MATHEMATICAL FRAMEWORK FOR UNCERTAINTY PROPAGATION IN GEOMETRIC NETWORKS

Project Functional Specifications

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Contents

1 Mathematical Summary

This work develops a mathematical framework for **propagating uncertainty through geometric relationships** in systems composed of rigid components, such as robots, tracking systems, and anatomical models. Each geometric relationship is represented by a **nominal rigid-body transformation** together with an associated uncertainty description. The objective is to determine how uncertainty accumulates and transforms when such relationships are composed along a network.

Rigid-body transformations do not form a vector space, and therefore uncertainty cannot be added to them directly. To address this, uncertainty is modeled as a **small perturbation expressed in the Lie algebra associated with rigid-body motion**, which is a six-dimensional vector space representing infinitesimal translations and rotations. These perturbations are mapped to physically valid rigid transformations using the exponential map, ensuring that all composed transformations remain consistent with rigid-body geometry.

Uncertainty is assumed to follow a **multivariate Gaussian distribution** in this six-dimensional perturbation space. This assumption enables uncertainty to be represented compactly using covariance matrices, supports first-order analytical propagation through geometric composition, and allows validation using Monte Carlo simulation. Together, these modeling choices provide a consistent and practical mathematical basis for uncertainty propagation in geometric networks.

2 Methods: Mathematical Framework for Uncertainty Propagation

2.1 Objective and Scope

We consider systems composed of multiple geometric components—such as robots, tracking devices, tools, and anatomical models—whose spatial relationships are described by rigid-body transformations. These relationships are not exact and are subject to uncertainty arising from calibration error, sensor noise, modeling assumptions, or manufacturing tolerances.

The objective of this framework is to provide a **mathematically consistent method for propagating uncertainty** through such geometric relationships. Specifically, given:

- a network of rigid transformations with known nominal geometry, and
- a selected uncertainty model associated with each transformation,

the framework computes the **nominal transformation and associated uncertainty** between any two nodes in the network.

This work addresses **uncertainty propagation only**. It does not perform estimation, optimization, or inference.

2.2 Nominal Geometry

A geometric relationship between two coordinate frames is represented by a rigid-body transformation belonging to the space $SE(3)$. Each transformation consists of:

- a rotation component, and
- a translation component.

2.3 Nominal Transformation: F_{nom}

For each geometric relationship, we define a **nominal transformation**, denoted by F_{nom}.

F_{nom} represents:

- the best deterministic estimate of the geometry,
- the mean or reference configuration,
- a transformation with no stochastic component.

Typical sources of F_{nom} include robot forward kinematics, calibration procedures, tracking system outputs, image-to-model registration, and CAD or CT-based geometric models.

2.4 Uncertainty Representation

Each geometric relationship is modeled as the pair

$$F = \{F_{\text{nom}}, C\},$$

where:

- $F_{\text{nom}} \in SE(3)$ is the nominal transformation,
- $C \in \mathbb{R}^{6\times 6}$ is a covariance matrix describing uncertainty.

The covariance is **selected** based on sensor specifications, tolerance analysis, calibration accuracy, or modeling assumptions. The framework does not estimate C.

Uncertainty is represented by a perturbation variable

$$\vec{\eta} \in \mathbb{R}^6, \quad \vec{\eta} \sim \mathcal{N}(0, C).$$

The perturbation vector is decomposed as

$$\vec{\eta} = \begin{bmatrix} \vec{\alpha} \\ \vec{\epsilon} \end{bmatrix},$$

where $\vec{\alpha} \in \mathbb{R}^3$ represents rotational uncertainty and $\vec{\epsilon} \in \mathbb{R}^3$ represents translational uncertainty.

Perturbations are assumed to be sufficiently small for first-order approximations to be valid.

2.5 Left-Multiplicative Perturbation Model

This framework adopts a **left-multiplicative perturbation** convention. The true transformation associated with F is modeled as

$$T = \exp(\vec{\eta}) \circ F_{\text{nom}}.$$

Under this convention:

- perturbations are expressed in the parent (global) coordinate frame,
- all uncertainty is represented in a consistent coordinate system,
- composed uncertainties admit a clear first-order interpretation.

All subsequent derivations assume this convention.

2.6 3-DOF Points

In addition to coordinate frames, the geometric network may include 3-degree-of-freedom point nodes. A point node represents a fixed location in $\mathbb{R}^3$ expressed in a specified reference frame.

