SIMULATION-BASED UNCERTAINTY PROPAGATION IN GEOMETRIC NETWORKS FOR SURGICAL ROBOTICS

Project Plan Report

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1 Motivation

Surgical robotic systems rely on multiple interacting geometric components, including robot kinematic chains, tracking systems, surgical tools, sensors, and anatomical models. Each component introduces uncertainty due to calibration error, measurement noise, modeling assumptions, and manufacturing tolerances. These uncertainties interact through geometric composition: transformations are chained, measurements are fused, and points of interest must be expressed consistently across multiple coordinate frames.

Core problem: There is no unified, general framework to model and query uncertainty propagation across an arbitrary geometric network. This project builds a principled simulation framework that connects geometry and probability, enabling end-to-end uncertainty queries in complex robotic systems.

2 System Abstraction: Network View

The system is modeled as a directed geometric network:

- **Nodes** represent geometric entities, including coordinate frames (robot base, links, trackers, cameras) and 3-DOF points (tool tips, fiducials, anatomical landmarks), also there are lots of other nodes might be included, optical tracker, EM tracker, ultrasound, camera, etc.
- **Edges** represent uncertain geometric relationships such as rigid-body transformations, kinematic relationships, and sensor measurements.

Under this abstraction, the central computational task is:

> *Given any two nodes in the network, compute the nominal geometric relationship between them and the associated propagated uncertainty.*

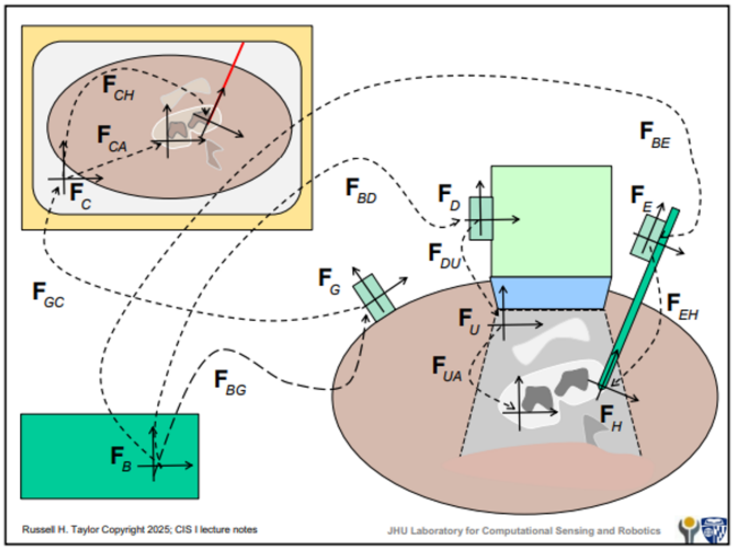

Figure 1: Example of a geometric transformation network in a computer-integrated surgery system (got from CIS I course materials). Frames represent imaging, tracking, robot, tool, and anatomy coordinate systems. Edges F_{XY} denote rigid-body transformations between frames. Multiple paths and closed loops arise naturally in such systems.

3 Project Goal and Significance

3.1 Goal

The goal of this project is to develop a general simulation framework that models and propagates uncertainty through a network of geometric relationships, enabling uncertainty queries between arbitrary nodes.

3.2 Significance

Modern surgical robotics systems rely on complex networks of geometric transformations linking imaging, tracking, robot kinematics, tools, and anatomy. Each component introduces uncertainty, yet there is currently no unified framework for systematically propagating and analyzing uncertainty across an entire geometric system.

This project develops a mathematically rigorous uncertainty propagation framework for interconnected rigid-body networks. By modeling 6-DOF pose uncertainty in $SE(3)$ and supporting both open-chain and closed-loop structures, the framework enables quantitative estimation of how local uncertainty affects task-level quantities such as tool tip position, relative distances, and anatomical localization.

Beyond theoretical analysis, this work will be developed into a **design tool** for robotic system evaluation. The software will allow users to define geometric networks, assign uncertainty models, and compute propagated covariance to compare system architectures before physical implementation. This enables sensitivity analysis, calibration evaluation, and early-stage risk assessment.

Additionally, the framework will serve as a **teaching tool** for future Computer-Integrated

Surgery (CIS) courses. It will provide a structured computational environment where students can visualize uncertainty flow through geometric systems and connect multivariate Gaussian theory with real robotic applications. As such, the project contributes not only to research methodology but also to long-term educational infrastructure in surgical robotics.

3.3 Deliverables

The project deliverables are structured in three tiers: minimum, expected, and maximum outcomes.

