Homework Assignment 5 – 601.455/655 Fall 2022 (Circle One)

Instructions and Score Sheet (hand in with answers)

Name	Name
Email	Email
Other contact information (optional)	Other contact information (optional)
Signature (required) I/We have followed the rules in completing this assignment ______________________________	Signature (required) I/We have followed the rules in completing this assignment ______________________________

Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.

1. You are to work **alone** or **in teams of two** and are **not to discuss the problems with anyone** other than the TAs or the instructor.
2. It is otherwise open book, notes, and web. But you should cite any references you consult.
3. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
4. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
5. Sign and hand in the score sheet as the first sheet of your assignment.

NOTE: This assignment has a total of 120 points. However, at most 100 points will count toward your course grade.

Scenario

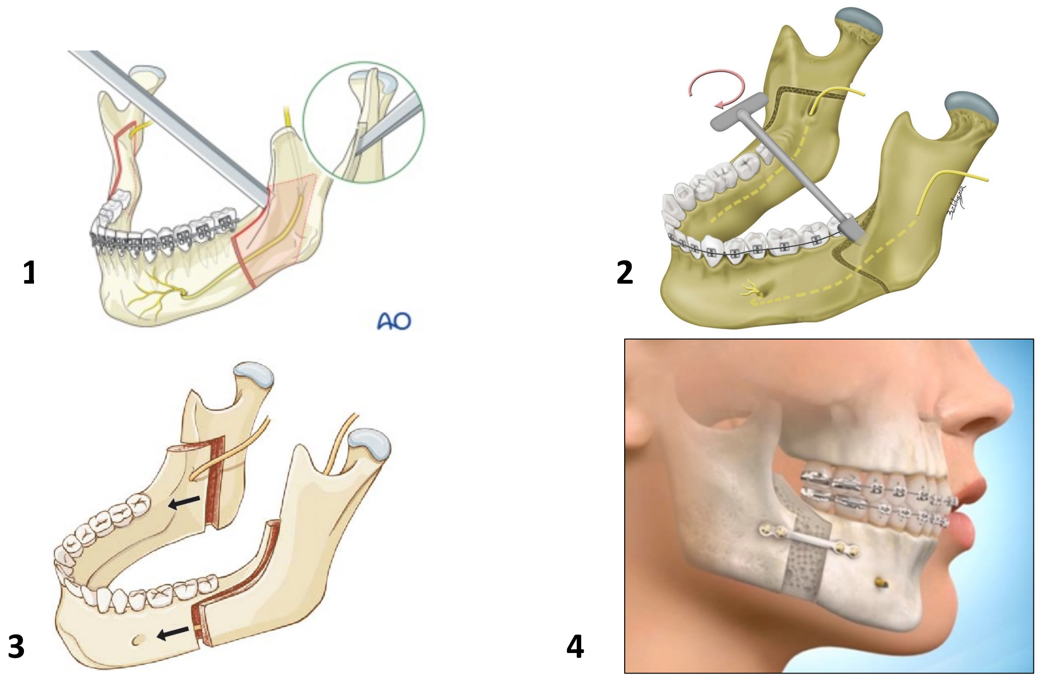


Figure 1: Mandibular Osteotomy

Consider the mandibular osteotomy procedure illustrated in Figure 1. Planar cuts are made in the mandible (i.e., the lower jaw) to split it into a front and back section. The back section is then translated forward and resecured to the back section by screws and metal plates. This procedure requires high precision and great care in order not to damage delicate nerves within the jaw. The cuts need to be planar and parallel in order to facilitate sliding the jaw forward.

For this assignment, we will assume that a CT-based presurgical plan has been made. The planning information for one of the cuts is shown in Figure 2. The planning system has defined a mandibular coordinate system $\mathbf{F}_{mandible}$ associated with the mandible. The cut is to be made in the XY plane of a cut coordinate system $\mathbf{F}_{mc}$ defined relative to $\mathbf{F}_{mandible}$. The cutting tool will be a drill-like rotary cutter, in which both the side of the cutter will be used to make the cut. The planning system has specified a path $\vec{\mathbf{p}}_{cut}(\vec{\zeta},\lambda)$ in XY plane of the cut coordinate system that the tip of the cutter is to follow, where $\vec{\zeta}$ are parameters describing the path and λ represents a displacement along the path. The cutter has a cylindrical shape with a radius r_{drill} and a length L_{drill}.