An uncertain point is represented as

$$p = \{p_{\text{nom}}, C_p\},$$

where $p_{\text{nom}} \in \mathbb{R}^3$ is the nominal point location and $C_p \in \mathbb{R}^{3\times 3}$ is the associated covariance matrix.

Point uncertainty may arise from sources such as segmentation error, feature localization noise, or measurement uncertainty. Point nodes allow the framework to represent geometric entities that do not carry orientation, such as landmarks, fiducials, or anatomical features.

When a point is transformed under an uncertain rigid-body transformation, its uncertainty is propagated using a first-order approximation that accounts for both pose uncertainty and point uncertainty. The specific propagation expressions are derived using the Jacobian of the point transformation with respect to the pose perturbation $\vec{\eta}$ and the point coordinates.

2.7 Composition of Uncertain Transformations

Given two nominal transformations

$$F_{\text{nom},ab}, \quad F_{\text{nom},bc},$$

the nominal composed transformation is

$$F_{\text{nom},ac} = F_{\text{nom},ab} \circ F_{\text{nom},bc}.$$

Let the corresponding uncertain relationships be

$$F_{ab} = \{F_{\text{nom},ab}, C_{ab}\}, \quad F_{bc} = \{F_{\text{nom},bc}, C_{bc}\}.$$

Under the left-multiplicative perturbation model, the composed perturbation satisfies the first-order approximation

$$\vec{\eta}_{ac} \approx \vec{\eta}_{ab} + \mathrm{Ad}_{F_{\text{nom},ab}}\, \vec{\eta}_{bc}.$$

Assuming independence, the covariance propagates as

$$C_{ac} \approx C_{ab} + \mathrm{Ad}_{F_{\text{nom},ab}}\, C_{bc}\, \mathrm{Ad}^{\top}_{F_{\text{nom},ab}}.$$

2.8 Path Composition in a Geometric Network

The system is represented as a directed graph whose nodes correspond to coordinate frames or points and whose edges correspond to uncertain geometric relationships.

Given a path

$$v_0 \to v_1 \to \cdots \to v_N,$$

the nominal transformation is

$$F_{\text{nom},st} = \prod_{k=0}^{N-1} F_{\text{nom},v_k v_{k+1}}.$$

Define the prefix transformation

$$F_{\text{nom},0k} = \prod_{i=0}^{k-1} F_{\text{nom},v_i v_{i+1}}.$$

The composed perturbation is approximated by

$$\vec{\eta}_{st} \approx \sum_{k=0}^{N-1} \mathrm{Ad}_{F_{\mathrm{nom},0k}}\, \vec{\eta}_{v_k v_{k+1}},$$

and the corresponding covariance is

$$C_{st} \approx \sum_{k=0}^{N-1} \mathrm{Ad}_{F_{\mathrm{nom},0k}}\, C_{v_k v_{k+1}}\, \mathrm{Ad}^{\top}_{F_{\mathrm{nom},0k}}.$$

3 Closed-Loop Constraints and Residual Uncertainty Estimation

3.1 Residual Transformation Model

Let a closed loop induce a constraint of the form

$$T_{\mathrm{res}} \;\approx\; T_k, \tag{1}$$

where

$$T_{\mathrm{res}} = \exp(\vec{\eta}_{\mathrm{res}})\, F_{\mathrm{res}}, \tag{2}$$

$$T_k = \exp(\vec{\eta}_k)\, F_k. \tag{3}$$

Here, F_{res} denotes an unknown or nominal residual transformation, and F_k is a known transformation obtained by composing a sequence of uncertain geometric relationships along a closed loop. The perturbations $\vec{\eta}_{\mathrm{res}}, \vec{\eta}_k \in \mathbb{R}^6$ are modeled as zero-mean Gaussian random variables with covariances C_{res} and C_k, respectively.

Define the loop residual in the Lie algebra as

$$\vec{r} \;\triangleq\; \log\!\left(F_{\mathrm{res}}^{-1} T_k\right). \tag{4}$$

3.2 First-Order Linearization

Assuming small perturbations and linearizing about $\vec{\eta}_{\mathrm{res}} = \vec{\eta}_k = 0$, the residual may be approximated to first order as

$$\vec{r} \;\approx\; A\,\vec{\eta}_k \;-\; \vec{\eta}_{\mathrm{res}} \;+\; \vec{b}, \tag{5}$$

where

- $A \in \mathbb{R}^{6\times 6}$ is an adjoint-based mapping determined by the nominal transformations (e.g., $A = \mathrm{Ad}_{F_{\mathrm{res}}^{-1} F_k}$),
- $\vec{b} = \log(F_{\mathrm{res}}^{-1} F_k)$ represents the nominal loop mismatch.

