Simulation-Based Uncertainty Propagation in Geometric Networks for Surgical Robotics

X.M. Christine Zhu

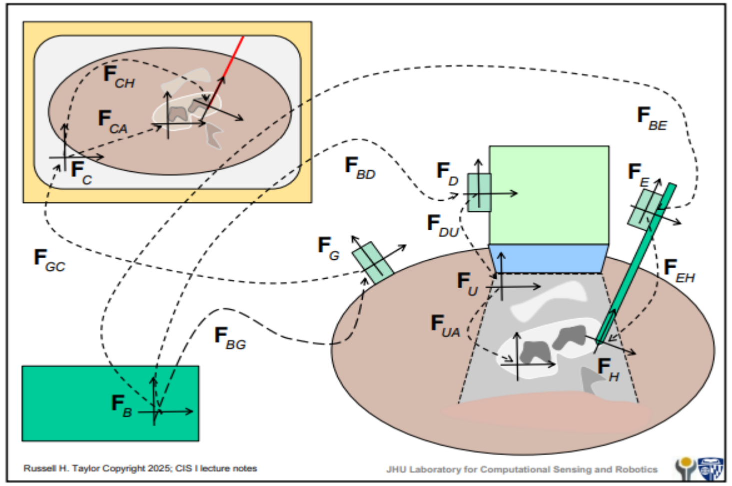
Mentor: Dr. Russell H. Taylor

Johns Hopkins University

Reference:
Barfoot, T. D., and Furgale, P. T., “Associating uncertainty with three-dimensional poses for use in estimation problems,” *IEEE Transactions on Robotics*, vol. 30, no. 3, pp. 679–693, Jun. 2014, doi: 10.1109/TRO.2014.2298059.

MOTIVATION & GOAL

- Surgical robotic systems rely on **multiple geometric components**:
  - Robots, tracking systems, anatomy, sensors, etc.
- Each component introduces **uncertainty**:
  - Calibration error
  - Measurement noise
  - Kinematic modeling error
- These uncertainties **interact** through geometric composition



Note from 2025 CIS I course

Core problem:

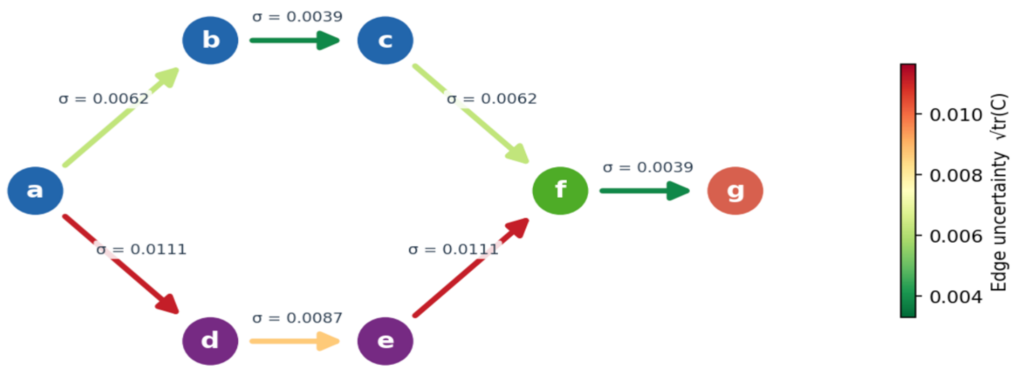
There is no unified, general framework to **model and query uncertainty propagation across an arbitrary geometric network**.

Project Goal:

Develop a **general simulation framework** that models and propagates uncertainty through a network of geometric relationships, enabling uncertainty queries between any two nodes.

Background & Relevance

- In surgical robotics, a robot arm is a **chain of coordinate frames** — each joint, sensor, and rigid link introduces measurement noise.
- The classical approach attaches a covariance matrix to a pose written as a position vector and Euler angles, then propagates it through linearized Jacobians. This breaks down because the rotation group SO(3) is a **nonlinear manifold** — a Gaussian distribution defined on Euler-angle coordinates is geometrically incorrect and degrades with large rotations.
- This paper directly solves that problem, and its solution is the **exact mathematical foundation** of our project



T. D. Barfoot and P. T. Furgale, "Associating Uncertainty With Three-Dimensional Poses for Use in Estimation Problems," *IEEE Transactions on Robotics*, vol. 30, no. 3, pp. 679–693, June 2014. DOI: 10.1109/TRO.2014.2298059

Paper’s Goals & Contributions

Central goal:

Given a 4×4 homogeneous transformation matrix T in SE(3), derive a rigorous and practical way to attach a covariance matrix and propagate it through rigid-body compositions.