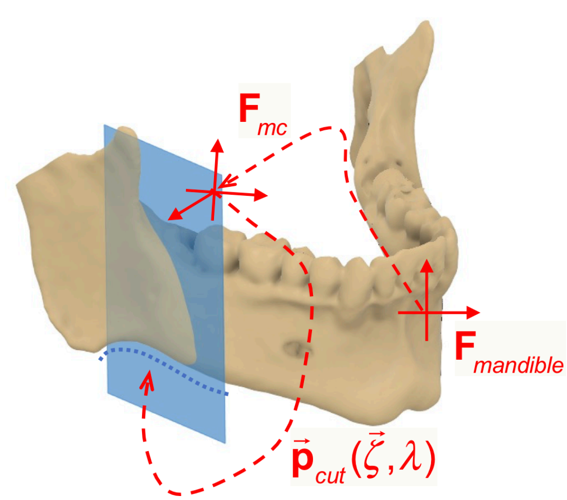


Figure 2: Planned Cut

During the cut, the direction $\vec{\mathbf{d}}_c$ of the cutter shaft must stay within a specified angle $\theta_{\max}$ of the desired shaft direction $\vec{\mathbf{d}}_{des}$. (Here, $\vec{\mathbf{d}}_c$ and $\vec{\mathbf{d}}_{des}$ are unit vectors). For safety reasons, all parts of the cutter must stay within a defined distance ρ_Z of the XY plane of $\mathbf{F}_{mc}$. Also, the tip of the cutter must stay within a projected distance ρ_{XY} of the nominal path $\vec{\mathbf{p}}_{cut}(\vec{\zeta},\lambda)$. I.e., the projected position of the tip onto the XY plane of $\mathbf{F}_{mc}$ must be within ρ_{XY} of the nominal path $\vec{\mathbf{p}}_{cut}(\vec{\zeta},\lambda)$.

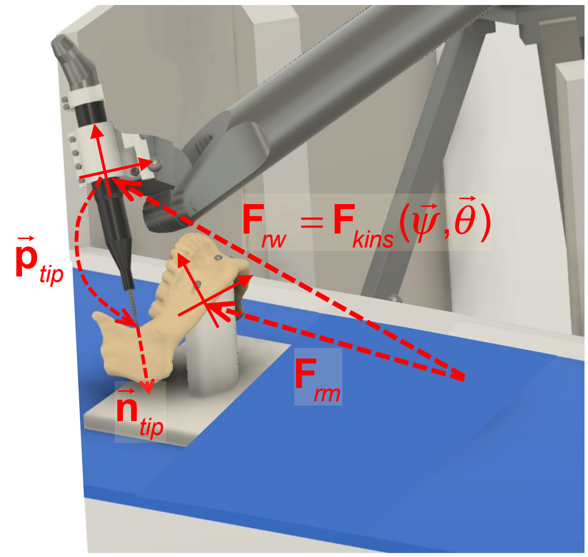


Figure 3: Robotic Cutting

We wish to use the hand-over-hand surgical robot shown in Figure 3 to assist in performing the cuts. Here, the position of the robot's tool holder relative to its base is $\mathbf{F}_{rw}(\vec{\theta}) = \mathbf{F}_{kins}(\vec{\psi},\vec{\theta})$, where $\vec{\psi}$ are structural parameters associated with the robot design, and $\vec{\theta}$ are the positions of the joints of the robot. The robot controller is also able to compute the Jacobean $\mathbf{J}_{kins}(\vec{\theta}) = \begin{bmatrix} \mathbf{J}_{\vec{\alpha}}(\vec{\theta}) \\ \mathbf{J}_{\vec{\varepsilon}}(\vec{\theta}) \end{bmatrix}$ such that

$$\mathbf{F}_{rw}(\vec{\theta}+\Delta\vec{\theta}) \approx \left[\mathbf{I}+sk(\vec{\alpha}),\vec{\varepsilon}\right]\bullet\mathbf{F}_{rw}(\vec{\theta})$$

where

$$\begin{bmatrix} \vec{\alpha}_{rw} \\ \vec{\varepsilon}_{rw} \end{bmatrix} = \vec{\eta}_{rw} = \mathbf{J}_{kins}(\vec{\theta})\Delta\vec{\theta}$$