Minimum

- Complete mathematical formulation for uncertainty propagation in rigid-body geometric networks.
- Core data structures:
  - Uncertain Frame (pose uncertainty in $SE(3)$)
  - Uncertain Point
- Basic subroutines for composition and covariance propagation.
- Support for open-chain network structures.
- Simple analytical test cases demonstrating correctness.

Expected

- Fully functional network system for uncertainty propagation.
- Support for querying uncertainty between arbitrary nodes.
- Command-line or script-based network specification.
- Complete documentation and illustrative examples.

Maximum

- Monte Carlo simulation framework for validation and comparison.
- AMBF-based visualization for geometric uncertainty behavior.
- Graphical user interface (GUI) for interactive network specification and uncertainty querying.

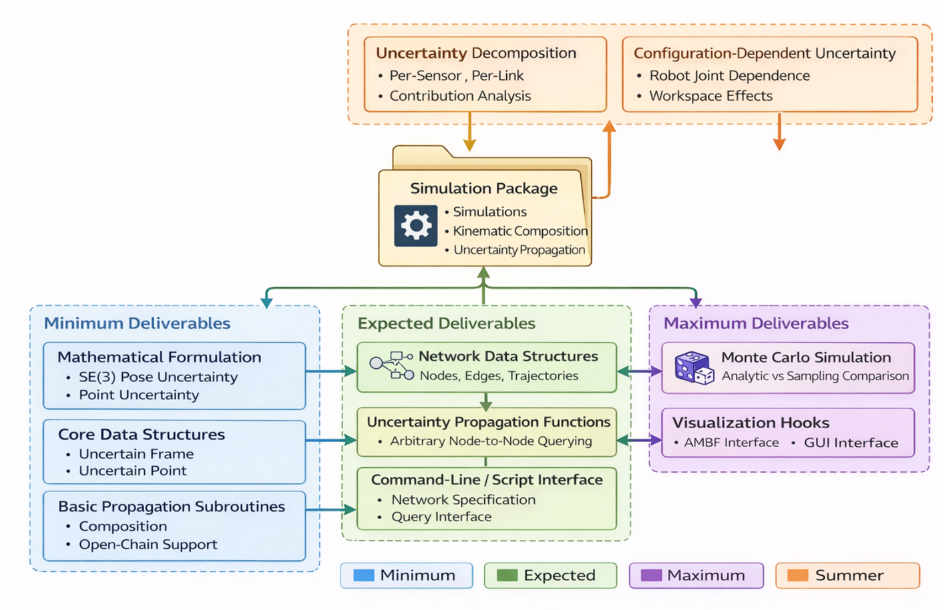

Figure 2: Plan and deliverables of this project.

4 Technical Approach and Development Plan

The technical approach consists of three major layers:

- **Uncertainty-aware geometric primitives** for frames and points.
- **Composable operators** for transformation, composition, and uncertainty propagation.
- **Network-level querying** to compute uncertainty between arbitrary nodes.

Development will proceed in progressive phases:

- **Phase 1: Mathematical & software foundations** (core representations and operators).
- **Phase 2: Network & simulation** (graph structure, path composition, query interface).
- **Phase 3: AMBF & visualization** (workflow demonstration and visualization hooks).

5 Mathematical Foundations

This project focuses on *uncertainty propagation*. It does not perform estimation, optimization, or inference.

5.1 Nominal Geometry

Geometric relationships between coordinate frames are represented by rigid-body transformations in $SE(3)$. Each relationship is associated with a nominal transformation

$$F_{\text{nom}} \in SE(3),$$

which represents the best deterministic estimate from kinematics, calibration, registration, or known CAD/CT models.

5.2 Pose Uncertainty Representation

Because rigid-body transformations do not form a vector space, uncertainty is represented as a small perturbation in the Lie algebra:

$$\vec{\eta} \in \mathbb{R}^6, \qquad \vec{\eta} \sim \mathcal{N}(0, C).$$

We use the decomposition

$$\vec{\eta} = \begin{bmatrix} \vec{\alpha} \\ \vec{\epsilon} \end{bmatrix},$$

where $\vec{\alpha} \in \mathbb{R}^3$ is rotational error and $\vec{\epsilon} \in \mathbb{R}^3$ is translational error.

Principle 5.1. Perturbation convention (left-multiplicative). The true transformation is modeled as

$$T = \exp(\vec{\eta}) \circ F_{\text{nom}}.$$

Under this convention, perturbations are expressed in the parent (global) coordinate frame, and first-order propagation admits a consistent interpretation across compositions.

