Force Distribution Model

1. Overview of the Force Sensing Tool
   The force sensing tool contains three FBG sensors, one near the inner membrane of the eyeball and the other two outside the eyeball. The placement of the sensors are fixed as shown in Figure 2. And the FBG sensors are approximated as force/torque sensors limited in two directions (x and y). We denote the measurement from the sensor $\Delta S_j = [\Delta s_{j1}, \Delta s_{j2}, \Delta s_{j3}]^T$, where $j = I, II, III$.

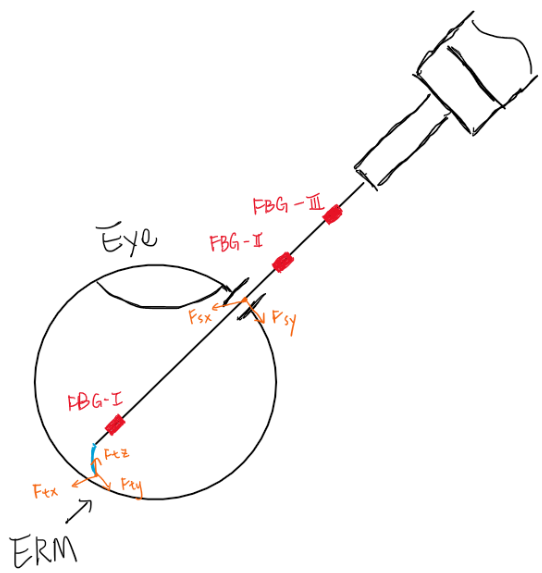

Figure 1: FBG Sensors and Forces at ERM and Sclerotomy

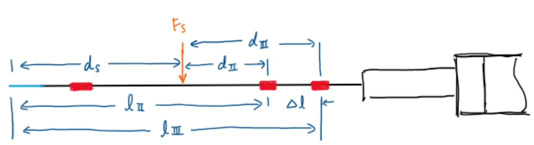

Figure 2: Positions of the FBG Sensors and Forces

2. Force on the Tip

   The force induced on FBG I is proportional to the sensor readings:

$$\Delta S_I = K_I F_I, \; F_I = [F_{Ix}, F_{Iy}]^T$$

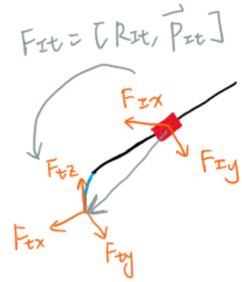

Figure 3: Force on Tip

From the kinematics model, we can obtain the transformation matrix from FBG I to snake tip $F_{It} = [R_{It}, p_{It}]$. From FEA analysis, the force at the tip F_t should equal to F_I. Using the information above, we can obtain the following equations:

$$\vec{F_t^I} \cdot \vec{x_I} = (R_{It} \cdot \vec{F_t}) \cdot \vec{x_I} = F_{Ix} \tag{1}$$

$$\vec{F_t^I} \cdot \vec{y_I} = (R_{It} \cdot \vec{F_t}) \cdot \vec{y_I} = F_{Iy} \tag{2}$$

These two equations could be then expanded into:

$$F_{tx} \cdot R_{It,11} + F_{ty} \cdot R_{It,12} + F_{tz} \cdot R_{It,13} = F_{Ix} \tag{3}$$

$$F_{tx} \cdot R_{It,21} + F_{ty} \cdot R_{It,22} + F_{tz} \cdot R_{It,23} = F_{Iy} \tag{4}$$

Note that $R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$.
To simplify the problem, we can assume that the twist of the tool shaft is minimal, which is equivalent to saying that the torque around z axis of the tool is close to 0. This condition can be put into the following equation:

$$(\vec{p}_{It} \times \vec{F_t^I})_3 = (\vec{p}_{It} \times (R_{It} \cdot \vec{F_t}))_3 = 0 \tag{5}$$

$\Rightarrow$

$$p_{It,1} \cdot (R_{It} \cdot \vec{F_t})_2 - p_{It,2} \cdot (R_{It} \cdot \vec{F_t})_1 = 0 \tag{6}$$

Combining equations (3),(4),(6), we get a system of three equations with three unknowns so that we can solve for forces at the tip of the snake.

3. Force at the Sclerotomy
The torque induced at the sensor FBG II and III should be proportional to the sensor readings:

$$\Delta S_j = K_j \tau_j, \ \tau_j = [\tau_{jx}, \tau_{jy}]^T, \ j = II, III$$

Both forces at the tip and near the sclerotomy contribute to the torque at the sensors II and III. So we have the following equations:

$$\vec{\tau}_j = \vec{\tau}_t^j + \vec{\tau}_s^j, \ j = II, III \tag{7}$$

Similarly to the precious section, we can find the transformation matrix from the FBG sensors to tip, $F_{IIt} = [R_{IIt}, p_{IIt}]$ and $F_{IIIt} = [R_{IIIt}, p_{IIIt}]$. We could then find:

$$\vec{\tau}_t^j = \vec{p}_{jt} \times (R_{jt} \cdot \vec{F_t}) \tag{8}$$

From equation 7 and 8, we know $\vec{\tau}_s^j$. And we can find the forces at the sclerotomy from the equations below.

$$F_{sy} = \frac{\tau_{s,1}^{III} - \tau_{s,1}^{II}}{\Delta l} \tag{9}$$

$$F_{sx} = \frac{\tau_{s,2}^{II} - \tau_{s,2}^{III}}{\Delta l} \tag{10}$$

We can then find the distance from the sclerotomy to the sensors by the equation below. Note that τ_s^j ans F_s both have two elements.

$$d_j = \frac{||\tau_s^j||}{||F_s||} \tag{11}$$

References