For convenience, we will adopt the notation $\Delta\mathbf{F}(\vec{\eta}) = \Delta\mathbf{F}(\vec{\alpha},\vec{\varepsilon}) = [\Delta\mathbf{R}(\vec{\alpha}),\vec{\varepsilon}] \approx [\mathbf{I}+sk(\vec{\alpha}),\vec{\varepsilon}]$ where $\vec{\eta} = \begin{bmatrix} \vec{\alpha} \\ \vec{\varepsilon} \end{bmatrix}$.

The position of the tip of the cutter is located at $\vec{\mathbf{p}}_{tip}$ relative to the tool holder, i.e., $\mathbf{F}_{rw}\bullet\vec{\mathbf{p}}_{tip}$ relative to the robot. Similarly, the direction of the cutter shaft relative to the tool holder is $\vec{\mathbf{n}}_{tip}$. As shown in the figure, $\vec{\mathbf{n}}_{tip}$ points "away" from the drill body, i.e., in the direction from the drill body toward the tip.

A registration process has been performed, so that the transformation $\mathbf{F}_{rm}$ between robot and mandible coordinates is known.

Question 1

A. **(5 points)** Give expressions for the position $\vec{\mathbf{p}}_{rt}(\vec{\theta})$ of the drill tip and the direction $\vec{\mathbf{n}}_{rt}(\vec{\theta})$ of the cutter shaft relative to robot coordinates as a function of $\vec{\theta}$.

B. **(5 points)** Give expressions for the position of the $\vec{\mathbf{p}}_{ct}(\vec{\theta})$ drill tip and the direction $\vec{\mathbf{n}}_{ct}(\vec{\theta})$ of the cutter shaft relative to the cut coordinate system as a function of $\vec{\theta}$.

C. **(10 points)** Write linear inequalities that express the constraint that all parts of the cutter must stay within a defined distance ρ_Z of the XY plane of $\mathbf{F}_{mc}$ when the robot joints are at $\vec{\theta}$. **Hint:** Note that the cutter lies between the points $\vec{\mathbf{p}}_{tip}$ and $\vec{\mathbf{p}}_{tip} - L_{cutter}\vec{\mathbf{n}}_{tip}$ relative to the tool holder. **Bonus possibility (worth 5 points):** As stated, the problem simply asks for constraints on $\vec{\theta}$. For an extra 5 points, produce constraints by a mid-level controller optimizer to compute constraints on the incremental motion $\Delta\vec{\theta}$.

D. **(10 points)** Suppose that $\vec{\mathbf{p}}_{cut}(\lambda)$ is a straight line in the XY plane of $\mathbf{F}_{mc}$ with the form $\vec{\mathbf{p}}_{cut}(\lambda) = \vec{\mathbf{a}}_{cut} + \lambda\vec{\mathbf{d}}_{cut}$, where $\left\|\vec{\mathbf{d}}\right\| = 1$. Write linear inequalities that express the constraint that the tip of the cutter must stay within a projected distance ρ_{XY} of the nominal path $\vec{\mathbf{p}}_{cut}(\vec{\zeta},\lambda)$. **Clarification:** Here we are looking for constraints on $\Delta\vec{\theta}$ that can be used in a mid-level controller optimizer, together with a procedure for computing any necessary intermediate variables. **Hint:** The first step is to find the closest point $\vec{\mathbf{c}}$ on the line to the current position $\vec{\mathbf{p}}_{ct}(\vec{\theta})$ and then generating constraints on the incremental motion $\Delta\vec{\theta}$.