If the nominal geometry satisfies $F_{\mathrm{res}} = F_k$, then $\vec{b} = \vec{0}$.

The loop constraint is enforced by requiring $\vec{r} \approx \vec{0}$.

3.3 Gaussian Conditioning and Posterior Covariance

Define the stacked perturbation vector

$$\vec{x} = \begin{bmatrix} \vec{\eta}_{\mathrm{res}} \\ \vec{\eta}_k \end{bmatrix}, \qquad \vec{x} \sim \mathcal{N}(\vec{0}, C_0), \tag{6}$$

with prior covariance

$$C_0 = \begin{bmatrix} C_{\mathrm{res}} & 0 \\ 0 & C_k \end{bmatrix}. \tag{7}$$

The linearized loop constraint may be written in standard linear-Gaussian form as

$$\vec{0} = H\,\vec{x} + \vec{\nu}, \tag{8}$$

where

$$H = \begin{bmatrix} -I & A \end{bmatrix}, \qquad \vec{\nu} \sim \mathcal{N}(\vec{0}, C_\nu), \tag{9}$$

and C_ν represents optional modeling or tolerance uncertainty associated with the constraint.

Conditioning on the loop constraint yields the posterior covariance

$$C_{\mathrm{post}}^{-1} = C_0^{-1} + H^\top C_\nu^{-1} H. \tag{10}$$

The posterior covariance of the residual perturbation is obtained as the corresponding block:

$$\mathrm{cov}(\vec{\eta}_{\mathrm{res}}) = (C_{\mathrm{post}})_{\mathrm{res,res}}\,. \tag{11}$$

This formulation naturally generalizes to multiple loop constraints by summing the corresponding information contributions.

3.4 Rotation-Only and Translation-Only Constraints

In some applications, loop constraints may involve only rotational or translational components. Let

$$\vec{\eta} = \begin{bmatrix} \vec{\alpha} \\ \vec{\epsilon} \end{bmatrix}, \tag{12}$$

where $\vec{\alpha} \in \mathbb{R}^3$ and $\vec{\epsilon} \in \mathbb{R}^3$ denote rotational and translational perturbations, respectively.

Rotation-only constraints are enforced by applying the selection matrix

$$S_\alpha = \begin{bmatrix} I_3 & 0 \end{bmatrix}, \tag{13}$$

leading to the reduced constraint

$$\vec{0} = S_\alpha \vec{r} = S_\alpha (A\,\vec{\eta}_k - \vec{\eta}_{\mathrm{res}}) + \vec{\nu}_\alpha. \tag{14}$$

An analogous construction using $S_\epsilon = \begin{bmatrix} 0 & I_3 \end{bmatrix}$ applies for translation-only constraints.

The corresponding posterior covariances are obtained by conditioning with the reduced measurement matrix.

3.5 Monte Carlo Validation

To assess the validity of the first-order approximation, Monte Carlo simulation is employed. Perturbations are sampled from $\mathcal{N}(0, C)$, exact transformations are constructed using the exponential map, and compositions are performed exactly along the chosen path.

Monte Carlo samples are mapped back to perturbation coordinates relative to $F_{\text{nom},st}$ using the logarithmic map, and the resulting sample covariance is compared to the analytical result.

4 Notation Cheat Sheet

This section provides a concise summary of the notation used throughout this document for rigid-body geometry and uncertainty propagation. All notation follows the conventions used in CIS I and related course materials.

4.1 Rigid-Body Geometry

- $SE(3)$: Space of rigid-body transformations (rotation and translation).
- $F \in SE(3)$: Rigid-body transformation between two coordinate frames.
- F_{nom}: Nominal (deterministic) rigid-body transformation.
- $\circ$: Composition operator for rigid-body transformations.

4.2 Pose Perturbation Representation

- $\vec{\eta} \in \mathbb{R}^6$: Six-degree-of-freedom pose perturbation vector.

  The perturbation vector is decomposed as

$$\vec{\eta} = \begin{bmatrix} \vec{\alpha} \\ \vec{\epsilon} \end{bmatrix},$$

where

- $\vec{\alpha} \in \mathbb{R}^3$: rotational error,
- $\vec{\epsilon} \in \mathbb{R}^3$: translational error.

  Perturbations are assumed to be small, enabling first-order approximations.

4.3 Uncertainty and Probability

- $\mathcal{N}(\mu, C)$: Multivariate Gaussian distribution with mean μ and covariance C.
- $\vec{\eta} \sim \mathcal{N}(0, C)$: Gaussian model of pose uncertainty.
- $C \in \mathbb{R}^{6\times 6}$: Pose covariance matrix.