5.3 Composition of Uncertain Transformations

Let

$$F_{ab} = \{F_{\text{nom},ab}, C_{ab}\}, \qquad F_{bc} = \{F_{\text{nom},bc}, C_{bc}\}.$$

The nominal composed transform is

$$F_{\text{nom},ac} = F_{\text{nom},ab} \circ F_{\text{nom},bc}.$$

Proposition 5.2. First-order covariance propagation under composition. Assuming independence and a left-multiplicative perturbation model,

$$C_{ac} \approx C_{ab} + \text{Ad}_{F_{\text{nom},ab}}\, C_{bc}\, \text{Ad}_{F_{\text{nom},ab}}^{\top},$$

where Ad_F denotes the adjoint operator associated with $F \in SE(3)$.

5.4 Uncertain 3-DOF Points

A 3-DOF point is represented as

$$p = \{p_{\text{nom}}, C_p\}, \qquad p_{\text{nom}} \in \mathbb{R}^3, \quad C_p \in \mathbb{R}^{3\times 3}.$$

Proposition 5.3. First-order point covariance propagation. When a point is transformed under an uncertain pose, the propagated point covariance is approximated by

$$C_{p'} \approx J_{\vec{\eta}}\, C_{\eta\eta}\, J_{\vec{\eta}}^{\top} + J_p\, C_p\, J_p^{\top},$$

capturing contributions from both pose uncertainty and intrinsic point uncertainty.

5.5 Network-Level Propagation

For a path in the network

$$v_0 \to v_1 \to \cdots \to v_N,$$

uncertainty is propagated by accumulating adjoint-weighted contributions along the path, enabling uncertainty queries between arbitrary nodes. Monte Carlo simulation will be used to validate the first-order analytical result.

6 Implementation Plan

6.1 Programming Language and Design Philosophy

The framework will be implemented in **Python**, prioritizing clarity, modularity, and rapid prototyping. The implementation will follow an object-oriented design to support extensibility, testing, and clean interfaces.

6.2 Core Classes and Data Structures

The planned class structure includes:

- `UncertainTransform`: stores (F_{nom}, C) with $C \in \mathbb{R}^{6\times 6}$ and implements composition and adjoint-based propagation.
- `UncertainPoint`: stores (p_{nom}, C_p) with $C_p \in \mathbb{R}^{3\times 3}$ and supports transformation under uncertain poses.
- `GeometricNode`: abstract base class for frame-nodes and point-nodes.
- `GeometricEdge`: represents uncertain geometric relationships (edges) with nominal geometry and covariance.
- `GeometricNetwork`: graph container supporting path discovery and node-to-node uncertainty queries.
- May have more, will add more during our implementation.

6.3 Propagation and Validation Modules

The codebase will include:

- **Analytical propagation** using first-order approximations (adjoint/Jacobian-based).
- **Monte Carlo validation** by sampling perturbations, composing exact transforms via exponential maps, and comparing sample covariances to analytical results.

6.4 Interfaces and Visualization

A script-based interface will support:

- network specification (nodes, edges, covariances),
- uncertainty query definition (source node, target node),
- exporting results in reproducible formats.

Visualization hooks will be developed for AMBF-based demonstrations and interactive inspection.

7 Timeline and Milestones

The project is structured into progressive phases:

- **Weeks 1–2:** Establish uncertainty modeling, define covariance representations, validate with simple examples.
- **Weeks 3–4:** Implement uncertainty-aware geometric primitives; validate propagation for kinematic chains.
- **Weeks 5–6:** Support general geometric networks; enable node-to-node uncertainty queries.
- **Weeks 7–8:** Validate propagation; implement Monte Carlo sampling.
- **Weeks 9–10:** Develop a usable interface (script/CLI); validate user workflow on representative examples.
- **Weeks 11–12:** Integrate with AMBF for visualization; finalize documentation.

8 Management and Data Practices

Regular standing meetings will be used to review progress, discuss modeling assumptions, and adjust scope as needed. Every Friday afternoon 1:30-2:30 with Dr. Taylor at his office. Project artifacts (code, configuration files, simulated data, and experiment results) will be maintained in a structured GitHub repository under version control. Intermediate outputs (e.g., Monte Carlo samples and propagated covariances) will be stored in reproducible formats for analysis and validation.

9 Expected Outcomes

By the end of this project, the expected outcomes are:

- a reusable simulation framework for uncertainty propagation in geometric networks;
- validated analytical and Monte Carlo uncertainty propagation pipelines;
- a script-based querying workflow and optional visualization hooks;
- a foundation suitable for future extensions and research publication.