Question 2

The basic mid-level control loop of the robot runs as a sample interval of ΔT seconds (e.g., $\Delta T = 0.005$ seconds). The joints of the robot are subject to positional, velocity, and acceleration limits:

$$-\vec{\theta}_{\min} \le \vec{\theta} \le \vec{\theta}_{\max}$$
$$-\dot{\vec{\theta}}_{\min} \le \dot{\vec{\theta}} \le \dot{\vec{\theta}}_{\max}$$
$$-\ddot{\vec{\theta}}_{\min} \le \ddot{\vec{\theta}} \le \ddot{\vec{\theta}}_{\max}$$

The robot is equipped with a force sensor that enables the robot to sense forces exerted by the surgeon's hand on the drill. For the purposes of this exercise we will assume that the forces exerted by the drill on the mandible can be ignored, so long as the cutter spins at a very fast speed and the translational motion of all parts of the drill cutter are slow enough. The robot will use constrained resolved rate control. The basic loop is as follows:

Step 1. Read the robot joint positions $\vec{\theta}$ and velocities $\dot{\vec{\theta}}$, along with the forces and torques exerted on the drill by the surgeon. Resolve these into a force/torque vector $\vec{\varphi}$.

Step 2. Perform basic safety checks.

Step 3. Solve the constrained optimization problem

$$\underset{\Delta\vec{\theta}}{\arg\min} \left\| \vec{\eta}_{rhs} - \mathbf{C}\vec{\varphi} \right\|^2$$
$$\vec{\eta}_{rw} = \mathbf{J}(\vec{\theta})\Delta\vec{\theta}$$
$$\vec{\eta}_{rhs} = \mathbf{M}(\mathbf{F}(\vec{\theta}))\vec{\eta}_{rw}$$
... other constraints here on $\vec{\eta}$ and $\Delta\vec{\theta}$

Step 4. Output joint velocity commands $\dot{\vec{\theta}}_{cmd} = \Delta\vec{\theta} / \Delta T$. Then sleep until the next time interval

The next questions concern how to write the "other constraints" to obtain our desired behavior.

A. **(10 points)** Write an expression for the matrix $\mathbf{M}(\mathbf{F}(\vec{\theta}))$ such that $\mathbf{F}(\vec{\theta})\Delta\mathbf{F}(\vec{\eta}_{rhs}) = \Delta\mathbf{F}(\vec{\eta}_{rw})\mathbf{F}(\vec{\theta})$, where $\vec{\eta}_{rhs} = \mathbf{M}(\mathbf{F}(\vec{\theta}))\vec{\eta}_{rw}$.

B. **(10 points)** Write constraints on $\Delta\vec{\theta}$ that enforce the joint position, velocity and acceleration limits. **Hint:** These may also involve $\vec{\theta}$.

C. **(10 points)** Write linear constraints that enforce the requirement that all parts of the cutter must stay within a defined distance ρ_Z of the XY plane of $\mathbf{F}_{mc}$ when the robot joints move to a new position $\vec{\theta} + \Delta\vec{\theta}$. Here you are likely to use $\vec{\eta}_{rw}$ and its elements $\vec{\alpha}_{rw}$ and $\vec{\varepsilon}_{rw}$

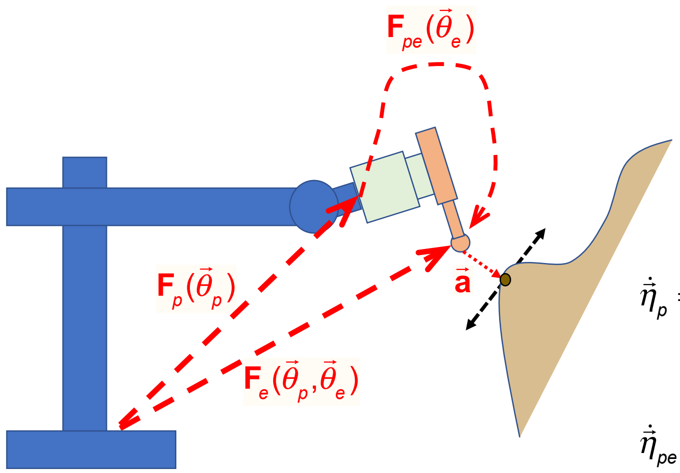
D. **(10 points)** Suppose that $\vec{\mathbf{p}}_{cut}(\lambda)$ is a straight line in the XY plane of $\mathbf{F}_{mc}$ with the form $\vec{\mathbf{p}}_{cut}(\lambda) = \vec{\mathbf{a}}_{cut} + \lambda\vec{\mathbf{d}}_{cut}$. Write linear constraints that enforce the requirement that that the tip of the cutter must stay within a projected distance ρ_{XY} of the nominal path $\vec{\mathbf{p}}_{cut}(\vec{\zeta},\lambda)$.