  The pose covariance is structured as

$$C_{\eta\eta} = \begin{bmatrix} C_{\alpha\alpha} & C_{\alpha\epsilon} \\ C_{\epsilon\alpha} & C_{\epsilon\epsilon} \end{bmatrix},$$

where diagonal blocks represent rotational and translational uncertainty, and off-diagonal blocks represent correlation between them.

4.4 Point Uncertainty

- $p_{\text{nom}} \in \mathbb{R}^3$: Nominal point location.
- $\delta p \in \mathbb{R}^3$: Point perturbation.

Point uncertainty is modeled as

$$p = p_{\text{nom}} + \delta p, \qquad \delta p \sim \mathcal{N}(0, C_p),$$

where $C_p \in \mathbb{R}^{3\times 3}$ is the point covariance matrix.

When a point is transformed under an uncertain rigid-body transformation, the propagated point covariance is approximated by

$$C_{p'} \approx J_{\vec{\eta}}\, C_{\eta\eta}\, J_{\vec{\eta}}^{\top} + J_p\, C_p\, J_p^{\top}.$$

4.5 Jacobians

- $J_{\vec{\eta}}$: Jacobian of the point transformation with respect to pose perturbation $\vec{\eta}$.
- J_p: Jacobian of the point transformation with respect to point coordinates.

4.6 Perturbation Convention

A left-multiplicative perturbation model is used throughout this work:

$$T = \exp(\vec{\eta}) \circ F_{\text{nom}}.$$

Under this convention, perturbations are expressed in the parent (global) coordinate frame, and uncertainty propagation is performed consistently in this frame.

4.7 Adjoint Operator

- $\mathrm{Ad}_F \in \mathbb{R}^{6\times 6}$: Adjoint operator associated with transformation $F \in SE(3)$.

The adjoint operator maps pose perturbations between coordinate frames during transformation composition.

4.8 Composition of Uncertain Transformations

Given two uncertain transformations with nominal components $F_{\text{nom},ab}$ and $F_{\text{nom},bc}$, the composed perturbation is approximated as

$$\vec{\eta}_{ac} \approx \vec{\eta}_{ab} + \mathrm{Ad}_{F_{\text{nom},ab}}\, \vec{\eta}_{bc}.$$

Assuming independence, the corresponding covariance propagates as

$$C_{ac} \approx C_{ab} + \mathrm{Ad}_{F_{\text{nom},ab}}\, C_{bc}\, \mathrm{Ad}_{F_{\text{nom},ab}}^{\top}.$$

4.9 Geometric Network Notation

- v_i: Node in a geometric network (coordinate frame or point).

For a path

$$v_0 \to v_1 \to \cdots \to v_N,$$

the prefix transformation is defined as

$$F_{\mathrm{nom},0k} = \prod_{i=0}^{k-1} F_{\mathrm{nom},v_i v_{i+1}},$$

and the propagated covariance along the path is

$$C_{st} \approx \sum_{k=0}^{N-1} \mathrm{Ad}_{F_{\mathrm{nom},0k}} \, C_{v_k v_{k+1}} \, \mathrm{Ad}_{F_{\mathrm{nom},0k}}^{\top}.$$

4.10 Monte Carlo Validation

Monte Carlo simulation is used to validate the first-order approximation. Perturbations are sampled from $\mathcal{N}(0, C)$, exact transformations are constructed using the exponential map, and the resulting sample covariance is compared with the analytical covariance.

5 Note

5.1 Distribution Assumptions and Extensibility

In this work, pose uncertainty is initially modeled using multivariate Gaussian distributions, as this choice supports compact representation, first-order analytical propagation, and efficient Monte Carlo validation.

While the current formulation assumes Gaussian uncertainty, the overall framework is designed to be distribution-agnostic at the architectural level. In particular, uncertainty representations are treated abstractly, allowing alternative distribution models to be incorporated in the future without altering the underlying geometric composition structure.

This design choice supports potential extensions to non-Gaussian uncertainty models, such as bounded uncertainty, heavy-tailed distributions, or sample-based representations, while preserving the same geometric network formulation.

5.2 Closed-loop constraints

Closed-loop constraints introduce statistical dependence among previously independent geometric relationships. The conditioning framework described here provides a mathematically consistent method for enforcing loop consistency while preserving the Lie-group structure of rigid-body transformations. Forward uncertainty propagation and closed-loop inference are thus unified within a single perturbation-based framework.