Key contribution:

The paper establishes a principled framework for associating Gaussian uncertainty with 3D poses on $SE(3)$,and shows how to propagate and fuse that uncertainty accurately in estimation problems.c

Experimental Validation in paper

- This paper does **not** present a physical robot or clinical experiment.
- Instead, it validates the proposed $SE(3)$uncertainty framework through **three simulation-based estimation studies**, always comparing analytic approximations against **Monte Carlo ground truth**.

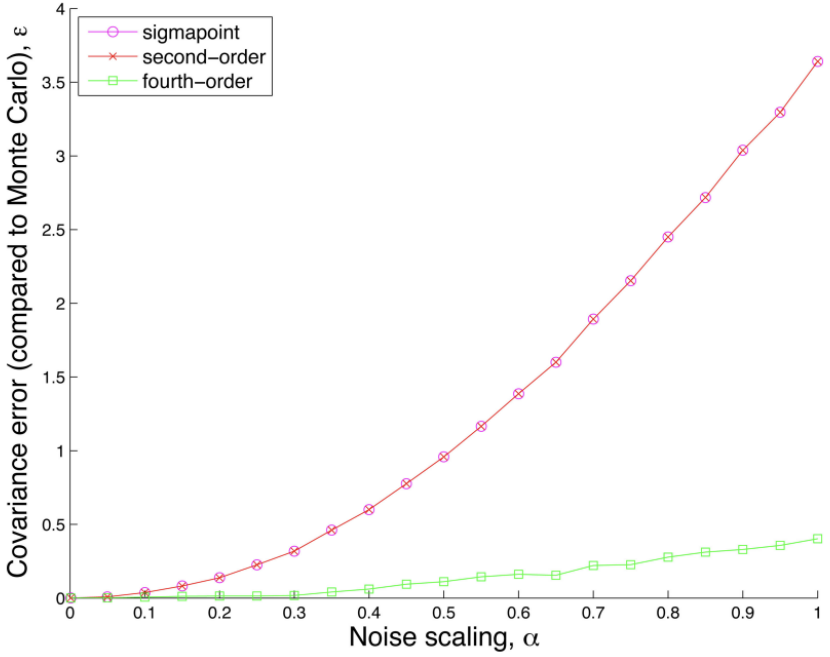
Main Takeaway

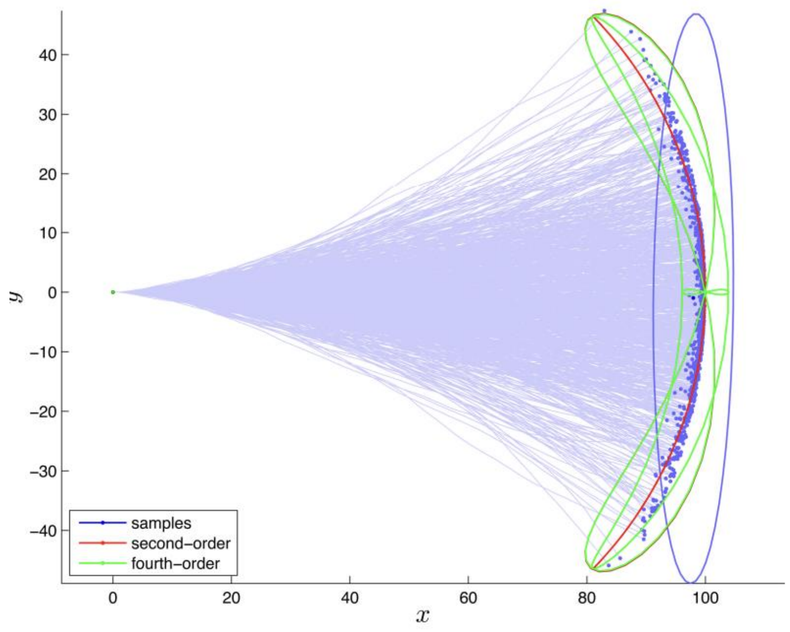
- The experiments show that uncertainty in 3D poses should be handled using the **geometry of** $SE(3)$,and that **higher-order or sampling-based methods** can give much better accuracy than simple first-order approximations.

Compound Pose Experiment

- Two uncertain poses are composed, and covariance is propagated through the pose chain.
- Methods compared: **second-order**, **fourth-order**, and **sigma-point** approximations versus **1,000,000-sample Monte Carlo**.

Result: all methods work well for small uncertainty, but as uncertainty increases, the **fourth-order approximation is most accurate**.



