}}_i = \bar{\mathbf{a}}_i - \bar{\mathbf{a}} \quad \tilde{\mathbf{b}}_i = \bar{\mathbf{b}}_i - \bar{\mathbf{b}}$$

Step 2: Estimate scale

$$\sigma = \frac{\sum_i \|\tilde{\mathbf{b}}_i\|}{\sum_i \|\tilde{\mathbf{a}}_i\|}$$

Step 3: Find \mathbf{R} that minimizes

$$\sum_i (\mathbf{R} \cdot (\sigma \tilde{\mathbf{a}}_i) - \tilde{\mathbf{b}}_i)^2$$

Step 4: Find $\bar{\mathbf{p}}$

$$\bar{\mathbf{p}} = \bar{\mathbf{b}} - \mathbf{R} \cdot \bar{\mathbf{a}}$$

Step 5: Desired transformation is

$$\mathbf{F} = \text{SimilarityFrame}(\sigma, \mathbf{R}, \bar{\mathbf{p}})$$

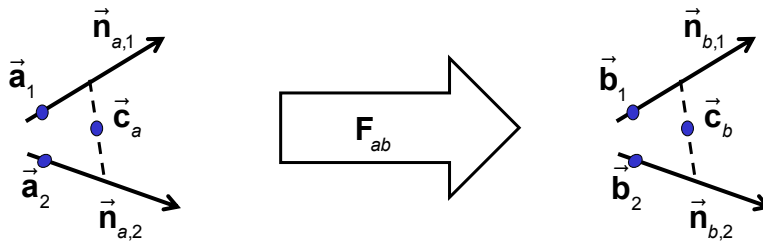
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Registration from line pairs



Approach 1:

Compute $\mathbf{F}_a = [\mathbf{R}_a, \vec{\mathbf{c}}_a]$ from line pair a

Compute $\mathbf{F}_b = [\mathbf{R}_b, \vec{\mathbf{c}}_b]$ from line pair b

$$\mathbf{F}_{ab} = \mathbf{F}_a^{-1} \mathbf{F}_b$$

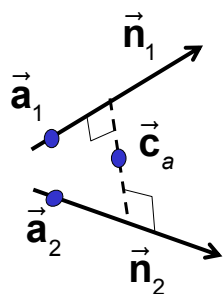
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Coordinate system from line pairs



$$\mathbf{R}_a = \left[\begin{array}{l} \vec{x}_1 = \vec{n}_1 \quad \vec{y}_1 = \frac{\vec{n}_1 \times \vec{n}_2}{\|\vec{n}_1 \times \vec{n}_2\|} \quad \vec{z}_1 = \vec{x}_1 \times \vec{y}_1 \end{array} \right]$$

\vec{c}_a = midpoint between the two lines

To get the midpoint:

$$\text{Solve } \begin{bmatrix} \vec{n}_1 & -\vec{n}_2 \end{bmatrix} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} \approx [\vec{a}_2 - \vec{a}_1]$$

$$\text{Then } \vec{c}_a = \frac{(\vec{a}_1 + \lambda \vec{n}_1) + (\vec{a}_2 + \nu \vec{n}_2)}{2}$$

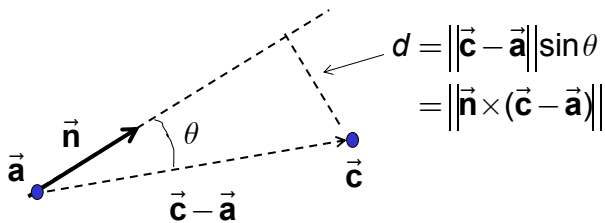
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Distance of a point from a line



So, to find the closest point to multiple lines

$$\vec{c} = \operatorname{argmin} \sum_k d_k^2$$

Solve this problem in a least squares sense:

$$\vec{n}_k \times (\vec{c} - \vec{a}_k) \approx \vec{0} \text{ for } k = 1, \dots, n$$

Equivalently, solve

$$\vec{n}_k \times \vec{c} \approx \vec{n}_k \times \vec{a}_k \text{ for } k = 1, \dots, n$$

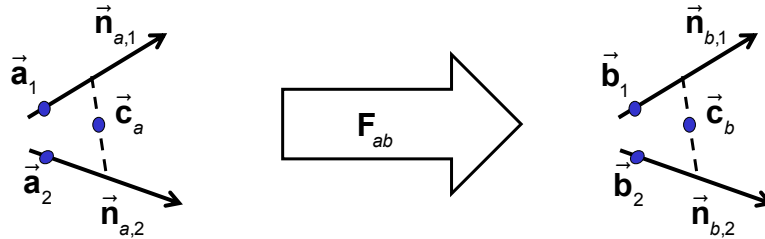
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Registration from multiple corresponding lines