E. (10 points) Suppose that $\vec{\mathbf{p}}_{cut}(\lambda)$ is defined by a sequence of straight line segments with vertices $P_{cut} = \{\cdots,\vec{\mathbf{a}}_i,\vec{\mathbf{a}}_{i+1},\cdots\}$. How would you modify your answer to Question 2D?

F. (10 points) Write a set of quadratic constraints that enforce the requirement that no part of the cutter can move faster than speed $\sigma_{\max}$.

G. (10 points) How would you linearize these constraints to guarantee that the maximum speed of the cutter is no more than $(1+\gamma)\sigma_{\max}$?

Question 3 (20 points)



$$\dot{\vec{\eta}}_p = \begin{bmatrix} \dot{\vec{\alpha}}_p \\ \dot{\vec{\varepsilon}}_p \end{bmatrix} = \mathbf{J}_p(\vec{\theta}_p)\dot{\vec{\theta}}_p \quad \left|\dot{\vec{\theta}}_p\right| \le \vec{\sigma}_p$$

$$\dot{\vec{\eta}}_{pe} = \begin{bmatrix} \dot{\vec{\alpha}}_{pe} \\ \dot{\vec{\varepsilon}}_{pe} \end{bmatrix} = \mathbf{J}_{pe}(\vec{\theta}_e)\dot{\vec{\theta}}_e \quad \left|\dot{\vec{\theta}}_e\right| \le \vec{\sigma}_e$$

$$\left|\vec{\theta}_e\right| \le \vec{\phi}_e \quad \vec{\sigma}_p << \vec{\sigma}_e$$

Figure 4: Kinematically redundant robot scenario.

Consider the kinematically redundant robot shown in Figure 4. The robot has a moderately slow, but very strong and precise proximal robot and a much faster fine positioning robot carrying a surgical tool. The pose of the tool's coordinate system is given by

$$\mathbf{F}_e(\vec{\theta}_p,\vec{\theta}_{pe}) = \mathbf{F}_p(\vec{\theta}_p)\mathbf{F}_{pe}(\vec{\theta}_e)$$

where $\vec{\theta}_p$ and $\vec{\theta}_e$ are the joints of the proximal and fine positioning robots, respectively. The controller can compute the forward kinematics of both robots in order to obtain $\mathbf{F}_p(\vec{\theta}_p)$ and $\mathbf{F}_{pe}(\vec{\theta}_e)$. It also can compute Jacobeans such that

$$\mathbf{F}_p(\vec{\theta}_p + \Delta\vec{\theta}_p) = \Delta\mathbf{F}(\vec{\theta}_p + \Delta\vec{\theta}_p)\mathbf{F}_p(\vec{\theta}_p) \quad \text{where } \Delta\mathbf{F}_p(\vec{\theta}_p + \Delta\vec{\theta}_p) \approx \mathbf{J}_p(\vec{\theta}_p)\Delta\vec{\theta}_p$$
$$\mathbf{F}_{pe}(\vec{\theta}_e + \Delta\vec{\theta}_e) = \mathbf{F}_{pe}(\vec{\theta}_e)\Delta\mathbf{F}_{pe}(\vec{\theta}_e + \Delta\vec{\theta}_e) \quad \text{where } \Delta\mathbf{F}_{pe}(\vec{\theta}_e + \Delta\vec{\theta}_p) \approx \mathbf{J}_{pe}(\vec{\theta}_e)\Delta\vec{\theta}_e$$

For the purposes of this problem, you can assume that there are no relevant travel limits on $\vec{\theta}_p$ but that there are limits $-\vec{\phi}_e \le \vec{\theta}_e \le \vec{\phi}_e$ on the travel of the fine positioning robot's joints. The speed limits on the joints are given by $-\vec{\sigma}_p \le \dot{\vec{\theta}} \le \vec{\sigma}_p$ and $-\vec{\sigma}_e \le \dot{\vec{\theta}} \le \vec{\sigma}_e$. For instance, the furthest that the joints of the proximal robot can move in time ΔT would be $\Delta T\vec{\sigma}_p$.