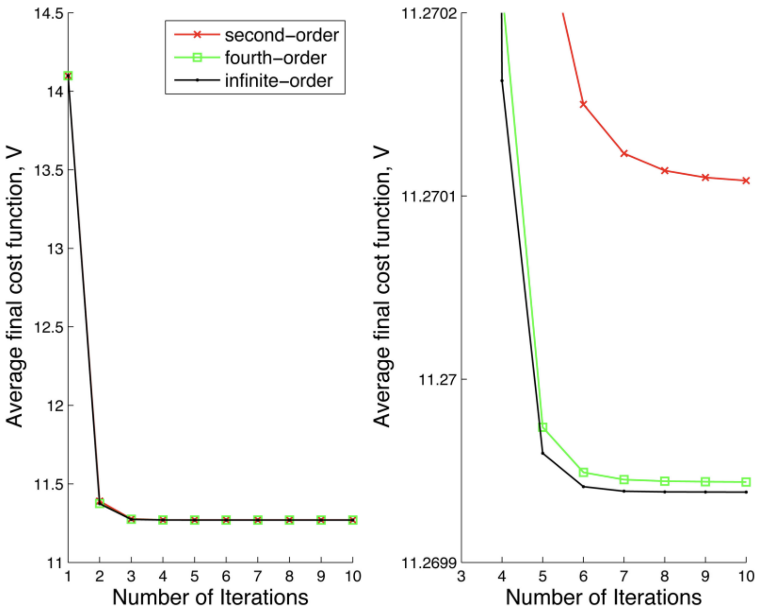


Barfoot, T. D., and Furgale, P. T., "Associating uncertainty with three-dimensional poses for use in estimation problems," *IEEE Transactions on Robotics*, vol. 30, no. 3, pp. 679–693, Jun. 2014, doi: 10.1109/TRO.2014.2298059.

Pose Fusion Experiment

- Several noisy pose estimates of the same true pose are fused using an **iterative Gauss–Newton method on $SE(3)$.**
- Performance is evaluated using **final cost** and **RMS pose error** over many random trials.

Result: more iterations and a better Jacobian inverse approximation improve performance, with benefits largely saturating after a few terms.

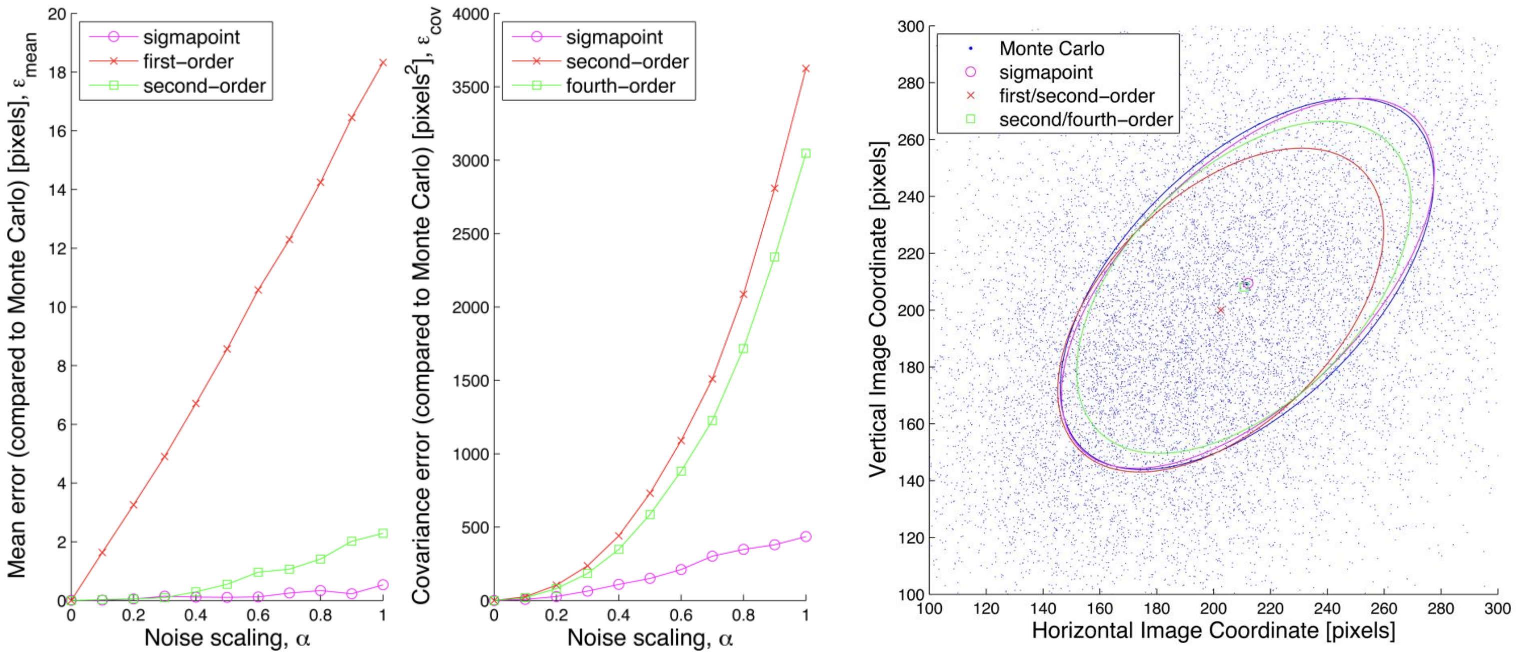


Barfoot, T. D., and Furgale, P. T., "Associating uncertainty with three-dimensional poses for use in estimation problems," *IEEE Transactions on Robotics*, vol. 30, no. 3, pp. 679–693, Jun. 2014, doi: 10.1109/TRO.2014.2298059.