Approach 2:

$$\text{Solve } \{ \dots, \mathbf{R}_{ab} \vec{n}_{b,k} \approx \vec{n}_{a,k}, \dots \} \text{ for } \mathbf{R}_{ab}$$

$$\text{Solve } \{ \dots, \vec{n}_{a,k} \times \vec{c}_a \approx \vec{n}_{a,k} \times \vec{a}_k, \dots \} \text{ for } \vec{c}_a$$

$$\text{Solve } \{ \dots, \vec{n}_{b,k} \times \vec{c}_b \approx \vec{n}_{b,k} \times \vec{b}_k, \dots \} \text{ for } \vec{c}_b$$

$$\vec{p}_{ab} = \vec{c}_a - \mathbf{R}_{ab} \vec{c}_b$$

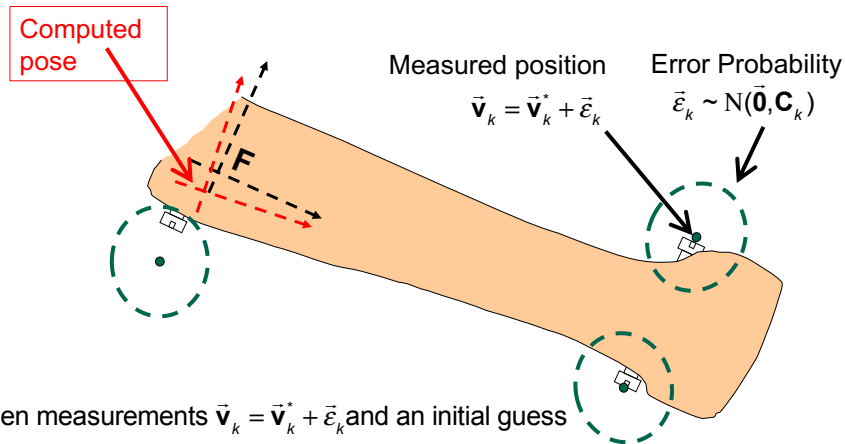
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Probabilistic Estimation



Given measurements $\vec{v}_k = \vec{v}_k^* + \vec{e}_k$ and an initial guess for \mathbf{F} , where $\mathbf{F}^* = \Delta \mathbf{F}(\eta_F) \mathbf{F}$, where $\vec{e}_k \sim N(\vec{0}, \mathbf{C}_k)$ and $\vec{\eta}_F = [\vec{\alpha}_F^T, \vec{\varepsilon}_F^T]^T \sim N(\vec{\mu}_F, \mathbf{C}_F)$, we want to improve our estimate of \mathbf{F} and determine a the probability distribution for the corresponding $\vec{\eta}_F$

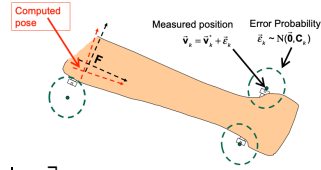
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Probabilistic Estimation



Recall that $\mathbf{v}_k + \bar{\boldsymbol{\varepsilon}}_k = \Delta\mathbf{F}(\bar{\boldsymbol{\eta}}_F)\mathbf{F}\bar{\mathbf{b}}_k$, so that

$$\bar{\boldsymbol{\varepsilon}}_k = \bar{\boldsymbol{\alpha}}_k \times \mathbf{F}\bar{\mathbf{b}}_k + \bar{\boldsymbol{\varepsilon}}_k = \mathbf{A}_k \bar{\boldsymbol{\eta}}_k, \text{ where } \mathbf{A}_k = \begin{bmatrix} -sk(\mathbf{F}\bar{\mathbf{b}}_k) & \mathbf{I} \end{bmatrix}$$

If we assume that the $\bar{\boldsymbol{\varepsilon}}_k$ are independent, then

$$pr(\mathbf{E}=[\bar{\boldsymbol{\varepsilon}}_1, \dots, \bar{\boldsymbol{\varepsilon}}_m] | \bar{\boldsymbol{\eta}}_F) = \prod_k pr(\bar{\boldsymbol{\varepsilon}}_k | \bar{\boldsymbol{\eta}}_F) = \prod_k \frac{\exp(-\bar{\boldsymbol{\eta}}_F^T \mathbf{G}_k^{-1} \bar{\boldsymbol{\eta}}_F / 2)}{\sqrt{(2\pi)^n |\mathbf{G}_k|}}$$

$$\text{where } \mathbf{G}_k = \mathbf{A}_k^T \mathbf{C}_k \mathbf{A}_k \text{ and } \mathbf{A}_k = \begin{bmatrix} -sk(\mathbf{F}\bar{\mathbf{b}}_k) & \mathbf{I} \end{bmatrix}$$

$$L(\mathbf{E} | \bar{\boldsymbol{\eta}}_F) = \log(pr(\mathbf{E} | \bar{\boldsymbol{\eta}}_F)) = -\sum_k \bar{\boldsymbol{\eta}}_F^T \mathbf{G}_k^{-1} \bar{\boldsymbol{\eta}}_F / 2 - \text{constant}$$