The tool is equipped with a sensor that is able to track the displacement $\vec{\mathbf{a}}$ from the tool to an anatomic feature of interest. This feature is on an anatomic object that is moving erratically relative to the robot. The average motion speed is well within the capabilities of the proximal robot to keep up, but there is a superimposed faster motion. The magnitude of this faster motion is within the work volume of the fine positioning robot and the fine positioning robot has sufficient speed to track it, so long as the feature of interest stays within the work volume of the fine positioning robot. Unfortunately, the magnitude of the slower anatomic motion is rather greater than the fine positioning robot's work volume, although. It is within the work volume of the proximal robot.

The mid-level has the following (simplified) control loop:

Step 1. Read the robot state, including the current positions $[\vec{\theta}_p, \vec{\theta}_e]$ and velocities $[\dot{\vec{\theta}}_p, \dot{\vec{\theta}}_e]$.

Step 2. Read the anatomy sensor value $\vec{\mathbf{a}}$.

Step 3. Compute $\mathbf{F}_p(\vec{\theta}_p)$, $\mathbf{F}_{pe}(\vec{\theta}_e)$, $\mathbf{J}_p(\vec{\theta}_p)$, and $\mathbf{J}_e(\vec{\theta}_e)$.

Step 4. Formulate and solve an optimization problem to compute incremental joint motions $\Delta\vec{\theta}_p$ and $\Delta\vec{\theta}_e$. The desired behavior is as follows:

- Minimize the distance $\|\vec{\mathbf{a}}\|$ of the tool from the anatomic object (most important goal).
- Keep the joints $\vec{\theta}_e$ as close as possible to their midpoint as possible (desirable but not crucial, in order to preserve flexibility in responding to unpredictable motions).
- Respect all joint and velocity limits.

Step 5. Output velocities $[\dot{\vec{\theta}}_p = \Delta\vec{\theta}_p / \Delta T, \dot{\vec{\theta}}_e = \Delta\vec{\theta}_e / \Delta T]$ to the joint level controllers.

Your problem (20 points) is to provide the formulation for the optimization problem in Step 4.

Hint: One useful first step is to compute a combined Jacobean for $\mathbf{F}_e$

$$\begin{aligned}
\mathbf{F}_e(\vec{\theta}_p,\vec{\theta}_e)\Delta\mathbf{F}_e(\vec{\theta}_p,\vec{\theta}_e) &= \Delta\mathbf{F}(\vec{\theta}_p+\Delta\vec{\theta}_p)\mathbf{F}_p(\vec{\theta}_p)\mathbf{F}_{pe}(\vec{\theta}_e)\Delta\mathbf{F}_{pe}(\vec{\theta}_e+\Delta\vec{\theta}_e) \\
&= \Delta\mathbf{F}(\vec{\theta}_p+\Delta\vec{\theta}_p)\mathbf{F}_e(\vec{\theta}_p,\vec{\theta}_e)\Delta\mathbf{F}_{pe}(\vec{\theta}_e+\Delta\vec{\theta}_e)
\end{aligned}$$

$$\Delta\mathbf{F}_e(\vec{\theta}_p,\vec{\theta}_e) \approx [\mathbf{I}+sk(\vec{\alpha}_e),\vec{\varepsilon}_e]$$

$$\vec{\eta}_e = \begin{bmatrix} \vec{\alpha}_e \\ \vec{\varepsilon}_e \end{bmatrix} = \begin{bmatrix} \mathbf{J}_e^{(\vec{\alpha})}(\vec{\theta}_p,\vec{\theta}_e) \\ \mathbf{J}_e^{(\vec{\varepsilon})}(\vec{\theta}_p,\vec{\theta}_e) \end{bmatrix} \begin{bmatrix} \Delta\vec{\theta}_p \\ \Delta\vec{\theta}_e \end{bmatrix} = \mathbf{J}_e(\vec{\theta}_p,\vec{\theta}_e) \begin{bmatrix} \Delta\vec{\theta}_p \\ \Delta\vec{\theta}_e \end{bmatrix}$$