Stereo Camera Experiment

- Pose and landmark uncertainty are propagated through a **nonlinear stereo camera model**.
- Methods are again compared with a large Monte Carlo reference.

Result: the **sigma-point method best matches Monte Carlo** for both projected mean and covariance under strong nonlinearity.



Barfoot, T. D., and Furgale, P. T., “Associating uncertainty with three-dimensional poses for use in estimation problems,” *IEEE Transactions on Robotics*, vol. 30, no. 3, pp. 679–693, Jun. 2014, doi: 10.1109/TRO.2014.2298059.

Results

Method	Small uncertainty	Medium	Large
First-order	~1% error	~3%	~15%
Second-order	~0.3%	~1%	~3%
Sigma-point	~0.4%	~1%	~4%

Key findings:

- For small perturbations (sub-mm, sub-degree), first-order and sigma-point methods both match Monte Carlo to within ~1% — sufficient for surgical robotics
- Second-order correction improves accuracy significantly at large uncertainties
- Sigma-point and second-order methods are proved to be **algebraically equivalent** for pose compounding — not just empirically similar

How I Use These Results

Paper result	How it is used in my project
$T=\exp(\delta\xi^\wedge)T_{nominal}, \delta\xi \sim N(0,\Sigma)$	UncertainTransform(F_{nom}, C) — left-perturbation convention throughout
$\Sigma_c = \Sigma_a + Ad(T_a)\cdot\Sigma_b\cdot Ad(T_a)^T$	Chain propagation in query() / query_frame(), validated in validate_open_chain_mc.py
Information-form fusion Σ_{fused} = inverse($\Sigma\Sigma_{k^{-1}}$)	**Extended** to S-matrix for paths sharing physical edges — prevents double-counting of shared uncertainty, the main gap left open by the paper
$C_{post} = (C0^{-1} + H^T C_{noise^{-1}} H)^{-1}$	Loop closure conditioning in condition_on_loop(), validated in validate_closed_loop_mc.py

The **S-matrix extension** is the primary novel contribution of my project beyond Barfoot & Furgale: when two paths share a physical edge, their uncertainties are correlated and the simple information-form sum overcounts the shared uncertainty. The S-matrix captures these off-diagonal cross-covariances and gives the correct fused result.

Strengths & Weaknesses

Strengths

- Mathematically rigorous — no gimbal lock, no coordinate singularities, valid for any rotation magnitude
- Closed-form formulas that are directly implementable — now the standard reference in robotics state estimation
- Naturally extends to kinematic chains of any length via the adjoint composition rule
- Consistent with Kalman filter theory — unifies pose uncertainty with decades of prior estimation work
- Forms the backbone of Barfoot's textbook *State Estimation for Robotics*

Strengths & Weaknesses

Weaknesses

- Gaussian approximation still breaks down for very large uncertainties (near-90° rotation variance)
- Does not handle joint distributions of correlated poses — Mangelson et al. (TRO 2020) identified this gap, which is exactly what the S-matrix in this project addresses
- Assumes Gaussian noise throughout — cannot represent multimodal or heavy-tailed errors from backlash, friction, or calibration drift common in real surgical instruments
- Left vs. right perturbation convention ambiguity causes inconsistencies across the literature

Key Takeaways

- Pose uncertainty must live in the Lie algebra se(3) to be geometrically correct — this is the foundational insight of the paper
- The adjoint map is the SE(3) predict step — it is the correct and complete tool for propagating covariance through rigid-body chains
- Information-form fusion is the SE(3) update step — optimal when paths are independent; this project extends it to the correlated case
- First-order linearization is sufficient for surgical robotics applications (sub-mm, sub-degree regime)
- The paper leaves the shared-edge correlation problem open — the S-matrix in this project is the direct fix
- The experiments show that uncertainty in 3D poses should be handled using the **geometry of $SE(3)$** ,and that **higher-order or sampling-based methods** can give much better accuracy than simple first-order approximations.

Reference

- T. D. Barfoot and P. T. Furgale, "Associating Uncertainty With Three-Dimensional Poses for Use in Estimation Problems," *IEEE Transactions on Robotics*, vol. 30, no. 3, pp. 679–693, June 2014. DOI: 10.1109/TRO.2014.2298059

Thank You!

X.M. Christine Zhu
xzhu83@jhu.edu