Find the value $\bar{\boldsymbol{\eta}}_F^\#$ that produces most likely value $\mathbf{E}^\#$ for $\mathbf{E} | (\bar{\boldsymbol{\eta}}_F = \bar{\boldsymbol{\eta}}_F^\#)$

$$\bar{\boldsymbol{\eta}}_F^\# = \underset{\bar{\boldsymbol{\eta}}_F}{\text{argmax}} \left(-\sum_k \bar{\boldsymbol{\eta}}_F^T \mathbf{G}_k^{-1} \bar{\boldsymbol{\eta}}_F \right) = \underset{\bar{\boldsymbol{\eta}}_F}{\text{argmin}} \sum_k \bar{\boldsymbol{\eta}}_F^T \mathbf{G}_k^{-1} \bar{\boldsymbol{\eta}}_F = \underset{\bar{\boldsymbol{\eta}}_F}{\text{argmin}} \bar{\boldsymbol{\eta}}_F^T \left(\sum_k \mathbf{G}_k^{-1} \right) \bar{\boldsymbol{\eta}}_F$$

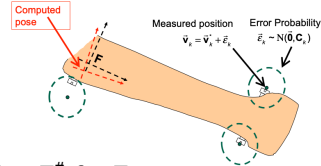
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Probabilistic Estimation



Continuing from $\bar{\boldsymbol{\eta}}_F^\# = \underset{\bar{\boldsymbol{\eta}}_F}{\text{argmin}} \bar{\boldsymbol{\eta}}_F^T \left(\sum_k \mathbf{G}_k^{-1} \right) \bar{\boldsymbol{\eta}}_F$

We can use this value to produce a most likely value $\mathbf{F}^\#$ for \mathbf{F}

$$\mathbf{F}^\# = \Delta\mathbf{F}(\bar{\boldsymbol{\eta}}_F^\#)\mathbf{F} = [\mathbf{R}(\bar{\boldsymbol{\alpha}}^\#), \bar{\boldsymbol{\varepsilon}}^\#] \bullet [\mathbf{R}, \bar{\mathbf{p}}] = [\mathbf{R}(\bar{\boldsymbol{\alpha}}^\#)\mathbf{R}, \mathbf{R}(\bar{\boldsymbol{\alpha}}^\#)\bar{\mathbf{p}} + \bar{\boldsymbol{\varepsilon}}^\#]$$

Remember that $\mathbf{R}(\bar{\boldsymbol{\alpha}}^\#) \neq \mathbf{I} + sk(\bar{\boldsymbol{\alpha}}^\#)$

If we now update $\mathbf{F} \leftarrow \mathbf{F}^\#$, we want to know how confident we can be in this new estimated value. We can redefine $\bar{\boldsymbol{\eta}}_F$ so that

$$\mathbf{F}^* = \Delta\mathbf{F}(\bar{\boldsymbol{\eta}}_F)\mathbf{F} \text{ where } \bar{\boldsymbol{\eta}}_F \sim \mathbf{N}(\bar{\mathbf{0}}, \mathbf{C}_F)$$

$$\mathbf{C}_F = \left(\sum_k \mathbf{G}_k^{-1} \right)^{-1} = \mathbf{Q}_F \boldsymbol{\Lambda}_F^2 \mathbf{Q}_F^T \text{ where } \boldsymbol{\Lambda}_F^2 \text{ is diagonal and } \mathbf{Q}_F \mathbf{Q}_F^T = \mathbf{I}$$

$$pr(\bar{\boldsymbol{\eta}}_F) = \frac{\exp(-\bar{\boldsymbol{\eta}}_F^T \mathbf{C}_F^{-1} \bar{\boldsymbol{\eta}}_F / 2)}{\sqrt{(2\pi)^n |\mathbf{C}_F|}} = \frac{\exp(-\bar{\boldsymbol{\eta}}_F^T \mathbf{C}_F^{-1} \bar{\boldsymbol{\eta}}_F / 2)}{\sqrt{(2\pi)^n |\boldsymbol{\Lambda}_F^2|}}$$

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